**Math 32A Note**

Sections 13.1, 13.2, 13.3

- Distinguish lines, segments, rays, and vectors
  - Line
  - Segment
  - Ray

  - no endpoint
  - 2 endpoints
  - 1 endpoint

→ How about vectors?

- Vector
  - magnitude → also called length
  - direction

Ex: Given 2 points P, Q. The vector from P to Q, denoted by \( \overrightarrow{PQ} \), has
  - magnitude = distance between P, Q
  - direction: from P to Q
    - basepoint
    - terminal point

* Rmk: Can denote \( \overrightarrow{PQ} \) by a single character \( \vec{v} = \overrightarrow{PQ} \).

* Rmk: 2 vectors are equal if they have the same lengths and the same direction.
Components of $P$ and $Q$:
Suppose $P = (x_p, y_p)$, $Q = (x_a, y_a)$, then
$$
\overrightarrow{PQ} = \langle x_t - x_p, y_t - y_p \rangle
$$

$x$-component $y$-component

Length:
Length of the vector $\vec{v} = \langle a, b \rangle$ is
$$
\| \vec{v} \| = \sqrt{a^2 + b^2}
$$
(this is the distance between the basepoint and the terminal)

Zero vector: $\vec{0} = \langle 0, 0 \rangle$ (zero length, no direction)

Vector algebra:
- Add/Subtract vectors: component-wise
  $\langle 1, 2 \rangle + \langle 0, 3 \rangle = \langle 1+0, 2+3 \rangle = \langle 1, 5 \rangle$
  $\langle 0, -1 \rangle - \langle -2, 1 \rangle = \langle 0-(-2), -1-1 \rangle = \langle 2, -2 \rangle$

- Multiply a vector by a scalar (a number):
  $\lambda \vec{v} = \lambda \langle a, b \rangle = \langle \lambda a, \lambda b \rangle$

  Note: $\| \lambda \vec{v} \| = |\lambda| \cdot \| \vec{v} \|$
**Basic properties of vector algebra:**
Review textbook Page 645.

**Parallelogram law:**
Used to add 2 vectors geometrically.

Example: \( \vec{v} \)

**Step 1:** Move (translate) one vector such that their base points coincide.

**Step 2:** Draw a parallelogram.

The sum (or resultant) is the diagonal vector (whose basepoint is the same as the basepoints of \( \vec{v}, \vec{w} \)).

**Question:** How to subtract 2 vectors geometrically?

\[ \vec{v} - \vec{w} = \vec{v} + (-\vec{w}) \]

**Linear combination** of \( \vec{v} \) and \( \vec{w} \) is a vector of the form \( m\vec{v} + n\vec{w} \)
where \( m, n \) are scalars.
Ex: \(3\vec{v} - \vec{w}, \quad 2\vec{v} + \vec{w}, \quad -\vec{v} + 3\vec{w}\), so on...

**Question**: given 2 vectors \(\vec{v}\) and \(\vec{w}\), can a vector \(\vec{u}\) be written as a linear combination of \(\vec{v}\) and \(\vec{w}\)?

Ex: \(\vec{v} = \langle1, 2\rangle\), \(\vec{w} = \langle-2, 1\rangle\) and \(\vec{u} = \langle4, 3\rangle\)

Write \(\vec{u}\) as a linear combination of \(\vec{v}\) and \(\vec{w}\).

\[\vec{u} = m\vec{v} + n\vec{w}\]

\[\langle4, 3\rangle = m\langle1, 2\rangle + n\langle-2, 1\rangle\]

\[= \langle m, 2m \rangle + \langle -2n, n \rangle\]

\[= \langle m - 2n, 2m + n \rangle\]

So \[\begin{cases} m - 2n = 4 \\ 2m + n = 3 \end{cases}\]

First equation implies \(m = 2n + 4\). Plug into the 2nd.

\[2(2n+4) + n = 3\]

So \(n = -1\).

So \(m = 2n + 4 = 2(-1) + 4 = 2\).

Thus \(\vec{u} = 2\vec{v} - \vec{w}\).

**Unit vector**: vector of length 1

- Special unit vectors: \(\hat{i} = \langle1, 0\rangle\), \(\hat{j} = \langle0, 1\rangle\)

- Unit vector in the same direction of \(\vec{v}\):
  \[\hat{\vec{v}} = \frac{\vec{v}}{||\vec{v}||}\] if \(\vec{v} \neq \vec{0}\).
1. Triangle inequality
\[ \| \vec{v} + \vec{w} \| \leq \| \vec{v} \| + \| \vec{w} \| \]
Equality holds if \( \vec{v} = \vec{0} \), or \( \vec{w} = \vec{0} \), or \( \vec{v} = \lambda \vec{w} \), \( \lambda > 0 \).

