ON THE DISTRIBUTION OF THE ZEROS OF THE FUNCTION $\zeta(s)$ AND ITS ARITHMETIC CONSEQUENCES[1]

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I. On the zeros of the function ζ and some analogous functions.

1. The function $\zeta(s)$ of Riemann is defined, when the real part of s is greater than 1, by the equation

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right), \tag{1}$$

where p designates successively the different prime numbers; the logarithms are natural logarithms. It is holomorphic in the entire plane, except at the point $s = 1$, which is a simple pole. It isn't zero for values of s whose real part is more than 1, since the right-hand side of the equation 1 is finite. But it admits an infinity of imaginary zeros with real part included between 0 and 1. Stieltjes has proven, in accordance with the estimates of Riemann, that the zeros all have the form $\frac{1}{2} + ti$ (the number t being real); but the proof has never been published, and it hasn't even been established that the function ζ has no zeros on the line[2] $\mathrm{Re}(s) = 1$.

It is this last conclusion that I myself propose to prove.

2. First of all, let s tend to 1 by decreasing real values. The logarithm of $\zeta(s)$, or, to within a finite quantity, the series

$$S = \sum_p \frac{1}{p^s} \tag{2}$$

tends to infinity like $-\log(s-1)$.

Now replace s by $s + ti$ and suppose that the limit point $1 + it$ is a zero of ζ. Then in that case, the real part of $\log(s + ti)$, or in other words (to within a finite quantity), the sum

$$P = \sum_p \frac{1}{p^s} \cos(t \log p) \tag{3}$$

must tend to infinity by negative values like $\log(s-1)$, in other words like $-S$, when s tends to 1 (t remaining fixed).

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[1]The fundamental results of the present Memoire were communications to The Acadamy of Sciences, during the 22 June 1896 session.

[2]$\mathrm{Re}(s)$ designates, as usual, the real part of s.

3. That stated, let α be an angle that we assume to be small; amongst the different prime numbers, distinguish two categories:

1. Those that satisfy, for some integer value of k, the double inequality

$$\frac{(2k+1)\pi - \alpha}{t} \leq \log p \leq \frac{(2k+1)\pi + \alpha}{t}. \tag{4}$$

The part of the sums S_n and P_n (in other words, the series (2) and (3) limited to their n first terms) corresponding to this first category of prime numbers will be designated by S'_n and P'_n.

2. The remaining prime numbers, in other words, those that do not satisfy the double inequality (4) for any value of k, will give, in the sums S_n and P_n, the parts S''_n and P''_n.

Consider the ratio $\rho_n = \frac{S'_n}{S_n}$, which is contained between 0 and 1: when n tends to infinity, the ratio has either a limit or is of bounded oscillation. *If $\zeta(1+it)$ is zero, this ratio or these bounds must tend to 1 with s.* In other words, ρ being some number smaller than 1, we can make a connection between any real value of s greater than 1, but sufficiently close to 1, and a value of n after which

$$\rho_n > \rho. \tag{5}$$

We can, in effect, obviously write:

$$P'_n \geq -S'_n \geq -\rho_n S_n,$$
$$P''_n \geq -S''_n \cos\alpha \geq -(1-\rho_n) S_n \cos\alpha$$

(the inequalities have their algebraic meaning). Therefore if we have

$$\rho_n \leq \rho,$$

it results in

$$P_n = -\theta S_n \text{ (not my fault)}$$

where $\theta = \rho + (1-\rho)\cos\alpha$ is a fixed number smaller than 1; and if that holds for an infinity of values of n, we may pass to the limit and write

$$P \geq -\theta S,$$

which is in contradiction with the hypothesis that $\zeta(1+ti) = 0$, by the remarks under the preceding section.

The equality $\zeta(1+it) = 0$ requires thus indeed that the ρ_n tend to 1 with s.

4. Change then t to $2t$, in the series (3) and set Q to be the new series so obtained: the terms that form, in the series (3), the sums $P'_n, P''_n, P_n = P'_n + P''_n$ give, in this new series, respectively the sums $Q'_n, Q''_n, Q_n = Q'_n + Q''_n$ and we have, this time,

$$Q'_n \geq S'_n \cos 2\alpha \geq \rho_n S_n \cos 2\alpha,$$
$$Q''_n \geq -S''_n \geq -(1-\rho_n) S_n,$$

and, as a consequence,

$$Q_n \geq S_n(\rho_n \cos 2\alpha - (1-\rho_n)):$$

where, in return for the inequality (5) being assumed true for n sufficiently large,

$$Q_n \geq \theta' S_n,$$

θ' designates the number $\rho\cos 2\alpha - (1-\rho)$, which is positive if we have $1 > \rho > \frac{1}{1+\cos 2\alpha}$.

Now this gives $Q \geq \theta' S$ and, as a result, Q increases indefinitely by positive values, in such a way that the limit point $1 + 2ti$ is an infinite point of ζ: we know that this cannot occur.

The impossibility of the hypothesis $\zeta(1+ti) = 0$ is therefore evident.

