Projective Modules over Local Rings

We need the well-known (whose easy proof we omit)

Modular Law. *Let* $A \subset B$ *and* C *be* R*-modules. Then*
(1) $A \cap (B + C) = B + (A \cap C)$.
(2) *If* $A + C = B + C$ *and* $A \cap C = B \cap C$ *then* $A = B$.

We write $\oplus$ to represent internal direct sum. The key result is the following

Proposition. (Kaplansky) *Let* $C = \bigoplus_{i \in I} C_i$ *in* ${}_R\mathfrak{M}$ *with each* C_i, $i \in I$, *countably generated. Let* M *be a direct summand of* C. *Then* $M = \bigoplus_{j \in J} M_j$ *with each* M_j *countably generated. In particular, any projective* R*-module is a direct sum of countably generated projective* R*-modules.*

Proof. Since any free R-module is the direct sum of a number of copies of R itself, the last statement follows easily from the first.

Write $C = M \oplus N$ in ${}_R\mathfrak{M}$. Let

$$\mathcal{R} := \{B \,|\, B \subset C \text{ in } {}_R\mathfrak{M} \text{ with } \pi_M(B) \subset B, \pi_N(B) \subset B \text{ and } B = \bigoplus_{j \in J} M_j \text{ some } J \subset I\}.$$

Here π_M and π_N are the projections. Note that if $B \in \mathcal{R}$ then B is a direct summand of C and $B = \pi_M(B) \oplus \pi_N(B)$.

Claim. Let $k \in I$. Then there exists a countable subset $J \subset I$ such that $W = \bigoplus_{j \in J} C_j$ lies in $\mathcal{R}$ and $k \in J$. In particular, W is countably generated:

Define countable subsets J_n of I and R-modules W_n as follows:

$J_o = \{k\}$ and $W_o = C_k$
If $J_n \subset I$ countable has been defined let $W_n = \bigoplus_{j \in J_n} C_j \subset C$, a countably generated submodule. In particular, $\pi_M(W_n)$ and $\pi_N(W_n)$ are also countable generated. Thus there exists a countable subset J_{n+1} of I containing J_n and satisfying $\pi_M(W_n) \subset W_{n+1}$ and $\pi_N(W_n) \subset W_{n+1}$.

Let $J = \cup J_n$ and $W = \bigoplus_{j \in J} C_j$. Then countable J and $W \in \mathcal{R}$ satisfy the claim.

Let

$$\mathcal{S} := \{B \,|\, B \in \mathcal{R} \text{ and } \pi_M(B) \text{ a direct sum of countably generated modules}\}.$$

Note that $0 \in \mathcal{S}$. Next let

$$\mathcal{T} := \{(B, \bigoplus_{i \in L} M_i) \,|\, B \in \mathcal{S} \text{ each } M_i \text{ is countably generated and } \pi_M(B) = \bigoplus_{i \in L} M_i\}.$$

Partially order $\mathcal{T}$ by $(B, \bigoplus_{i \in L} M_i) \leq (B', \bigoplus_{i \in L'} M_i')$ if $B \subset B'$, $L \subset L'$, and $M_i = M_i'$ if $i \in L$.

By Zorn's Lemma, there exists $(A, \bigoplus_{i \in L} M_i) \in \mathcal{T}$ maximal.

Claim. $A = C$ (and hence we are done):

Suppose not, i.e., $A < C$. Then there exists a $k \in I$ such that $C_k \not\subset A$. Let $W = \bigoplus_{j \in J} C_i \in \mathcal{R}$ be the countably generated module of the previous claim with $k \in J$, a countable subset of I. Write $B = A + W$ and check that $B \in \mathcal{R}$. Since $\pi_M(A)$ is a direct summand of A and A is a direct summand of C, we have $\pi_M(A)$ is a direct summand of C. Since $\pi_M(A) \subset \pi_M(B)$, it follows by the Modular Law that $\pi_M(B) = \pi_M(A) \oplus P$ for some P. Similarly, $\pi_N(B) = \pi_N(A) \oplus Q$ for some Q. Consequently,

$$B = \pi_M(B) \oplus \pi_N(B) = \pi_M(A) \oplus \pi_N(A) \oplus P \oplus Q = A \oplus P \oplus Q.$$

Since $B/A \cong (A + W)/A \cong W/(W \cap A)$ is countably generated and $B/A \cong P \oplus Q$, we conclude that P is countably generated. Thus $(A, \bigoplus_{i \in L} M_i) < (B, \bigoplus_{i \in L} M_i \oplus P)$ in $\mathcal{T}$. This contradicts maximality so $A = C$ as needed. □