2. Vectors in 3D:
Almost everything is the same as in 2D:
- Length: \( \vec{v} = \langle a, b, c \rangle \)
  \[ \| \vec{v} \| = \sqrt{a^2 + b^2 + c^2} \]
- Addition/subtraction: component-wise as in 2D
- Multiply by a scalar: same as in 2D.

3. Equation of a sphere:
\[ (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \]
Center: \( (a, b, c) \) and radius = \( R \).

4. Equation of a cylinder:
\[ (x-a)^2 + (y-b)^2 = R^2 \]
Ex: Draw $(x-1)^2 + y^2 = 1$, $z \geq 1$.

Draw the base first $(x-1)^2 + y^2 = 1$.

Now since $z \geq 1$, only parts where the height $\geq 1$ is drawn.

Standard basis vectors in $\mathbb{R}^3$.

$$
\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle
$$

Parametric equation of a line.

- Equation such that each component depends on a parameter $t$.

- A point $P = (x_0, y_0, z_0)$ can also be seen as a vector $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. 
Equation of a line thru a point \( P = (x_0, y_0, z_0) \) in the direction of \( \vec{v} = \langle a, b, c \rangle \) is:

\[
\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle
\]

**Ex:** Find a parametric equation of a line going thru \( P = (0,1,-1) \) with direction \( \vec{v} = (1,2,1) \).

\[
\vec{r}(t) = \langle 0, 1, -1 \rangle + t\langle 1, 2, 1 \rangle = \langle t, 1+2t, -1+t \rangle
\]

So, \( x(t) = t, \ y(t) = 1+2t, \ z(t) = -1+t \).

**Question:** Is this parametrization unique?

\[
\rightarrow \text{No! Since } \vec{v} \text{ can be replaced by any vectors parallel to } \vec{v} \text{ (not } 0). \]

**Intersection of 2 lines:**

**Ex:**

\[
\vec{r}_1(t) = \langle 0, 0, 1 \rangle + t\langle -1, 2, 3 \rangle
\]

\[
\vec{r}_2(t) = \langle -6, 0, 3 \rangle + t\langle -2, 2, 2 \rangle
\]

Set \( \vec{r}_1(t) = \vec{r}_2(t) \Rightarrow \begin{cases} -t = 2t - 6 \\ 2t = 2t \\ 3t + 1 = 2t + 3 \end{cases} \)

All 3 equations have the same solution \( t = 2 \).

So \( \vec{r}_1, \vec{r}_2 \) intersect. The intersection is:

\[
\langle 0, 0, 1 \rangle + 2 \langle -1, 2, 3 \rangle = \langle -2, 4, 7 \rangle.
\]
**Dot product:**

\[ \vec{v} = \langle v_1, v_2, v_3 \rangle, \quad \vec{w} = \langle w_1, w_2, w_3 \rangle \]

\[ \vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3 \]

**Remark:** Dot product of 2 vectors is a **scalar**.

**Properties**

1. \( \vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \) (commutative)
2. \( (\lambda \vec{v}) \cdot \vec{w} = \lambda (\vec{v} \cdot \vec{w}) \)
3. \( \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \) (distributive)
4. \( \vec{v} \cdot \vec{v} = ||\vec{v}||^2 \)
5. \( \vec{v} \cdot \vec{w} = ||\vec{v}|| \cdot ||\vec{w}|| \cdot \cos \theta \)

(\( \theta \) is the angle between \( \vec{v}, \vec{w} \))

\[ 0 \leq \theta \leq \pi \]

**Find angle between 2 vectors:**

\[ \cos \theta = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \cdot ||\vec{w}||}, \quad \text{so} \quad \theta = \cos^{-1} \left( \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \cdot ||\vec{w}||} \right) \]

**Orthogonal:** 2 vectors are perpendicular (or orthogonal) if their angle is \( \frac{\pi}{2} \). Denote \( \vec{v} \perp \vec{w} \).

**Note:** \( \vec{v} \perp \vec{w} \) if and only if \( \vec{v} \cdot \vec{w} = 0 \).
The angle between \( \vec{v}, \vec{w} \) is obtuse \( (\frac{\pi}{2} < \theta \leq \pi) \) if \( \vec{v} \cdot \vec{w} < 0 \).

**Projections:**

\[ \vec{u}_{||v} = \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}. \]

Denoted by \( \text{proj}_v \vec{u} \).

Same idea, define \( \vec{u}_{\perp v} \) which is orthogonal to \( \vec{v} \).

We have

\[ \vec{u} = \vec{u}_{||v} + \vec{u}_{\perp v}. \]

(Decomposition of \( \vec{u} \) w.r.t. \( \vec{v} \))