5. It is remarkable that this proof is based only on simple properties of $\zeta(s)$: to summarize, it is, in effect, exclusively formed from the following remarks: 1. the logarithm of our function is expressible as a series of the form $\sum a_n e^{-\lambda_n s}$, the numbers a_n are all positive; 2. the function is uniform on the right of the boundary of convergence of this series and that single simple pole is not present there.

Any function satisfying these conditions is therefore different from 0 on the boundary line.

Thus, in the preceeding proof, it was uniquely to simplify the writing that we have reduced the right side of the equation (1) to the series S: the proof itself is equally applicable to the whole of $\log \zeta(s)$. Similarly, the prime numbers having been distributed in any fashion into two categories, the numbers in the first category designated by p', those of the second by p'', if the function represented (when the real part of s is greater than 1) by the infinite product

$$f(s) = \frac{1}{\prod_{p'}\left(1 - \frac{1}{p'^s}\right)\prod_{p''}\left(1 + \frac{1}{p''^s}\right)} \tag{6}$$

is holomorphic on the line $\mathrm{Re}(s) = 1$, then it is different from 0 on this line[3].

In effect, the logarithm of the product

$$f(s)\zeta(s) = \frac{1}{\prod_{p'}\left(1 - \frac{1}{p'^s}\right)^2\prod_{p''}\left(1 - \frac{1}{p''^{2s}}\right)}$$

is represented by the series $\sum a_n e^{-\lambda_n s}$ with positive coefficients; this product then satisfies the conditions indicated above.

This case is, for example, that of the function of Schlömilch

$$\sum \frac{(-1)^n}{(2n+1)^s} = \prod_p \frac{1}{1 + \frac{(-1)^{\frac{p+1}{2}}}{p^s}}. \tag{7}$$

6. More generally, we will expand the proposition above to the series introduced in Arithmetique by Dirichlet, and to which we immediately recall, in the completion of certain points, the principle properties.

These series belong to the category of series of the form $\sum \frac{a_n}{n^s}$ *periodic*, in other words which the coefficients a_n themselves repeat every k. Such series are evidently linear combinations of the k functions

$$\begin{aligned}
\xi_1(s) &= \frac{1}{1^s} + \frac{1}{(k+1)^s} + \frac{1}{(2k+1)^s} + \dots,\\
\xi_2(s) &= \frac{1}{2^s} + \frac{1}{(k+2)^s} + \frac{1}{(2k+2)^s} + \dots,\\
&\vdots\\
\xi_k(s) &= \frac{1}{k^s} + \frac{1}{(k+k)^s} + \frac{1}{(2k+k)^s} + \dots,
\end{aligned}$$

studied by MM. Hurwitz[4] and Cahen[5]. These functions are uniform in the entire plane, with a single simple pole at $s = 1$ and corresponding residue $\frac{1}{k}$, thus it results in the expression

$$\xi_r(s) = \frac{i}{2\pi}\Gamma(1-s)\int (-x)^{s-1}\frac{e^{(k-r)x}}{e^{kx}-1}dx, \tag{8}$$

[3] It may be defined at the point $s = 1$, but this circumstance is not presented in the following

[4] *Zeitschrift für Mathematik und Physik*, v. XXVII, p. 86-102; 1882

[5] *Thèse de Doctorat.* 1891 and *Annals de l'École Normale supérieure*, series 3, v. XI.

where the integral is being evaluated along the contour C running from $+\infty$ and ending there after being rotated in the trigonometric direction around the origin, and $-x$ is taken to have (for x real and positive) the argument $-i\pi$ in the first part of the path of integration, and, as a result, $+i\pi$ in the second.

The integral in the preceeding formula represents an entire function of s, and the general theorems given in my memmoire *On the properties of entire functions*[6] allows one to determine its order. To this effect, one can, for example, divide the contour C in two parts: the one C' leaving the point $x = 1$ and returning there after circling around the origin; the other C'' includes the two lines from 1 to $+\infty$; the integral evaluated on these two lines is the same as the integral

$$\int_1^\infty x^{s-1} \frac{e^{(k-r)x}}{e^{kx}-1} dx \tag{9}$$

except for the exponential factors $e^{-i\pi s}$ for the first and $e^{i\pi s}$ for the second. Now, the coefficient of s^n in the integral (9), which has for its value

$$\frac{1}{n!} \int_1^\infty (\log x)^n \frac{e^{(k-r)x}}{e^{kx}-1} \frac{dx}{x},$$

is (since r is a whole number greater than 0) is, at most, comparable to the corresponding coefficient of the function

$$Q(s) = \int_1^\infty x^{s-1} e^{-x} dx$$

that occurs in the study of the function Γ and for which the order of magnitude as s tends to infinity is the same as that of Γ. As for the integral evaluated along C', the coefficient of s^n, which has for its value

$$\frac{1}{n!} \int_{C'} (\log x)^n \frac{e^{(k-r)x}}{e^{kx}-1} \frac{dx}{x},$$

is at most of order $\frac{K^n}{1 \cdot 2 \cdots n}$, we designate by K the maximum modulus of $\log x$ on the contour in question. We see therefore that the function considered is of order 1: the number of zeros of this function, contained in the circle of radius R, is of the order of $R \log R$.

