Math 110A Take-home Midterm

Do all ten problems below. You may use references (but not other people), although you should make a serious effort to do each problem by yourself. If you copy a proof from a reference, it must be fully sited. If the reference's proof uses results that we have not done, you must do those results also. You must also fill in the details that the reference omits. This is especially true if you use the text for the class.

Below G is always a group and H a subgroup.

1. Suppose that for three consecutive integers i, $(ab)^i = a^i b^i$ for all $a, b \in G$. Show that G is abelian.

2. Show the following:
   a. We know that if G is abelian then every subgroup is normal. Is the converse true (i.e., if every subgroup of G is normal then G is abelian)? Prove or give a counterexample (with details).
   b. Find three distinct groups K, H, G such that $K \lhd H$ and $H \lhd G$ but K is not normal in G.

3. A *commutator* of G is an element of the form $xyx^{-1}y^{-1}$ where $x, y \in G$. Let G' be the subgroup of G generated by all commutators, i.e., every element of G' is the **product** of commutators. We call G' the *commutator* or *derived subgroup* of G. It is also denoted $[G : G]$. Show all the following are true.
   a. $G' \lhd G$.
   b. G/G' is abelian.
   c. If $N \lhd G$ and G/N is abelian then $G' \subset N$.
   d. If $G' \subset H \subset G$ then $H \lhd G$.

4. Recall that

$$Aut(G) = \{\sigma : G \to G \mid \sigma \text{ is an isomorphism}\}.$$

   A subgroup H of G is called *characteristic* in G and denoted $H \lhd\lhd G$ if for every $\sigma \in Aut(G)$, the restriction $\sigma|_H$ lies in $Aut(H)$. Show the following:
   a. If $K \lhd\lhd H$ and $H \lhd G$ then $K \lhd G$.
   b. $Z(G) \lhd\lhd G$.
   c. $G' \lhd\lhd G$.
   d. Inductively define $G^{(n)}$ as follows: $G^{(1)} = G'$. Having defined $G^{(n)}$ define $G^{(n+1)} := (G^{(n)})'$. Then $G^{(n+1)} \lhd\lhd G$.

5. A group G is called *solvable* if there exist subgroups $N_i \subset G$, $i = 1, \ldots, r$, some r satisfying all of the following
   i. $N_i \lhd N_{i+1}$ for $i = 1, \ldots, r-1$.
   ii. $1 = N_1$ and $G = N_r$.
   iii. N_{i+1}/N_i is abelian.

   Show all of the following:
   a. A subgroup of a solvable group is solvable.
   b. The homomorphic image of a solvable group is solvable.
   c. If $N \lhd G$ and both N and G/N are solvable then so is G.
   [If you look this up, try to find the proof that uses the homomorphism theorems. You will need to know how to prove similar results where the use of the homomorphism theorems must be necessary.]

6. Show each of the following (You may assume the Sylow theorems without proving them.):
   a. Any p-group is solvable.
   b. If $|G| = pq, p^2q, p^2q^2$ or pqr with p, q, r primes then G is solvable.

7. Let G be a finite group, $H < G$ a subgroup. Suppose that H is a p-group, p a prime, and $p|[G : H]$. Show that $p|[N_G(H) : H]$ and $H < N_G(H)$. In particular, if G is a p-group and $H < G$ then $H < N_G(H)$.

8. Suppose that H is a proper subgroup of a finite group G. Show that $G \neq \bigcup_{g \in G} gHg^{-1}$.

9. Let S be a G-set. Suppose that both S and G are finite. If $x \in G$ let

$$F_x(S) = F_{\langle x \rangle}(S) := \{s \in S \mid x \cdot s = s\}.$$

   Show the number of orbits N of this action satisfies

$$N = \frac{1}{|G|} \sum_{x \in G} |F_x(S)|.$$

10. Let p be an odd prime. Classify all groups G of order $2p$ up to isomorphism.