Homework #4

Problem 1. let V be a vector space over F. Let $T : F \to V$ be a linear map. Let $v = T(1)$. Show that $T(\alpha) = \alpha v$ for any $\alpha \in F$.

Problem 2. Let $T : V \to W$ and $S : W \to X$ be linear transformations. Show that $S \circ T : V \to X$ is a linear transformation.

Problem 3. Let $T : V \to W$ be linear. Let $Z \subset W$ be a subspace. Show that the *inverse image* of Z

$$T^{-1}(Z) := \{v \in V \,|\, T(v) \in Z\}$$

is a subspace of V.

Problem 4 If $T : V \to W$ is a linear transformation with a finite dimensional image, we write $\mathrm{rank}(T)$ for $\dim \mathrm{im}(T)$. Let $S : U \to V$ and $T : V \to W$ be linear maps of vector spaces. Suppose that both T and S have a finite dimensional image. Show all of the following:

i. $\mathrm{rank}(T \circ S) \leq \min(\mathrm{rank}(T), \mathrm{rank}(S))$.
ii. $\mathrm{rank}(T \circ S) = \mathrm{rank}(S)$ if and only if $\mathrm{im}(S) \cap \ker(T) = 0$
iii. $\mathrm{rank}(T \circ S) = \mathrm{rank}(T)$ if and only if $\mathrm{im}(S) + \ker(T) = V$

Problem 5. Let V and W be vector spaces over F. Show that

$$L(V, W) := \{T \,|\, T : V \to W \text{ is linear }\}$$

is a vector space over F. If V and W are both finite dimensional vector spaces, conjecture what the dimension of $L(V, W)$ is. Prove your conjecture.

Problems 6,7,8. Do three problems from Section 3.1 p. 73-4

Problems 9, 10. Do two problems from Section 3.2 p. 83-4