7. When we change s to $1-s$, the new value of the function ξ is expressed as a function of the old by the relations established by M. Hurwitz[7] and which may be taken to be the form[8]

$$\frac{1}{\Gamma(1-s)} \sum_{l=1}^{k} \sigma^{lr} \xi_l(s) = \left(\frac{2\pi}{k}\right)^{s-1} \left(e^{(s-1)\frac{i\pi}{2}} \xi_{k-r}(1-s) + e^{-(s-1)\frac{i\pi}{2}} \xi_r(1-s)\right), (r = 1, 2, \ldots, k), \tag{10}$$

where σ designates $e^{\frac{2i\pi}{k}}$.

8. To define his series, Dirichlet[9] takes the decomposition of the number k in prime factors

$$k = 2^\lambda p^\omega p'^{\omega'} \cdots \qquad (\lambda \geq 0; \omega, \omega', \ldots > 0), \tag{11}$$

and, to all integers n prime with k, assigns the corresponding indices

$$\alpha, \beta, \gamma, \gamma', \ldots$$

[6] *Journal de M. Jordan*, series 4, v. IX; 1893

[7] Hurwitz, *loc. cit.*, p. 93.

[8] Cahen, *loc. cit.*, no. 47, 53.

[9] *Abhandlungen der Berl. Acad.*, 1837; translated by Terquem, *Journal de Lioville*, 1st series, v. IV; 1839. We conform to the notation employed in the *Vorlesungen über Zahlentheorie*, edited by Dedekind, 1863 edition, supplement VI.

defined by the congruences

$$(12)\qquad \begin{cases} n \equiv (-1)^{\alpha} 5^{\beta} & (\text{mod } 2^{\lambda}), \\ n \equiv g^{\gamma} & (\text{mod } p^{\omega}), \\ n \equiv g'^{\gamma'} & (\text{mod } p'^{\omega'}), \\ \dots & \dots, \end{cases}$$

where $g, g', \dots$ are the primitive roots modulo $p^{\omega}, p'^{\omega'}, \dots$ respectively. The numbers α and β are in this way defined modulo a and b: the numbers a and b both having the value 1 if $\lambda = 0, 1$, and take the values $a = 2$ and $b = \frac{1}{2}\varphi(2^{\lambda})$ if $\lambda \geq 2$. Similarly, the numbers $\gamma, \gamma', \dots$ are defined relative to the moduli

$$c = \varphi(p^{\omega}), c' = \varphi(p'^{\omega'}), \dots,$$

where φ is the well-known function expressing the count of numbers prime to a given integer and less than it.

Conversely, knowledge of the indices $\alpha, \beta, \gamma, \gamma', \dots$ makes known the number n, modulo k. In otherwords, to the $\varphi(k)$ values of n relatively prime to k and incongruent amongst themselves modulo k correspond, in a unique way, the

$$abcc' \dots = \varphi(k)$$

systems of values of $\alpha, \beta, \gamma, \gamma', \dots$ incongruent amongst themselves modulo $a, b, c, c', \dots$.

Denote by $\theta, \eta, \omega, \omega', \dots$ respectively an $a^{th}, b^{th}, c^{th}, c'^{th}, \dots$ root of unity, in other words, set

$$(13)\qquad \begin{cases} \theta = \pm 1, \\ \eta = e^{\frac{2i\pi\mu}{b}}, \\ \omega = e^{\frac{2i\pi\tau}{c}}, \\ \omega' = e^{\frac{2i\pi\tau'}{c'}}, \\ \dots. \end{cases}$$

Dirichlet introduced the function

$$\psi_{\nu}(n) = \begin{cases} 0 & \text{if } n \text{ is not prime with } k, \\ \theta^{\alpha}\eta^{\beta}\omega^{\gamma}\omega'^{\gamma'} \cdots & \text{if } n \text{ is prime with } k, \end{cases}$$

$\alpha, \beta, \gamma, \gamma', \dots$ are the indices of n (the index ν is for the purpose of distinguishing one of the $\varphi(k)$ functions ψ corresponding to the different possible choices of $\theta, \eta, \omega, \omega', \dots$).

He subsequently forms the series (periodic in the sense indicated above)

$$(14)\qquad L_{\nu}(s) = \sum_{n=1}^{\infty} \frac{\psi_{\nu}(n)}{n^s} = \sum_{r=1}^{k} \xi_r(s)\psi_{\nu}(r) \qquad (\nu = 1, 2, \dots, \varphi(k)),$$

equal to the infinite product

$$(15)\qquad L_{\nu}(s) = \prod \frac{1}{1 - \frac{\psi_{\nu}(q)}{q^s}},$$

in which q must be replaced successively by all the prime numbers.

