Determinants:

- 2x2: \( \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \)

- 3x3: \( \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix} \)

Cross product:

Given \( \vec{v} = <v_1, v_2, v_3> \), \( \vec{w} = <w_1, w_2, w_3> \). We define the cross product \( \vec{v} \times \vec{w} \) to be a vector

\[
\vec{v} \times \vec{w} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}
\]

\[
= \det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} \hat{i} - \det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} \hat{j} + \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \hat{k}
\]

Rule: Cross product of 2 vectors is a vector; Dot product of 2 vectors is a scalar.

Intuitively, \( \vec{v} \times \vec{w} \) is a 3D vector that is orthogonal to both \( \vec{v} \) and \( \vec{w} \). Such vector is not uniquely determined since \(- (\vec{v} \times \vec{w})\) also satisfies the condition. Therefore we...
need to introduce a way to determine the sign. We follow a right-hand rule:

\[ \overrightarrow{v} \times \overrightarrow{w} \]

Use your right hand and curl your fingers from \( \overrightarrow{v} \) to \( \overrightarrow{w} \), your thumb will be in the direction of \( \overrightarrow{v} \times \overrightarrow{w} \).

Properties of cross product:
1. \( \overrightarrow{v} \times \overrightarrow{w} \) is a vector.
2. \( \overrightarrow{v} \times \overrightarrow{w} \perp \overrightarrow{v} \) and \( \overrightarrow{v} \times \overrightarrow{w} \perp \overrightarrow{w} \)
3. \( \|\overrightarrow{v} \times \overrightarrow{w}\| = \|\overrightarrow{v}\| \cdot \|\overrightarrow{w}\| \cdot \sin \theta \)
4. \( \overrightarrow{w} \times \overrightarrow{v} = -\overrightarrow{v} \times \overrightarrow{w} \)
5. \( \overrightarrow{v} \times \overrightarrow{v} = \overrightarrow{0} \)
6. \( \overrightarrow{v} \times \overrightarrow{w} = \overrightarrow{0} \iff \overrightarrow{v} = \lambda \overrightarrow{w} \) or \( \overrightarrow{w} = \mu \overrightarrow{v} \)
7. \( \overrightarrow{v} \times (\overrightarrow{w} + \overrightarrow{u}) = \overrightarrow{v} \times \overrightarrow{w} + \overrightarrow{v} \times \overrightarrow{u} \)
8. \( (\overrightarrow{w} + \overrightarrow{u}) \times \overrightarrow{v} = \overrightarrow{w} \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{v} \)

Properties of dot product:
1. \( \overrightarrow{v} \cdot \overrightarrow{w} \) is a scalar.
2. No such property.
3. \( \overrightarrow{v} \cdot \overrightarrow{w} = \|\overrightarrow{v}\| \cdot \|\overrightarrow{w}\| \cdot \cos \theta \)
4. \( \overrightarrow{w} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{w} \)
5. \( \overrightarrow{v} \cdot \overrightarrow{v} = \|\overrightarrow{v}\|^2 \)
6. \( \overrightarrow{v} \cdot \overrightarrow{w} = 0 \iff \overrightarrow{v} = \overrightarrow{0} \) or \( \overrightarrow{w} = \overrightarrow{0} \)
7. \( \overrightarrow{v} \cdot (\overrightarrow{w} + \overrightarrow{u}) = \overrightarrow{v} \cdot \overrightarrow{w} + \overrightarrow{v} \cdot \overrightarrow{u} \)
8. \( (\overrightarrow{w} + \overrightarrow{u}) \cdot \overrightarrow{v} = \overrightarrow{w} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{v} \)

Special cross product:

\[
\begin{align*}
\hat{i} \times \hat{j} &= \hat{k} \\
\hat{j} \times \hat{k} &= \hat{i} \\
\hat{k} \times \hat{i} &= \hat{j}
\end{align*}
\]

(follow the arrows)
Area of parallelogram spanned by $\vec{v}$ and $\vec{w}$ is

\[ \text{Area}(P) = \| \vec{v} \times \vec{w} \| \]

Area of a parallelepiped $P$ spanned by $\vec{u}$, $\vec{v}$, $\vec{w}$ is

\[ V = | \vec{u} \cdot (\vec{v} \times \vec{w}) | = | \text{det}(\vec{u} \vec{v} \vec{w}) | \]

The quantity $\vec{u} \cdot (\vec{v} \times \vec{w})$ is called the vector triple product.

**Remark:** $\| \vec{v} \times \vec{w} \|^2 = \| \vec{v} \|^2 \| \vec{w} \|^2 - (\vec{v} \cdot \vec{w})^2$

Proof in textbook page 675.

