HW #5

1.(*) Let R be a commutative ring. Show that a polynomial $f = a_0 + a_1 t + \cdots + a_n t^n$ is a unit in $R[t]$ if and only if a_0 is a unit in R and a_i is nilpotent for every $i > 0$. Is the same result true for $R[[t]]$, the ring of formal power series defined by extending the operations of those on $R[t]$ (no convergence condition to worry)? Prove or give a counterexample.

2. Let F be a finite field with q elements. Then F is of characteristic p for some prime p, has $q = p^n$ elements for some positive integer n, and every element $\alpha \in F$ satisfies $\alpha^q = \alpha$.

3.(*) Let R be a commutative ring. If $f = a_0 + a_1 t + \cdots + a_n t^n$ is a polynomial in $R[t]$ define the *formal derivative* f' of f to be $f = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}$.
   a. Show the usual rules of differentiation hold.
   b. Suppose R is a field of characteristic zero. Show that a polynomial $f \in R[t]$ is divisible by the square of a non-constant polynomial in $R[t]$ if and only if f and f' are not relatively prime.

4. Let F be a subfield of the complex numbers $\mathbf{C}$. Let $f \in F[t]$ be an irreducible polynomial. Show that f has no multiple root in $\mathbf{C}$. [An element $a \in \mathbf{C}$ is called a multiple root of f if $f = (t-a)^m g$ for some g in $\mathbf{C}[t]$ and integer $m > 1$.]

5. Let F be a field and $f \in F[t]$ a polynomial of degree $n \geq 1$. Let $\bar{\ } : F[t] \to F[t]/(f)$ be the canonical epimorphism.
   (a) Show that $V = F[t]/(f)$ is a vector space over F of dimension n where $\alpha\bar{g} := \overline{\alpha g}$ for all $\alpha \in F$ and $g \in F[t]$. In particular, if $f \in F[t]$ is irreducible then f has a root in a field K containing F that is an n-dimensional F-vector space.
   (b) Show that there exists a field L containing F such that f factors into a product of linear polynomials in $L[t]$ and $\dim_F(L) \leq (\deg f)!$.