The series L_{ν} are distributed into three categories: the first includes the lone series L_1, which corresponds to

$$\theta = \eta = \omega = \omega' = \dots = 1;$$

the second includes all the series L, for which the numbers $\theta, \eta, \ldots$ are equal to $+1$ or to -1 (with the exception of L_1); the third, the series corresponding to the case where at least one of these numbers is imaginary. These last are pairs of conjugates; the series

$$L_\nu(s) = \sum_{r=1}^{k} \xi_r(s)\psi_\nu(r),$$

derived from the roots $\theta, \eta, \omega, \omega', \ldots$, is the conjugate of the series

$$L_{\nu'}(s) = \sum_{r=1}^{k} \frac{\xi_r(s)}{\psi_\nu(r)},$$

derived from the roots $\frac{1}{\theta}, \frac{1}{\eta}, \frac{1}{\omega}, \frac{1}{\omega'}, \ldots$.

The series L_1 admits, as a sole singularity, the simple pole $s = 1$. As for the other series L, they are holomorphic in the entire plane (because the sum $\frac{1}{k}\sum_r \psi_\nu(r)$ of the residues at the point $s = 1$ is zero). Dirichlet proved that they are all different from 0 for $s = 1$.

9. From the general relation (10), M. Hurwitz was able to deduce that certain series of the second category reproduce, to within a factor, through changing s to $1 - s$ in the fashion of the function ζ.

This proposition is a particular case of the theorem proved by M. Lipschitz[10], which is the following: *The series $L_\nu(s)$ is (to within an exponentially and trigonometric factor, analogous to that which is encountered in the related formula for the function ζ) changed to its conjugate by the changing of s to $1 - s$, under the following conditions:*

(1) $\lambda \geq 3$, *μ odd;*
(2) $\tau \neq p - 1$, *if* $\omega = 1$; *τ not divisible by p if* $\omega > 1$;
(3) $\tau' \neq p' - 1$, *if* $\omega' = 1$; *τ' not divisible by p' if* $\omega' > 1$; ...

This theorem gives us important information on the distribution of the zeros of $L_\nu(s)$. Since this function does not have any imaginary zeros with real part greater than s (not my fault), neither does it have any with real part negative: the imaginary zeros are contained in that same band as those of $\zeta(s)$. They are even, like those of $\zeta(s)$, disposed symmetrically by ratio to the line Re $(s) = \frac{1}{2}$, since to any zero α corresponds a zero α' (different or not of the first), such that α and $1 - \alpha'$ are imaginary conjugates.

Nonetheless, this conclusion has not yet been proven in the case where the relation of Lipschitz is not applicable; but we reduce the other cases to this one by the following remarks

(1) If a root ω, for example, is equal to 1, we have

$$L_\nu(s) = (1 - \psi'_\nu(p)p^{-s})L'_\nu(s),$$

the series L'_ν is constructed by removing the factor p^ω from the number k. The same circumstance is produced for the factor 2 when the exponent λ is equal to 1;

(2) If the integer τ is divisible by p^h, the series can be constructed by removing from k the part divisible by p^h, the primitive root g of p^ω is a primitive root of $p^{\omega-h}$. The new value of τ will not contain more than a factor of p. It is then the same for the factor 2 when the integer μ is even, and also when $\lambda = 2, \theta = 1$.

(3) The reasoning of the author is still valid for $\lambda = 2, \theta = -1$; take for the expression[11] $\left(\theta, \psi; e^{\frac{2ri\pi}{2^\lambda}}\right)$ the value $e^{\frac{2ri\pi}{4}} + \theta e^{\frac{-2ri\pi}{4}}$.

[10] *Journal de Crelle.* v. 105, p. 127-157.

[11] *Loc. cit.*, p. 144 formula (9). M. Lipschitz denotes by the letter ψ the quantity that we call η.

Our conclusion is therefore established for all the series L_ν. We could from there develop, on the distribution of the zeros of L_ν, an analogous theory to that of M. von Mangoldt[12]. The sole remark on which the author bases this, besides the properties common to $\zeta(s)$ and the series $L_\nu(s)$, is that the argument of $\zeta(s)$ remains finite when the fixed point s describes the line Re $(s) = a > 1$. Now this property applies equally to functions L_ν. We could thus complete the analysis presented to this consideration[13] by Piltz.

10. The fundamental equation utilized by Dirichlet to prove his theorem is

$$\sum_\nu \frac{\log L_\nu(s)}{\psi_\nu(m)} = \varphi(k)\left(\sum \frac{1}{q^s} + \frac{1}{2}\sum{}' \frac{1}{q^{2s}} + \frac{1}{3}\sum{}'' \frac{1}{q^{3s}} + \dots\right), \tag{16}$$

where m is any integer prime with k and where the symbols $\sum, \sum', \sum'', \dots$ extending, the first to the prime numbers q such that $q \equiv m \pmod k$, the second to the prime numbers q such that $q^2 \equiv m \pmod k$, etc. For $m = 1$, this gives

$$\log \prod_\nu L_\nu(s) = \varphi(k)\left(\sum \frac{1}{q^s} + \frac{1}{2}\sum{}' \frac{1}{q^{2s}} + \frac{1}{3}\sum{}'' \frac{1}{q^{3s}} + \dots\right).$$

Thus *the series of Dirichlet has no zero on the line* Re $(s) = 1$, because the function $\prod_\nu L_\nu(s)$ satisfies the conditions enumerated in 5.