Check right-hand system:

\[ \text{det}(\vec{u} \vec{v} \vec{w}) > 0 \iff \{\vec{u}, \vec{v}, \vec{w}\} \text{ is a right-handed system.} \]

Planes in 3D:

Given a plane $P$ in 3D and a point $P_0 = (x_0, y_0, z_0) \in P$. Let $\vec{n} = \langle a, b, c \rangle$ be a vector that is orthogonal to $P$. Such $\vec{n}$ is called a normal vector. We see that if $P$ is another point in $P$, then $\overrightarrow{PP_0} \cdot \vec{n} = 0$. This actually characterizes
the plane \( P \). Since \( \overrightarrow{PP_0} = \langle x-x_0, y-y_0, z-z_0 \rangle \), \( \vec{n} = \langle a, b, c \rangle \)

\[
0 = \overrightarrow{PP_0} \cdot \vec{n} = a(x-x_0) + b(y-y_0) + c(z-z_0),
\]

or

\[
(a(x-x_0) + b(y-y_0) + c(z-z_0) = 0)
\]

This is called the scalar form of the equation of \( P \).

The equivalent form

\[
\vec{n} \cdot \langle x, y, z \rangle = d \quad (d = \vec{n} \cdot \overrightarrow{PP_0})
\]

is called the vector form.

**Ex:** Find the equation of the plane thru \( P_0 = (0, 1, 2) \) with normal vector \( \vec{n} = \langle -1, 2, 1 \rangle \).

- **Scalar form:**
  
  \[-(x-0) + 2(y-1) + 1(z-2) = 0\]

- **Vector form:**
  
  \[
d = \vec{n} \cdot \overrightarrow{PP_0} = \langle -1, 2, 1 \rangle \cdot \langle 0, 1, 2 \rangle = 4
  \]
  
  So \( \langle -1, 2, 1 \rangle \cdot \langle x, y, z \rangle = 4 \); or
  
  \[-x + 2y + z = 4\]

**Def:**

Points that are on the same line is called collinear.

**Note:** \( P, Q, R \) are collinear \( \iff \overrightarrow{PA} \times \overrightarrow{PR} = \overrightarrow{0} \).
**Plane determined by 3 points:**

Remember that we need a point on the plane and a normal vector to find an equation for the plane. Here we have 3 points. Just pick any point. So we only need to find a normal vector. What do we find it?

\[ \overrightarrow{PQ} \times \overrightarrow{PR} \text{ is a normal vector.} \]

**Question:** any other choices? → Yes. \( \overrightarrow{QP} \times \overrightarrow{QR} \), ...

**Intersection of a plane and a para line:**

Ex: Given a plane: \( x - y + 2z = 1 \)

a line: \( \mathbf{r}(t) = \langle 0, 1, -1 \rangle + t\langle 2, -1, 0 \rangle \)

We can express the line as

\[ x = 2t, \quad y = 1 - t, \quad z = -1 \]

To find their intersection(s), plug these into the equation of the plane:

\[ (2t) - (1 - t) + 2(-1) = 1, \text{ which gives } t = \frac{4}{3}. \]

So \( x = 2t = \frac{8}{3}, \quad y = 1 - t = -\frac{1}{3}, \quad z = -1 \) are the coordinates of the intersection.
Question? What is the number of intersections can a line and a plane have?
- Either 1 intersection or infinitely many (happen when the line is on the plane).

[Image: Trace of the plane]

The intersection of a plane $P$ with a coordinate plane or a plane parallel to a coordinate plane is called a trace.

Ex: draw the plane $x - y + 2z = 1$ and find its traces.

Let $z = y = 0$, we get $x = 1$

$y = x = 0$, $z = \frac{1}{2}$

$x = z = 0$, $y = -1$

So the plane contains the shaded triangle.

Set $z = 0$, then $x - y = 1$. This is the trace on $xy$-plane.

Set $y = 0$, then $x + 2z = 1$. This is the trace on $xz$-plane.

Set $x = 0$, then $-y + 2z = 1$. This is the trace on $yz$-plane.
Quadric surfaces:

- Defined by a quadratic equation in 3 variables:
  \[ Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0 \]

- To draw them, figure out what their traces are. We do this by holding one of the x, y, z constant (in other words, we try to slice the surface vertically, horizontally,...)

- For descriptions and best pictures of special quadric surfaces, see textbook page 691.

Although it is possible to draw these surfaces by slicing, it is best to memorize their general formula and "looks" (even though you know the shapes of the traces, putting them together is quite complicated. Say the hyperbolic paraboloid for example).