Lemma 1. *Let $C \in {}_R\mathfrak{M}$ be countably generated. Suppose that for every direct summand B of C and $x \in B$, there exists a direct sum decomposition $B = F \oplus A$ in ${}_R\mathfrak{M}$ with F a free R-module containing x. Then C is R-free.*

Proof. Let $x_1, x_2, ..., x_n, ...$ be a countable generating set for C. We inductively define R-modules A_n and F_n with F_n a free R-module. Let

$F_o = 0$ and $A_o = C$

Having defined $F_1, ..., F_n, A_n$ such that $x_1, ..., x_n$ lie in $\bigoplus_{i=1}^n F_i$ and $C = \bigoplus_{i=1}^n F_i \oplus A_n$, define F_{n+1} and A_{n+1} as follows:

Write $x_{n+1} = f_n + a_n$ with $f_n \in \bigoplus_{i=1}^n F_i$ and $a_n \in A_n$. By hypothesis, $A_n = F_{n+1} \oplus A_{n+1}$ for some free R-module F_{n+1} containing a_n.

Then $C = \bigoplus_{i=1}^{n+1} F_i \oplus A_{n+1}$. It follows that $C = \bigoplus_{i=1}^{\infty} F_i$ is free. □

Lemma 2. *Let R be a local ring and P a projective R-module. Let $x \in P$. Then there exists a free R-module D containing x such that $P = D \oplus A$ in ${}_R\mathfrak{M}$.*

Proof. Let $P \oplus Q = F$ be free. Among all (ordered) bases of F choose a basis $\{f_i\}_{i \in I}$ for F in such a way that x is a linear combination of the fewest number of basis elements, $f_1, f_2, \ldots$. Thus $x = \displaystyle\sum_{i=1}^{n} a_i f_i$ with each a_i a non-zero element of R and n minimal. Let $F_{n+1} = \displaystyle\bigoplus_{i > n+1} Rf_i$. Suppose that $a_n = \displaystyle\sum_{i=1}^{n-1} a_i r_i$, a right R-linear combination of $a_1, ..., a_{n-1}$. Then

$$f_1 + r_1 f_n, f_2 + r_2 f_2, ..., f_{n-1} + r_{n-1} f_n, f_n, f_{n+1}, \cdots$$

is another basis for F and $x = \sum_{i=1}^{n-1} a_i(f_i + r_i f_n)$. This contradicts the choice of basis. It follows that no a_i is a right R-linear combination of the other a_j. Let $\pi_P : F \to P$ be the projection. Let $y_i = \pi_P(f_i)$. Then $x \in P$ so

$$\sum_{i=1}^{n} a_i f_i = x = \pi_P(x) = \sum_{i=1}^{n} a_i y_i.$$

Since the y_i lie in F,

$$y_i \equiv \sum_{j=1}^{n} c_{ij} f_j \ (\bmod F_{n+1})$$

some c_{ij} in R. It follows that

$$a_i = \sum_{j=1}^{n} a_j c_{ji}.$$

Since no a_j is a right linear combination of the others and $J(R) = R \setminus R^{\times}$, we see that c_{ij} lies in $J(R)$ for $i \neq j$ and $1 - c_{ii}$ lies in $J(R)$, i.e., c_{ii} does not lie in $J(R)$ so is a unit.

Claim. (c_{ij}) lies in $GL_n R$, i.e., is invertible.

The matrix (c_{ij}) in $\mathbf{M}_n R$, corresponds to an R-endomorphism $f : R^n \to R^n$. (Write f on the right so that you do not have to go to the opposite ring.) Let $\overline{R} = R/J(R)$, a division ring, and $^- : R \to \overline{R}$. Then $(\overline{c_{ij}})$ is invertible by the above, so the induced map $\overline{f} : \overline{R}^n \to \overline{R}^n$ is an automorphism. Since R^n is a fg R-projective, it follows that f is an isomorphism. Thus $y_1, ..., y_n$ form a basis for F/F_{n+1}. Let $D = \sum_{i=1}^{n} R y_i$, a free submodule of F, satisfying $D \oplus F_{n+1} = F$. Then $P = D \oplus (F_{n+1} \cap P)$ and $x \in D$ as needed □

Theorem. (Kaplansky) *If R is local then any projective R-module P is free.*

Proof. By the theorem we may assume that P is countably generated. Let $P = B \oplus C$ in ${}_R\mathfrak{M}$ with $x \in B$. So B is projective and $B = D \oplus A$ in ${}_R\mathfrak{M}$ for some free R-module D containing x by Lemma 2. It follows by Lemma 1 that P is free. □