II. Arithmetic consequences.

11. We are indeed far, as we see it, from having proven the assertion of Riemann-Stieltjes; we have not been able to similarly exclude the possibility of an infinity of zeros of $\zeta(s)$ arbitrarily close to the boundary line. Meanwhile, the result of which we have reached suffices, by itself, to demonstrate the principle arithmetic consequences that we have, until now, tried to draw from the properties of $\zeta(s)$.

Immediately we can remark that the equation

$$\sum_p \frac{1}{p^s} = -\log(s-1) + \text{a finite quantity}$$

already provides some information on the distribution of the prime numbers. Let, in effect, a be a number larger than 1, and denote by N_λ the number of prime numbers contained between a^λ and $a^{\lambda+1}$. The left side of the preceeding equation is contained between $\sum_\lambda \frac{N_\lambda}{a^{\lambda s}}$ and $\sum_\lambda \frac{N_\lambda}{a^{(\lambda+1)s}}$. Put $\frac{1}{a^{s-1}} = x$ and note that $s - 1 = \frac{\log \frac{1}{x}}{\log a}$ can here be replaced by $1 - x$, we can write, to within a finite quantity, for x smaller than 1, but tending to 1:

$$\sum \frac{N_\lambda}{a^\lambda} x^\lambda > \log(1-x) > \frac{x}{a}\sum \frac{N_\lambda}{a^\lambda} x^\lambda,$$

from which we deduce that, ϵ being an arbitrarily small positive number, we have an infinite number of times

$$N_\lambda > \frac{(1-\epsilon)a^\lambda}{\lambda}$$

and an infinite number of times

$$N_\lambda < \frac{(1+\epsilon)a^\lambda}{\lambda},$$

[12] *Journal de Crelle.* v. 144.

[13] *Habilitationschrift.* Iéna, 1884.

analogous conclusions to this that, for example, M. Poincaré gives in his *Memoire on the extension to the complex prime numbers the inequalities of M. Tchebicheff*[14], and it suffices, like those, to establish that if the ratio of number x to the sums of the logarithms of the prime numbers smaller than it has a limit, this limit cannot be 1.

The other inequalities may without doubt be drawn from the fact that, for a real number t different from 0, the quantity $\sum \frac{1}{p^s}\cos(t\log p)$ remains finite when s tends to 1.

12. In his previously cited memoire, M. Cahan presents a proof of the theorem stated by Halphen: *The sum of the logarithms of prime numbers less than x is asymptotic to x.* However, his reasoning depends on the proposition of Stieltjes on the reality of the roots of $\zeta\left(\frac{1}{2}+ti\right)=0$. We will see that by slightly modifying the analysis of the author we can establish the same result in total rigor.

To this effect, in place of the integral $\frac{1}{2i\pi}\int_{a-\infty i}^{a+\infty i}\frac{x^z}{z}dz$, equal to 1 or to 0 as x is larger or smaller than 1, we consider the more general integral

$$J_\mu = \frac{1}{2i\pi}\int_{a-\infty i}^{a+\infty i}\frac{x^z}{z^\mu}dz.$$

In this integral, like the first, x is a positive quantity just like a; μ is positive.

When μ is an integer, this integral is evalutated by the same method that J, or J after an integration by parts, is deduced from the identity

$$\frac{1}{z^\mu} = \frac{(-1)^{\mu-1}}{\Gamma(\mu)}\frac{d^{\mu-1}}{dz^{\mu-1}}\left(\frac{1}{z}\right).$$

All of the parts disappear at infinity when integrated and it yields

$$J_\mu = \begin{cases} 0, \text{if } x<1 \\ \frac{1}{\Gamma(\mu)}\log^{\mu-1}x \text{ if } x>1. \end{cases} \tag{17}$$

The same formula can be proven for μ not an integer, in which case it is understood that z^μ must have the determination that is real and positive for $z=0$. For $x<1$, we integrate along the rectangle having one of its sides on the line $\mathrm{Re}(z)=a$ and situated in the region $\mathrm{Re}(z)>a$, the second side of the rectangle tends to infinity like the μ'^{th} power $(0<\mu'<\mu)$ of the first. The result is then evident.

