Homework #5

Read Hoffman and Kunz Sections 3.5 and 3.6. We shall go over material from these sections in the 4:00 session Monday.

Problem 1. For each of the following linear transformations T, compute the **rank** of $T :=$ dim (im T), the **nullity** of $T :=$ dim (ker T) [Use the Dimension Theorem], and find the matrix representation relative to the standard bases. Also tell which of these are monomorphisms, epimorphisms, or isomorphisms.

(a) $T : \mathbf{R}^3 \to \mathbf{R}^4$ defined by $T(x, y, z) = (x - z, x + z, 2x + 2z, z)$.

(b) $T : \mathbf{R}^4 \to \mathbf{R}^2$ defined by $T(x, y, z, w) = (x + y - z - w, x + z)$.

(c) $T : \mathbf{R}^4 \to \mathbf{R}$ defined by $T(x, y, z, w) = 0$ for all $(x, y, z, w) \in \mathbf{R}^4$.

(d) $T : \mathbf{R} \to \mathbf{R}$ defined by $T(x) = 3x$.

(e) $T : \mathbf{R}^5 \to \mathbf{R}^5$ defined by $T(v) = -v$ for all $v \in \mathbf{R}^5$.

Problem 2. Let $V = \mathbf{C}^2$. Let $\mathcal{S} = \{e_1, e_2\}$, $\mathcal{B} = \{(1, i), (-i, 2)\}$, $\mathcal{C} = \{(-1, 2), (1, i)\}$ be bases for V and let $T : V \to V$ be defined by $T(x, y) = (x - y, x + y)$. Compute the **rank** of $T :=$ dim (im T) and the **nullity** of $T :=$ dim (ker T). Is T a monomorphism?, an epimorphism?, an isomorphism? Then determine the following matrix representations of T.

(a) $[T]_{\mathcal{S}}$

(b) $[T]_{\mathcal{S},\mathcal{B}}$

(c) $[T]_{\mathcal{B},\mathcal{S}}$

(d) $[T]_{\mathcal{B}}$

(e) $[T]_{\mathcal{C}}$

Problem 3. Let $\mathcal{S}_n$ be the standard basis for $\mathbf{R}^n$. Let $\mathcal{B} = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$, a basis for $\mathbf{R}^3$ and let $\mathcal{C} = \{(0, 1), (1, 0)\}$, a basis for $\mathbf{R}^2$. Let

$$T : \mathbf{R}^3 \to \mathbf{R}^2 \text{ be defined by } T(x, y, z) = (x + y, 2z - x).$$

Compute the **rank** of $T :=$ dim (im T) and the **nullity** of $T :=$ dim (ker T). Is T a monomorphism?, an epimorphism?, an isomorphism? Then determine the matrix representation $[T]_{\mathcal{B},\mathcal{C}}$ of T.

Problem 4. Let

$$T : \mathbf{R}^3 \to \mathbf{R}^3$$

be the linear transformation determined by

$$T(e_3) = 2e_1 + 3e_2 + 5e_3, \quad T(e_2 + e_3) = e_1 \quad \text{and } T(e_1 + e_2 + e_3) = e_2 - e_3.$$

Compute the **rank** of $T := \dim\,(\text{im } T)$ and the **nullity** of $T := \dim\,(\ker T)$. Is T a monomorphism?, an epimorphism?, an isomorphism? Then determine the matrix representation $[T]_{\mathcal{S}}$ of T.

Problem 5. Let $\mathcal{B} := \{1, t, t^2, t^3\}$, a basis for $\mathbf{R}[t]_3$. Let $T : \mathbf{R}[t]_3 \to \mathbf{R}[t]_3$ be the linear transformation determined by $T(f(t)) = f(t+1)$, where $f(t) \in \mathbf{R}[t]_3$. Determine the image and kernel of this map as well as the rank and nullity. Determine the matrix representation $[T]_{\mathcal{B}}$ of T.

Problem 6. Let $\mathcal{B} := \{1, t, t^2\}$ and $\mathcal{C} := \{1 + t, t, 1 + t^2\}$, both bases for $\mathbf{R}[t]_2$. Let $T : \mathbf{R}[t]_2 \to \mathbf{R}[t]_2$ be the linear transformation given by

$$T(\alpha_0 + \alpha_1 t + \alpha_2 t^2) = (\alpha_0 + \alpha_1) + \alpha_2 t^2.$$

Determine the following

(a) $[T]_{\mathcal{B}}$

(b) $[T]_{\mathcal{C}}$

(c) An invertible matrix C so that $C[T]_{\mathcal{B}}C^{-1} = [T]_{\mathcal{C}}$

Recall from calculus the following two facts:

a. Two non-zero vectors in $\mathbf{R}^3$ are perpendicular if and only if their dot product is zero.

b. The cross product of two linear independent vectors is perpendicular to the plane spanned by the two linearly independent vectors in $\mathbf{R}^3$.

Use these facts to do the following:

Problem 7. Let $v = (1, 1, 1), w = (3, 2, 1),$ and $x = (-1, 0, 1)$ in $\mathbf{R}^3$. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ the linear transformation defined as follows: First rotate in the plane perpendicular to v by an angle of θ degrees, followed by a flip in the plane perpendicular to w, then followed by rotating in the plane perpendicular to x by an angle of ϕ degrees. Determine $[T]_{\mathcal{S}}$. You do not have to multiply all the matrices together. [A flip in the plane perpendicular to w takes the w coordinate to its negative and fixes the plane perpendicular to w.]

Problem 8. Let V and W be fdvs over F both of dimension n. Let $T : V \to W$ be a linear transformation. Show that there exists a basis $\mathcal{B}$ for V and a basis $\mathcal{C}$ for W such that $[T]_{\mathcal{B},\mathcal{C}}$ is a diagonal matrix.

Problem 9. Hoffman, Kunze p. 105 Problem 3.5/5

Problem 10. Hoffman, Kunze p. 106 Problem 3.5/12