Homework #8

Problem 1. Let $V = C[1,3]$ with $\langle f, g\rangle = \int_1^3 fg$. Let $f(x) = \frac{1}{x}$. Show that the constant polynomial g nearest f is $g = \frac{1}{2}\ln 3$. Compute $||g - f||^2$ for this g.

Problem 2. Let $V = C[0, 2\pi]$ with $\langle f, g\rangle = \int_0^{2\pi} fg$. Let $W = \{1, \cos x, \sin x\}$. Let $f(x) = x$. Find $f(x)_W$.

Problem 3. Let $V = \mathbf{M}_n\mathbf{C}$ with $\langle A, B\rangle = \mathrm{tr}(AB^*)$. Let

$$S = \{\begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix} \mid \lambda_1, ..., \lambda_n \in \mathbf{C}\}$$

(i.e., the *diagonal matrices*). Determine $S^\perp$ and $\dim(S^\perp)$.

Problem 4. Let $V = C[-1,1]$ an inner product space via $\langle f, g\rangle = \int_{-1}^1 fg$. Let $W_{odd} := \{f \in V \mid f(x) = -f(-x)\}$. Show $W_{odd}^\perp = W_{even} := \{f \in V \mid f(x) = f(-x)\}$.

Problem 5. Let V be an inner product space and $W \subset V$ a finite dimensional subspace. Show that $\langle v_W, x\rangle = \langle v, x_W\rangle$ for all $v, x \in V$.

Problem 6. Let V be an inner product space and $S \subset V$ a subset. Show

a. $\mathrm{Span}(S) \subset (S^\perp)^\perp$.

b. $\mathrm{Span}(S) = (S^\perp)^\perp$ if V is finite dimensional.

Problem 7. Let V be a finite dimensional inner product space over F and W a subspace of V. Show that $P_W : V \to V$ defined by $v \to v_W$ is a linear operator and satisfies all of the following:

(i) $\mathrm{im}(P_W) = W$ and $\ker(P_W) = W^\perp$. In particular, $V = \mathrm{im}(P_W) \oplus \ker(P_W)$.

(ii) $P_W \circ P_W = P_W$.

(iii) If $W' \subset W^\perp$ is a subspace, then $P_W \circ P_{W'} = 0$.

(iv) $1_V = P_W + P_{W^\perp}$.

Problem 8. Let A be an $n \times n$ real matrix. Let $row(A)$ be the subspace of $\mathbf{R}^n$ spanned by the rows of A and $col(A)$ be the subspace of $\mathbf{R}^n$ spanned by the columns of A viewed in $\mathbf{R}^n$. Note that if R_i is the ith row of A and $v \in \mathbf{R}^{n\times 1}$, then $R_i v$ is the dot product of R_i and v. Show that $\ker A = (row(A))^\perp$.

Problem 9. Let A be an $m \times n$ real matrix. Show that $(A^tA)^t = A^tA$ and if A has rank n, then A^tA is an invertible $n \times n$ matrix (see Problem 8).

Problem 10. Let V be an inner product space over F. Let $S = \{v_1, ..., v_n\}$ be an orthogonal set. Suppose that no $v_i = 0$. Let $v \in V$. Prove Bessel's Inequality:

$$\sum_{i=1}^n \frac{|\langle v, v_i\rangle|^2}{||v_i||^2} \le ||v||^2.$$

Moreover, show equality holds if and only if

$$v = \sum_{i=1}^n \frac{\langle v, v_i\rangle}{||v_i||^2} v_i.$$