For $x>1$, we will begin by supposing $\mu<1$. We then integrate along the contour ABCDEFGHA (*fig.* 1) composed again of a rectangle having a side AB on the line $\mathrm{Re}(z)=a$, but situated in the region $\mathrm{Re}(z)<a$ and interrupted on its side DA by a bend that goes to the origin and returns following the negative part of the imaginary axis. If the side BC tends to infinity like the μ'^{th} power $(0<\mu'<\mu)$ of AB, the integral evaluated along this side which runs to infinity disappears and it results in

$$J_\mu = \frac{1}{2i\pi}\lim\left(\int_{HG}+\int_{FE}\right).$$

Now, on the path HG, the argument of z is $-\frac{i\pi}{2}$, and on the path FE, $\frac{3i\pi}{2}$. It becomes thus indeed

$$\begin{aligned} J_\mu &= \frac{e^{\frac{\mu i\pi}{2}}-e^{-\frac{3\mu i\pi}{2}}}{2\pi}\int_0^\infty\frac{x^{-it}dt}{t^\mu} \\ &= -\frac{e^{-\frac{\mu i\pi}{2}}\sin\mu\pi}{i\pi}\int_0^\infty\frac{\cos(t\log x)-i\sin(t\log x)}{t^\mu}dt = \frac{\log^{\mu-1}x}{\Gamma(\mu)}. \end{aligned}$$

[14]*Journal de M. Jordan*, series 4, v. VIII, 1892.

This formula, established for $\mu < 1$, extends to the case $\mu > 1$ by an integration by parts deduced from the identity

$$\frac{1}{z^{\mu+m}} = \frac{(-1)^m\Gamma(\mu)}{\Gamma(\mu+m)}\frac{d^m}{dz^m}\left(\frac{1}{z^\mu}\right).$$

13. Parrallel to the path followed by M. Cahen, we apply the formula (17) to the integral

$$\psi_\mu(x) = -\frac{1}{2i\pi}\int_{a-\infty i}^{a+\infty i}\frac{x^z}{z^\mu}\frac{\zeta'(z)}{\zeta(z)}dz, \tag{18}$$

where a is any number greater than 1. By virtue of the development

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_p \log p\left(\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right), \text{ (not my fault)}$$

our formula gives

$$\psi_\mu(x) = \frac{1}{\Gamma(\mu)}\left(\sum \log p \log^{\mu-1}\frac{x}{p} + \sum{}' \log p\log^{\mu-1}\frac{x}{p^2}+\dots\right), \tag{19}$$

the symbol $\sum$ extends to the prime numbers smaller than x, the symbol $\sum'$ extends to the prime numbers smaller than $x^{\frac{1}{2}}$, etc.

14. The advantage that we find to take $\mu > 1$ resides in the convergence of the series $\frac{1}{|\alpha|^\mu}$, where α denotes successively the zeros of $\zeta(z)$, convergences on which rests, as we shall see, the reasoning that will follow.

In this condition, in effect, we can separate from the set of roots α a number M of these quantities, which will be quite large, so the sum $\sum\frac{1}{|\alpha|^\mu}$, over the remaining roots, is smaller than any positive number ϵ. Since none of the α have real part equal to 1, we can (*fig.* 2) trace a parallel CD to the imaginary axis, leaving to its right the parallel $\mathrm{Re}(z) = 1$ and to the left the M first roots α. From the points C, D of this line, we bring out CE, DF parallel to the real axis, parallels which are understood to contain between them the M roots in question, but none of the other roots, and that we lengthen until E, F respectively encounter two lines OEG, OFH issuing from the origin and situated respectively in the two angles formed by the negative part of the real axis with the two directions of the imaginary axis. Finally, we close the contour of integration ABGECDFHA (*fig.* 2) by two variable parallels BG, AH to the real axis (parallels including, of course, CE and DF inside them), rejoining at A, B the line $\mathrm{Re}(z) = a$.

15. I say, in the first place, that we can remove the parallels BG, AH to infinity, in such a fashion that the part of the integral ψ_μ relative to these lines goes to zero.

We can follow, for that one step, as in my memoire on the properties of entire functions[15]. The method that follows differs slightly from that one; the change seems advantageous to me.

Let A be a number larger than unity. Trace along the parallels to the real axis to the distances of this axis represented by $A^3, A^6, \dots, A^{3\lambda}, \dots$. The number[16] of roots α, of which the coordinates are contained between $A^{3\lambda}$ and $A^{3\lambda+3}$ is at most equal to $K\lambda A^{3\lambda}$, the number

[15] *Loc. cit.* no 29 and following

[16] Every root is, of course, included a number of times equal to its order of multiplicity.

K being finite[17], and it is even *a fortiori* of the interval $\left(A^{3\lambda+1}, A^{3\lambda+2}\right)$; in this manner, if we arrange the roots α in increasing order of the coefficients of i, there exists at least two consecutive values for which the coefficients of i differ by a quantity larger than $\frac{A^{3\lambda+2}-A^{3\lambda+1}}{K\lambda A^{3\lambda}} = \frac{A(A-1)}{K\lambda}$.

We trace, at an equal distance between these two roots, a parallel to the real axis of which the ordinal is denoted by z_0, and the difference between z_0 and the ordinal of any root α is greater than $\frac{A(A-1)}{2K\lambda}$.

