Math 110AH Midterm Problem 5

(100 points) Write whether each of the following ten statements is true or false (or leave unanswered) and prove the result if true or give a counterexample if not.

Your solutions for Part 5 is due 11:59 PM Tuesday 8 Nov 22 on gradescope.

Instructions

- You must give a proof or counterexample to each statement including reasons.
- You must do it yourself.
- Although you may not use other people, you may use books or the web. However, you should make a serious attempt to do a problem before you use other allowable sources.
- If you use a book or the web, results used must be proved if not proven in class or prior homework. If you are copying (or essentially copying a proof) from a book or page(s) on the web, write the book and page number or web address and justify the details (some which may not be written down in the book or on the website). Cutting and pasting is not allowed.
- Sign that you followed the rules (if you did) as problem labeled (k) below.

Statements for Problem 5

(a) Suppose that $a,\ b,\ c \in \mathbb{Z}$ satisfy $(a, b) = 1$ and $c \mid ab$. Then $c \mid a$ or $c \mid b$.

(b) Let $m > 1$ be an integer not divisible by 17. Then the congruence $17x \equiv 7 \bmod m$ always has a solution in integers.

(c) Let a, b, c be positive integers. If the (diophantine) equation $ax + by = c$ has a solution (in integers), then it has only finitely many solutions in positive integers x, y.

(d) The group $\mathbb{Z}/m\mathbb{Z}$, $m > 1$, is cyclic under addition of order m and has $\phi(m)$ distinct (cyclic) generators, where ϕ is the Euler ϕ-function.

(e) Let G be a finite group and $H, K \subset G$ subgroups. Then $|HK| \mid |G|$.

(f) Let G be an abelian group. Then $\mathrm{Aut}(G)$ is abelian.

(g) Let G be a group and H a subgroup of G. Let $* : G \times H \to G$ be the map defined by $(g, h) \mapsto gh$. Then $\sim$ given by $g \sim g'$ if there exists an h in H satisfying $g' = gh$ is an equivalence relation on G with the equivalence class of g given by the coset gH.

(h) The group of real numbers $\mathbb{R}$ under addition contains the integers $\mathbb{Z}$ as a normal subgroup. Moreover, the set of cosets $\mathbb{R}/\mathbb{Z}$ is a group and $\mathbb{R}/\mathbb{Z}$ is isomorphic to the circle group $\mathbf{T} := \{z \in \mathbb{C} \mid |z| = 1\}$.

(i) Let G be a group of order pq with p and q primes (but not necessarily distinct). Then G contains a subgroup of order p or a subgroup of order q.

(j) Let G be a group with elements a, b in G of prime order p, q, respectively with $p \neq q$. Then the group $\langle a, b\rangle$ generated by a, b has order pq.

(k) Sign that you followed the rules (if you did) as problem labeled (k).