

## Commutative Algebra Problems

Prove the following

1. If  $\mathfrak{A} < R$  is an ideal then  $V(\mathfrak{A})$  is irreducible if and only if  $\sqrt{\mathfrak{A}} \in \text{Spec } R$ . In particular,  $\text{Spec } R$  is irreducible if and only if  $\text{nil } R \in \text{Spec } R$ .
2.  $\text{Spec } R$  is the disjoint union of two closed sets if and only if there exist an idempotent  $e \neq 0, 1$  in  $R$ . In particular, if  $R = (R, \mathfrak{m})$  is local then  $\text{Spec } R$  is connected.
3.  $V(\mathfrak{p}) \subset \text{Spec } R$  with  $\mathfrak{p} \in \text{Min } R$  are the maximal irreducible subspaces (called *irreducible components* of  $\text{Spec } R$ ). Then  $\text{Spec } R = \bigcup_{\text{Min } R} V(\mathfrak{p})$ . If  $R$  is noetherian then  $|\text{Min } R| < \infty$ .
4. Let  $\varphi : A \rightarrow B$  be a ring homomorphism. Let  $\mathfrak{P} \in \text{Spec } B$  and  $\mathfrak{p} = {}^a\varphi(\mathfrak{P})$ . Let  $\varphi_{\mathfrak{P}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{P}}$  the induced ring homomorphism given by  $\frac{a}{b} \mapsto \frac{\varphi(a)}{\varphi(b)}$ . Show that the following are equivalent:
  - (i)  $\varphi$  is flat.
  - (ii)  $\varphi_{\mathfrak{P}}$  is flat for all  $\mathfrak{P} \in \text{Spec } B$ .
  - (iii)  $\varphi_{\mathfrak{m}}$  is flat for all  $\mathfrak{m} \in \text{Max } B$ .
5. If  $A \subset_i B$  are domains with the same quotient field  $F$  then  $i$  is faithfully flat if and only if  $A = B$ . Suppose that  $A < F$ . Show that  $F$  is  $A$ -flat but not  $A$ -projective. [You can use that a projective module over a local ring is free.]
6. Let  $(B_{ij}, \psi_{ij})$  be a direct system of rings and  $B$  the direct limit. For each  $i$  let  $\varphi_i : A \rightarrow B_i$  be a ring homomorphism such that  $\psi_{ij} \circ \varphi_i = \varphi_j$  whenever  $i \leq j$  (i.e., a direct system of  $A$ -algebras). Then the  $\varphi_i$  induce  $\varphi : A \rightarrow B$ . Show that

$${}^a\varphi(\text{Spec } B) = \bigcap_i {}^a\varphi_i(\text{Spec } B_i).$$

[Let  $\mathfrak{p} \in \text{Spec } A$ . Then, using tensor products and direct limits commute,  ${}^a\varphi^{-1}(\mathfrak{p})$  is the spectrum of

$$B \otimes_A k(\mathfrak{p}) \cong \varinjlim (B_i \otimes_A k(\mathfrak{p}))$$

where  $k(\mathfrak{p})$  is the quotient field of  $A/\mathfrak{p}$ . Show  ${}^a\varphi^{-1}(\mathfrak{p}) = \emptyset$  if and only if  $B_i \otimes_A k(\mathfrak{p}) = 0$  for some  $i$ , i.e., if and only if  ${}^a\varphi_i^{-1}(\mathfrak{p}) = \emptyset$ .]

7. Let  $\varphi : A \rightarrow B$  be a ring homomorphism. Consider the following three statements:
  - I.  ${}^a\varphi$  is a closed map.
  - II.  $\varphi$  satisfies GU.
  - III. If  $\mathfrak{P} \in \text{Spec } B$  then the induced map  $\bar{\varphi} : A/{}^a\varphi(\mathfrak{P}) \rightarrow B/\mathfrak{P}$  satisfies LO.

Then (I)  $\Rightarrow$  (II)  $\Leftrightarrow$  (III).

If  $A$  is noetherian and  $B$  is of finite type over  $A$  then (II)  $\Rightarrow$  (I).

8. Let  $\varphi : A \rightarrow B$  be a ring homomorphism. Consider the following three statements:
  - I.  ${}^a\varphi$  is an open map.
  - II.  $\varphi$  satisfies GD.
  - III. If  $\mathfrak{P} \in \text{Spec } B$  then the induced map  $\tilde{\varphi} : A_{{}^a\varphi(\mathfrak{P})} \rightarrow B_{\mathfrak{P}}$  satisfies LO.