Now we have

$$\begin{aligned}\frac{\zeta'(z)}{\zeta(z)} &= \sum_\alpha \left(\frac{1}{z-\alpha} + \frac{1}{\alpha}\right) - \sum_\beta \left(\frac{1}{z-\beta} + \frac{1}{\beta}\right) - \frac{1}{z} + C \\ &= \sum_\alpha \frac{z}{\alpha(z-\alpha)} - \sum_\beta \frac{z}{\beta(z-\beta)} - \frac{1}{z} + C\end{aligned} \tag{20}$$

the α denotes the zeros, the β denotes the poles (real and negative) of ζ and C is a constant. When z varies on the segment BG of the parallel of the ordinal z_0, the ratio $\frac{z-\beta}{\beta}$ remains greater than a fixed number, independent of β, and it is likewise for the ratio $\frac{z-\alpha}{\alpha}$, if the ordinate of α is outside the interval $\left(A^{3\lambda}, A^{3\lambda+3}\right)$. The parts corresponding to the right side of the equation (20) give thus the product of z by a finite sum (since the sums $\sum \frac{1}{\alpha^2}$, $\sum \frac{1}{\beta^2}$ are finite).

As for the terms corresponding to the roots α contained between the parallels with ordinates $A^{3\lambda}$ and $A^{3\lambda+3}$ they give, after mentioning it is bounded by $K\lambda A^{3\lambda}\frac{2K\lambda A^3}{A(A-1)}$, a quantity of the form $K'z_0 \log z_0$ (where K' is a new finite number).

We have thus

$$\left|\frac{\zeta'(z)}{\zeta(z)}\right| < K' z_0 \log z_0;$$

which becomes in our integral

$$\left|\int_{BG} \frac{x^z}{z^\mu}\frac{\zeta'(z)}{\zeta(z)} dz\right| < \frac{K' \log z_0}{z_0^{\mu-1}} \int |x^z|\, dz,$$

a quantity infinitely small for z_0 infinite.

16. The integral evaluated along the indefinite line AB can thus be replaced by the integral along the indefinite contour HFDCEG, augmented by the sum of the residues relative to the pole $z = 1$ and to the zeros α not contained between the parallels CE, DF.

The residue relative to the pole $z = 1$ is $-x$.

The residues relative to the zeros α not contained between CE and DF are a sum less than ϵx, where ϵ can be chosen as small as we want, and that independently of x.

As for the integral evaluated along the contour HFDCBG (not my fault), this is infinitely small relative to x. That is evident for the finite part FDCE, where it suffices to remark that $\frac{1}{z^\mu}\frac{\zeta'(z)}{\zeta(z)}$ is finite. On the infinite parts EG, FH, the ratios $\left|\frac{z-\alpha}{\alpha}\right|$, $\left|\frac{z-\beta}{\beta}\right|$ are greater than a fixed number, and, consequently, the quantity $\left|\frac{1}{z}\frac{\zeta'(z)}{\zeta(z)}\right|$ is finite. The integral along these paths is therefore less than $K\int\left|\frac{x^z}{z^{\mu-1}}\right| |dz|$ (the number K is finite), that is to say, a finite quantity, decreasing when x grows.

[17] It is clear that we can dispense with the precautions that we take here and utilize the results obtained by M. von Mangoldt on the distribution of the quantities α; the method of the text has the advantage of it applies whenever we know the order of the function being studied.

$\psi_\mu(x)$ is thus asymptotic to x, since, to make the difference $(x - \psi_\mu(x))$ less than ηx, it suffices to choose $\epsilon < \frac{\eta}{2}$, then x quite large so that the integral $\int_{HFDCEG}$ is smaller than $\frac{\eta}{2}x$.

17. In the expression (19) of $\psi_\mu(x)$, we make an abstraction of the terms contained under the signs $\sum$ other than the first. The number of these signs is, in effect, less than $\frac{\log x}{\log 2}$, and the larger of these sums corresponds to the first, smaller itself than $\log \Gamma\left(1 + x^{\frac{1}{2}}\right) \log^{\mu-1} x$, consequently (to within a finite factor) to $x^{\frac{1}{2}} \log^\mu x$. We thus neglect a quantity less than $x^{\frac{1}{2}} \log^{\mu+1} x$; and the result obtained below can be expressed in this way: *the sum* $\frac{1}{\Gamma(\mu)} \sum \log p \log^{\mu-1} \frac{x}{p}$, *extended to the prime numbers less than x is asymptotic to x.*

This result (where it is understood that we must suppose $\mu > 1$) differs from that of Halphen: *the sum of the logarithms of the prime numbers less than x is asymptotic to x.* We shall see that it is included as a particular case.

18. For that, take $\mu = 2$, this gives

$$\sum_0^x \log p \log \frac{x}{p} = x(1 + \eta),$$

η being (for x quite large) smaller in absolute value than such numbers that we want.

In this relation, change x to $x(1 + h)$ and subtract term-by-term: it becomes

$$\sum_0^x \log p \log(1 + h) + \sum_x^{x(1+h)} \log p \log \frac{x(1 + h)}{p} = x(h + \eta),$$

an equality in which the symbol $\sum_\alpha^\beta F(p)$ denotes the sum of the values of the function F for prime numbers contained between α and β.

