Mathematics 110BH

HW #1

1. Let F be a field then $\mathbf{M}_n(F)$ is a simple ring. If $n > 1$ then $\mathbf{M}_n(F)$ is not a division ring.

2. Let R be a (commutative) ring then every ideal of $\mathbf{M}_n(R)$ has the form $\mathbf{M}_n(\mathfrak{A})$ (obvious notation) where $\mathfrak{A}$ is an ideal in R.

3. If R is a non-commutative ring satisfying $x^2 = x$ for all x in R then R is commutative.

4. (*) Let R be the set of all continuous functions $f : [0,1] \to \mathbf{R}$ then R is a ring under $+$ and $\cdot$ of functions. Show that any maximal ideal of R has the form $\{f \in R \,|\, f(a) = 0\}$ for some fixed a in $[0,1]$. [You need some analysis, namely, any closed interval I is *compact*, i.e., any open cover of I has a finite subcover. (If a, b lies in $(0,1)$ then the open interval (a,b) and the half open intervals $[0,a)$ and $(a,1]$ are all considered open intervals in $[0,1]$.)

5. Let R be a (commutative) ring. Prove the Binomial Theorem:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

6.(*) Let R be a (commutative) ring. An element x of R is called *nilpotent* if some power of x is zero. Prove
   a. If x is nilpotent then $1 + x$ is a unit.
   b. The set of nilpotent elements of R forms an ideal called the *nilradical* of R.
   c. Compute the nilradical of the rings: $\mathbf{Z}/12\mathbf{Z}$, $\mathbf{Z}/n\mathbf{Z}$, $n > 1$, and $\mathbf{Z}$.

7. Let R be a (commutative) ring. An element e of R is called an *idempotent* if $e^2 = e$. For example, if S is another ring the element $(1_r, 0_S)$ is an idempotent in the ring $R \times S$. The object of this exercise is to prove a converse. Let e be an idempotent of R. Then prove
   a. $e' := 1 - e$ is an idempotent of R.
   b. The principal ideal Re of R is a ring with identity $1_{Re} = e$. [Note that Re is not a subring of R since Re will not have the same identity as R unless $e = 1$.]
   c. R is ring isomorphic to $Re \times Re'$

8. Let R be a nonzero ring and

$$\mathrm{End}_{\mathbf{Z}}(R) := \{f : R \to R \mid f \text{ an additive group homomorphism}\}.$$

Show that $\mathrm{End}_{\mathbf{Z}}(R)$ is a ring with addition defined by $(f+g)$ given by $(f+g)(x) = f(x) + g(x)$ for all $x \in R$ and multiplication given by composition. Prove that the units in $\mathrm{End}_{\mathbf{Z}}(R)$ is $\mathrm{Aut}_{\mathbf{Z}}(R)$, the group of additive group automorphisms of R.