Homework #3

Problem 1. Consider the linear differential equation

$$Ly = y'' + ay' + by = 0$$

where a and b are real numbers. Show that the solution set $W = \{f \in C(-\infty, \infty) \mid Lf = 0\}$ is a real vector space of dimension at least two. [In fact, it is exactly two.] Generalize.

Problem 2. Let V be a finite dimensional vector space over F and W a subspace of V. Show $W = V$ if and only if $\dim(W) = \dim(V)$.

Problem 3. Let V be a finite dimensional vector space over F and $T : V \to V$ a linear map, i.e., $T(\alpha v + w) = \alpha T(v) + T(w)$ for all $\alpha \in F$ and all $v, w \in V$. Suppose that T is not injective. Show that $\dim\,(\ker(T)) > 0$ where $\ker(T) := \{v \in V \mid T(v) = 0\}$. Also show that T is not onto.

Problem 4. Let U and W be subspaces of a vector space V over F. Suppose that $\dim(U) + \dim(W) > \dim(V)$. Show $U \cap W \neq \{0\}$.

Problem 5. Let

$$\begin{aligned} W_1 =& \{A \in M_2(\mathbf{R}) \mid A = A^t\} \\ W_2 =& \{A \in M_2(\mathbf{R}) \mid A = -A^t\} \end{aligned}$$

Determine the dimensions of these real vector spaces by producing bases. Show that $M_2(\mathbf{R}) = W_1 + W_2$. What if we replace the $\mathbf{R}$ in all the above with $\mathbf{C}$ but still view everything as real vector spaces?

Problem 6. Let V be an n-dimensional space over F with $n > 0$. Let $v \neq 0$ in V be a fixed vector. Produce a linear map $f : V \to F$ such that $f(v) \neq 0$. (Recall that f *linear* means that $f(\alpha x + y) = \alpha f(x) + f(y)$.) Show that f is onto. Show that $\ker(f) := \{w \in V \mid f(w) = 0\}$ is $n - 1$ dimensional.
[Hint: If you have trouble doing this, think of the case that $V = F^n$.]

Problem 7. Let $\mathcal{S} := \{e_1, \ldots e_n\}$ be the standard basis for $\mathbf{R}^n$ and $\mathcal{B} := \{v_1, \ldots v_n\}$ be another basis. Suppose $v_i = (\alpha_{1i}, \ldots \alpha_{ni})$ for $1 \leq i \leq n$. Discuss how you would determine the coordinates of the e_i in the basis $\mathcal{B}$, i.e., how you would determine β_{ij} such that $e_i = \sum_j \beta_{ij} v_j$. More generally, discuss the case when V is a finite dimensional vector space over F with two bases $\mathcal{B}$ and $\mathcal{C}$, you know how to write elements in the $\mathcal{B}$ basis as linear combinations of vectors in the $\mathcal{C}$ basis and want to do the reverse.

Problems 8,9. Do two problems you did not do before from section 2.3 p 48.

Problem 10. Do a problem you did not do before from section 2.6 p 66.