For prime numbers in second sum, the quantity $\frac{x(1+h)}{p}$ is contained between 1 and $1 + h$: we can therefore write, on dividing by $\log(1 + h)$,

$$\text{(21)} \qquad \begin{aligned} \sum_0^x \log p &< x\frac{h + \eta}{\log(1 + h)} \\ \sum_0^{x(1+h)} \log p &> x\frac{h + \eta}{\log(1 + h)}. \end{aligned}$$

In this last, change x to $\frac{x}{1+h}$: it becomes

$$\text{(22)} \qquad \sum_0^x \log p > x\frac{h + \eta}{(1 + h)\log(1 + h)}.$$

The formulas (21) and (22) prove the statement of Halphen. We see, in effect, that $\sum_0^x \log p$ is contained between $x(1 + \rho)$ and $x(1 - \rho)$, if we have chosen h such that

$$1 - \frac{\rho}{2} < \frac{h}{(1 + h)\log(1 + h)} < \frac{h}{\log(1 + h)} < 1 + \frac{\rho}{2},$$

then x quite large so that $\eta < \frac{\rho}{2} \log(1 + h)$.

19. The results that preceed extend of themselves to the series of Dirichlet. We consider the integral

$$-\frac{1}{2i\pi}\int_{AB}\left(\sum_{\nu}\frac{1}{\psi_\nu(m)}\frac{L'_\nu(z)}{L_\nu(z)}\right)\frac{x^z}{z^\mu}dz, \tag{23}$$

where μ is a number larger than 1, the other letters having the same sense as in sections 8-10. This integral represents, to within a quantity infinitely small relative to x, the product of $\varphi(k)$ with the sum of the logarithms of the prime numbers congruent to m, modulo k and smaller than x, multiplied respectively by the corresponding values of $\log^{\mu-1}\frac{x}{p}$.

Now we can apply our reasoning to this integral exactly like we have done on the integral (18), since the properties of $\zeta(s)$, that we have utilized and that are related to the distribution of the zeros and order, have been proven for the series of Dirichlet. The quantity that appears under the symbol $\int$ in the integral (23) has a simple pole at $z = 1$ – the pole of $\frac{L'_\nu(z)}{L_\nu(z)}$ – with corresponding residue $-\frac{x}{\psi_1(m)} = -x$; the residues relative to the other poles give a sum that we can consider as negligible face-to-face with x, in the way that the integral is evaluated along the contour GECDFH, as it has been explained.

Thus the integral (23) is asymptotic to x. Along the same lines as the preceeding section we recognize that *the sum of the logarithms of the prime numbers smaller than x and contained in an arithmetic progression determined by common difference k is asymptotic to $\frac{x}{\varphi(k)}$*.

The general equation

$$\frac{1}{\Gamma(\mu)}\sum_0^x \log q \log^{\mu-1}\frac{x}{q} = \frac{x}{\varphi(k)}(1+\rho)$$

that, as we have come to see, includes the relation corresponding to $\mu = 1$, do not seem to be deducible conversely from this one; it is interesting to look for which pieces of information this equation supplies on the order of magnitude of ρ, that is to say, of the error committed by replacing $\sum_0^x \log q$ by its asymptotic value.

20. In conclusion, I note the possible application of the same method to the series of Weber[18] and Meyer[19], for which we extend the theorems of Dirichlet on arithmetic progressions to quadratic forms. After a proof that these series are uniform, the relation[20], analogous to that given previously in section 10, will prove that they do not cancel out on the line $\mathrm{Re}(s) = 1$.

In the case where the determinant is negative, and where we consider the sole quadratic form (without intervention of an arithmetic progression), a formula given by Weber[21] provides the necessary proof; at the same time, it makes known the order of these series and provides the relation corresponding to changing s to $1 - s$. The methods presented in the present memoire are thus presently applicable to this particular case.

Notes. – Throughout the correction from trials, I received communication of the researcher that M. de la Vallée-Poussin devoted to the same subject in the *Annales de la Société scientifique de Bruxelles*[22]. Our reasonning, findings of independent methods, are on some points the same: it is remarkable, in particular, to note that M. de la Vallée-Poussin as well must resort to use as an intermediary the fact that the function ζ has no roots of the form

[18] *Math. Annalen*, v. XX, p. 301

[19] *Journal de Crelle*, v. 103, p. 98; cf. Bachmann, *Analytische Zahlentheorie*, Ch. X; Leipzig, Teubner, 1894.

[20] Bachmann, *loc. cit.*, p. 291, line 6, formula (34).

[21] Bachmann, *loc. cit.*, p. 302, line 4.

[22] Volume XX, 2^{nd} part; 1896.

$1+ti$, although the methods of the proofs are entirely different. I believe that we may not refuse to my method the advantage of simplicity.

The criticisms, addressed by M. de la Vallée-Poussin to the proof based on the use of the integral $\int_{a-\infty i}^{a+\infty i} x^z \frac{dz}{z}$, are not relevant to ours, based on the integral

$$\int_{a-\infty i}^{a+\infty i} x^z \frac{dz}{z^\mu} \qquad (\mu > 1),$$

thanks to the fact that this last remains meaningful, even when we replace every element by its modulo.