Then (I)  $\Rightarrow$  (II)  $\Leftrightarrow$  (III).

[To prove (I)  $\Rightarrow$  (III), observe that  $B_{\mathfrak{q}}$  is the direct limit of the rings  $B_s$  where  $\mathfrak{q} \in D(s)$ . By Exercise 6,

$${}^a\varphi(\text{Spec } B_{\mathfrak{q}}) = \bigcap_{s \in D(\mathfrak{q})} {}^a\varphi_s(\text{Spec } B_s) = \bigcap_{s \in D(\mathfrak{q})} {}^a\varphi_s(Y_s),$$

where  $Y = \text{Spec } B$  and  $Y_s = \text{Spec } B_s$  with  $\varphi_s$  the composition of  $\varphi$  and the obvious localization map. Since  $Y_s$  is an open neighborhood of  $\mathfrak{q}$  in  $Y$  and since  ${}^a\varphi$  is open, it follows that  ${}^a\varphi(Y_s)$  is an open neighborhood of  $\mathfrak{p}$  in  $X = \text{Spec } A$  and therefore contains  $\text{Spec } A_{\mathfrak{p}}$ .]

9. Let  $\varphi : B \rightarrow B'$  be an integral homomorphism of  $A$ -algebras. If  $C$  is an  $A$ -algebra show that  $\varphi \otimes 1_C : B \otimes_A C \rightarrow B' \otimes_A C$  is integral.
10. Let  $A$  be a normal domain,  $F$  its quotient field. Let  $L/F$  be a finite galois extension. Let  $\mathfrak{P} \in \text{Spec } A_L$  lie over  $\mathfrak{p} \in \text{Spec } A$ . Let

$$G_{\mathfrak{P}} := \{\sigma \in G(L/F) \mid \sigma(\mathfrak{P}) = \mathfrak{P}\}$$

be the *Decomposition Group* of  $\mathfrak{P}$ . Show

- i. The natural map  $(A/\mathfrak{p})_{\mathfrak{p}} \rightarrow (A_L^{G_{\mathfrak{P}}}/(\mathfrak{P} \cap L^{G_{\mathfrak{P}}}))_{\mathfrak{p}}$  is an isomorphism.
- ii. The quotient field of  $L(\mathfrak{P})$  of  $A_L/\mathfrak{P}$  is a normal extension of the quotient field  $F(\mathfrak{p})$  of  $A/\mathfrak{p}$ .
- iii. If  $\bar{\cdot} : A_L \rightarrow A_L/\mathfrak{P}$  is the canonical map then we have a group epimorphism

$$G_{\mathfrak{P}} \rightarrow G(L(\mathfrak{P})/F(\mathfrak{p})) \text{ by } \sigma \mapsto \bar{\sigma}$$

$$\text{where } \bar{\sigma} : \frac{\bar{a}}{\bar{b}} \mapsto \frac{\overline{\sigma(a)}}{\overline{\sigma(b)}}.$$

11. Let  $M$  be a finitely generated  $R$ -module and  $S \subset R$  be a multiplicative set. Then  $\text{ann}_{S^{-1}R} S^{-1}M = S^{-1} \text{ann}_R M$ .
12. Let  $R$  be a domain with quotient field  $F$ . Suppose for all rings  $R \subset A \subset F$  that  $A$  is noetherian. Show that  $\dim R \leq 1$ .
13. Let  $\varphi : A \rightarrow B$  be a ring homomorphism and  $\mathfrak{p}$  a prime ideal in  $A$ . Show that the map

$$({}^a\varphi)^{-1}(\mathfrak{p}) \rightarrow \text{Spec}(B \otimes_A k(\mathfrak{p})) \quad \text{by} \quad \mathfrak{P} \mapsto \mathfrak{P}^* := \mathfrak{P}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$$

is a homeomorphism and

$$(B \otimes_A k(\mathfrak{p}))_{\mathfrak{P}^*} = B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}$$

14. Prove that any regular local ring is a domain.
15. Let  $A \subset B$  be noetherian domains with  $B$  a finitely generated  $A$ -algebra. Let  $\mathfrak{P} \in \text{Spec } B$  and  $\mathfrak{p} = A \cap \mathfrak{P}$ . Show

$$\dim B_{\mathfrak{P}} + \text{tr deg}_{A/\mathfrak{p}} B/\mathfrak{P} \leq \dim A_{\mathfrak{p}} + \text{tr deg}_A B.$$