Homework #2

Problem 1. Let $V \neq 0$ be a vector space over F. Suppose that V can be spanned by one vector. Show that any subspace W of V is either the zero subspace or all of V. Discuss what can happen if V can be spanned by two vectors.

Problem 2. Let $V \neq 0$ be a vector space over F. Suppose that W and X are subspaces of V. Is $W \cup X$ always a subspace of V? Prove if true and give a counterexample if not. If it is not true find a condition that will guarantee it to be true. Prove your assertion.

Definition. Let V and W be vector spaces over F. A map $T : V \to W$ is called **LINEAR** or a **LINEAR TRANSFORMATION** if for all $v_1, v_2 \in V$ and for all $\alpha \in F$, we have

$$T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2).$$

[We usually write Tv for $T(v)$ (out of laziness).]

If $T : V \to W$ is a linear transformation, let

$$\ker(T) = N(T) := \{v \in V \mid Tv = 0\}$$

called the **KERNEL** or **NULL SPACE** of T.

Example. Let A be an $m \times n$ matrix in $F^{m\times n}$. Then $A : F^{n\times 1} \to F^{m\times 1}$ by $v \mapsto Av$ (matrix multiplication) is a linear transformation, by rules of matrix addition and multiplication.

Problem 3. Let V and W be vector spaces over F. Let $T : V \to W$ be a linear transformation. Prove all of the following:

a. $\ker(T) \subset V$ and $\operatorname{im}(T) \subset W$ are subspaces.

b. Suppose that $\ker(T) = N(T) = \{0\}$. Suppose that $\{v_1, \dots, v_k\}$ is a linearly independent subset of V. Prove that $\{Tv_1, \dots, Tv_k\}$ is linearly independent in W.

c. Let A be an $m \times n$ matrix in $F^{m\times n}$. Suppose that $\ker(A) = N(A) = \{0\}$. Let $v_1, \dots, v_k \in F^{n\times 1}$ be linearly independent. Then $Av_1, \dots, Av_k$ are linearly independent in $F^{m\times 1}$.

d. Let A be an invertible $n \times n$ matrix in $\mathbf{M}_n(F)$. Let $v_1, \dots, v_k \in F^{n\times 1}$ be linearly independent. Then $Av_1, \dots, Av_k$ are also linearly independent in $F^{n\times 1}$. What would you conjecture if $k = n$. Can you prove your conjecture?

Problem 4. Let V be a vector space over F and $v_1, \dots, v_n \in V$. Suppose that $v_1 \in \operatorname{Span}(v_2, \dots, v_n)$. Show that $v_1, \dots, v_n$ are linearly dependent (assuming they are distinct). and $\operatorname{Span}(v_1, \dots, v_n) = \operatorname{Span}(v_2, \dots, v_n)$.

Problem 5. Suppose that $\alpha_1, \dots \alpha_n$ are distinct real numbers. Prove that the functions $e^{\alpha_1 x}, \dots, e^{\alpha_n x}$ are linearly independent.

Problem 6,7. Do two problems from Section 2.2 p. 39 that you did not do for HW #1

Problem 8,9. Do two problems from Section 2.3 p. 48

Problem 10. Do one problem from Section 2.6 p.66