Take Home Midterm and Homework #7

Instructions: This homework/midterm is open book. All proofs must be fully written or typed out. You may not cut and paste solutions that you did not write or type yourself. If you copy part of or the whole proof from a reference you must cite that reference when used. If results not proven in class or homework are used, you must prove these also. You may not help each other or have others help solve the problems for you.

In the problems below, all inner product spaces are defined over F equal to $\mathbb{R}$ or $\mathbb{C}$ (but you cannot assume which one unless it is given).

Problem 1. Let V be a finite dimensional vector space over F and $T : V \to V$ a linear operator.

(a) Show that there exists a positive integer N and $a_0, a_1, \ldots, a_N$ in F not all zero such that $a_N T^N + a_{N-1}T^{N-1} + \cdots + a_1 T + a_0 1_V$ is the zero linear operator, i.e., if $f = a_N t^N + a_{N-1}t^{N-1} + \cdots + a_1 t + a_0$, then $f(T) = 0$.

(b) Show that there is a unique monic polynomial $q \in F[t]$ such that $q(T) = 0$ and if $g \in F[t]$ is any polynomial satisfying $g(T) = 0$, then q divides g in $F[t]$, i.e., $g = qh$ for some $h \in F[t]$. (You may use the Division Algorithm without proving it.)

Problem 2. Let $V = \mathbb{R}^3$ under the dot product. Let $T : V \to V$ be rotation by the angle θ counterclockwise in the plane (through the origin) perpendicular to $(1, 2, 3)$, $S : V \to V$ the reflection with respect to the plane (through the origin) spanned by the vectors $(1, 0, 1), (1, 2, 1)$ (i.e., this plane is fixed and vectors perpendicular to it are send to their negative), and $U : V \to V$ rotation by the angle φ counterclockwise in the plane (through the origin perpendicular to $(1, 0, 1)$. Find the matrix representation of $U \circ S \circ T$ in the standard basis. [You do not have to multiply out the matrices but use Problem 8 to compute inverses.]

Problem 3. Let V be an inner product space. Prove that $d : V \times V \to \mathbb{R}$ defined by $d(v, w) = ||v - w||$ makes V into a **metric space**, i.e., for all $v, w, x \in V$, d satisfies all of the following:

(a) $d(v, w) \geq 0$ and equals zero if and only if $v = w$.

(b) $d(v, w) = d(w, v)$.

(c) (Triangle Inequality) $d(v, w) \leq d(v, x) + d(x, w)$.

Problem 4. Let $V = \mathbb{R}[x]$ be the space of polynomial functions. Make V into an inner product space by $\langle f, g\rangle = \int_{-1}^{1} fg$. Gram-Schmidt $\{1, x, x^2, x^3\}$ to an OR (then an ON) set.

Problem 5. Let V be a finite dimensional inner product space with an orthonormal basis $\{v_1, ..., v_n\}$. Let $v, w \in V$. Prove Parseval's Formula:

$$\langle v, w\rangle = \sum_{i=1}^{n} \langle v, v_i\rangle \overline{\langle w, v_i\rangle}.$$

In particular, the Pythagorean Theorem holds, viz.,

$$||v||^2 = \sum_{i=1}^{n} |\langle v, v_i\rangle|^2.$$

Problem 6. Let $T : V \to W$ be a linear transformation of finite dimensional inner product spaces of the same dimension with respective inner products $\langle\, , \,\rangle_V$ and $\langle\, , \,\rangle_W$.

(a) Show the following are equivalent:
  - (i) $\langle T(v_1), T(v_2)\rangle_W = \langle v_1, v_2\rangle_V$ for all $v_1, v_2 \in V$. We say that T **preserves the inner product**.
  - (ii) T is an isomorphism of vector spaces preserving inner products. We say that T is an **isometry**.
  - (iii) T takes every ON basis of V to an ON basis of W.
  - (iv) T takes some ON basis of V to an ON basis of W.
  - (v) $||Tv|| = ||v||$ for every $v \in V$.

(b) Show two finite dimensional inner product spaces over F are **isometric** (i.e., there exists an isometry between them) if and only if they have the same dimension.

Problem 7. Let V be a finite dimensional inner product space with an orthonormal basis $\{v_1, ..., v_n\}$ and $S, T : V \to V$ be two linear operators. Let $A = [T]_{\mathcal{B}}$ and $B = [S]_{\mathcal{B}}$ in $\mathbb{M}_n(F)$ viewed as operators on $F^{n\times 1}$ via the dot product (with $\mathcal{S}$ the standard basis). Then show all of the following:

(a) $\langle T(v_i), v_j\rangle = A_{ji} = Ae_i \cdot e_j$ and $\langle v_i, S(v_j)\rangle = (B^*)_{ji} = e_i \cdot Be_j$ for all $1 \le i, j \le n$.

(b) There exists a unique linear operator $T^* : V \to V$ such that $[T^*]_{\mathcal{B}} = A^*$ and it satisfies

$$\langle T(v_i), v_j\rangle = \langle v_i, T^*(v_j)\rangle \quad \text{for } i = 1, \ldots, n.$$

(c) The linear operator $T^* : V \to V$ in (b) satisfies

$$(*) \qquad \langle Tv, w\rangle = \langle v, T^*w\rangle \quad \text{for all } v, w \in V.$$

(d) T^* is the unique linear operator satisfying (*). T^* is called the **adjoint** of T.

(e) $T^{**} := (T^*)^* = T$.

Problem 8. Let $A \in \mathbb{M}_n(F)$ be a matrix whose columns form an ON basis for $F^{n\times 1}$ under the dot product. Show A is invertible and the inverse of A is A^*.

Problem 9. Show explicitly that the adjoint of a linear operator $T : V \to V$ with V a finite dimensional inner product space is independent of the orthonormal basis $\mathcal{B}$ used to construct it in Problem 7, i.e., if you compute $[T]_{\mathcal{C}}$ and $[T^*]_{\mathcal{C}}$ in another orthonormal basis $\mathcal{C}$ of V, then (b) of Problem 7 holds by for the elements in $\mathcal{B}$ by direct computation using the basis $\mathcal{C}$.

Problem 10. Let V be a finite dimension inner product space and W a subspace of V. We know that $V = W \perp W^\perp$, i.e., if $v \in V$, then there exist unique $w \in W$ and $w^\perp \in W^\perp$ satisfying $v = w + w^\perp$.

(a) Let $T : V \to V$ be the linear operator satisfying $v = w + w^\perp \mapsto w - w^\perp$. Prove that T is both hermitian and an isometry. (See Problem 6 for the definition of an isometry.)

(b) Let $T : V \to V$ be a linear operator that is both hermitian and an isometry. Show that there exists a subspace W of V such that $T(v) = w - w^\perp$ where $v = w + w^\perp$ with $w \in W$ and $w^\perp \in W^\perp$.