HW #5

1.(*) Prove the following Useful Counting Result. Let $H < G$ be a subgroup of a finite group G. Suppose that $|G|$ does not divide $[G : H]!$. Then G contains a proper normal subgroup N such that N is a subgroup of H. In particular, G is not simple.

2.(*) Let $f : A \to B$ be a set map. If $D \subset B$ is a subset then the *preimage* of D in A is the set $f^{-1}(D) := \{a \in A \mid f(a) \in D\}$. Prove the following
Properties of preimages. Let $f : A \to B$ be a set map and $C \subset A$ and $D \subset B$ subsets then
  (i) $C \subset f^{-1} \circ f(C)$ with equality if f is one to one.
  (ii) $f \circ f^{-1}(D) \subset D$ with equality if f is onto.

3.(*) Prove the following form of the Correspondence Principle:
Let $K \triangleleft G$ and $\phi : G \to G/K$ by $g \mapsto gK$. Let L be a subgroup of G/K. Then
  (i) There exists a subgroup H of G containing K with $L = H/K$.
  (ii) If $L \triangleleft G/K$ and H is as in (i), then $H \triangleleft G$.
  (iii) Suppose that H_1, H_2 are two subgroups of G containing K. If $H_1/K = H_2/K$ then $H_1 = H_2$.
  (iv) If G is a finite group and H is as in (i) then $[G : H] = [G/K : H/K] = [G/K : L]$ and $|H| = |K| \cdot |L|$.

4.(*) Let G be a group. Show all of the following:
  a. $Z(G)$ is a subgroup of G. Morevover, $Z(G) \triangleleft G$.
  b. G is abelian if and only if $Z(G) = G$. [Of course, you should have done (a) and (b) already.]
  c. If $a \in G$ let $Z_G(a) = \{x \in G \mid xa = ax\}$, the *centralizer* of a in G. Then $Z_G(a)$ is a subgroup and $Z(G) = \bigcap_{a \in G} Z_G(a)$.
  d. If $a \in G$ let $C(a) := \{xax^{-1} \mid x \in G\}$, the *conjugacy class* of a in G. Show that $a \in Z(G)$ if and only if $C(a) = \{a\}$ if and only if $|C(a)| = 1$ if and only if $G = Z_G(a)$.
  e. If G is a finite group then $a \in Z(G)$ if and only if $|Z_G(a)| = |G|$.

5. Let G be a G-set. If $s_1, s_2 \in G$ satisfy $s_1 = x \cdot s_2$ then $G_{s_1} = xG_{s_2}x^{-1}$