**Math 110A Take-home Midterm**

**Instructions:** Do all ten problems below. Parts in problems do NOT count equally.

You may use books, the web, and other (non-human) sources, although you should make a serious effort to do each problem by yourself without looking anything up.

If you copy a proof or part of a proof from any source, you must reference that source and the pages, web address (or description) that you are using. If the sources's proof uses results that we have not done, you must write up those results also. You must also fill in the details that the source leaves out. This includes the text for the class. Failure to site a source in a proof, even if a partially copied solution will result in a negative score equal to the value of that problem. You must write or type up your results. No cutting and pasting is allowed.

Below $G$ is always a group and $H$ a subgroup of $G$.

1. Suppose that for three consecutive integers $i$, $(ab)^i = a^i b^i$ for all $a, b \in G$. Show that $G$ is abelian.
2. Let

$Q := \langle -1, i, j, k \mid (-1)^2 = 1,\ i^2 = -1 = j^2,\ k = ij = -ji, \text{ and } -1 \text{ commutes with all } x \in Q\rangle.$

   Show $Q$ is a group of eight elements and has the following properties:
   ($a$) All subgroups of $Q$ are normal, but $Q$ is not abelian.
   ($b$) The group $Q$ contains a unique subgroup of order two.
   ($c$) There exist no proper subgroups $H$ and $K$ of $Q$ with at least one normal, such that $Q = HK$ and $H \cap K = 1$.
   ($d$) There exists no group monomorphism of $Q$ into $S_7$.
3. A *commutator* of $G$ is an element of the form $xyx^{-1}y^{-1}$ where $x, y \in G$. Let $G'$ be the subgroup of $G$ generated by all commutators, i.e., every element of $G'$ is a **product** of commutators and their inverses. We call $G'$ the *commutator* or *derived subgroup* of $G$. It is also denoted $[G, G]$. Show all the following are true.
   ($a$) $G' \triangleleft G$.
   ($b$) $G/G'$ is abelian.
   ($c$) If $N \triangleleft G$ and $G/N$ is abelian, then $G' \subset N$.
   ($d$) If $H \subset G$ is a subgroup satisfying $G' \subset H \subset G$, then $H \triangleleft G$.
4. Recall that

$$\mathrm{Aut}(G) = \{\sigma : G \to G \mid \sigma \text{ is an isomorphism}\}.$$

   A subgroup $H$ of $G$ is called *characteristic* in $G$ and denoted $H \triangleleft\triangleleft G$ if for every $\sigma \in \mathrm{Aut}(G)$, the restriction $\sigma|_H$ lies in $\mathrm{Aut}(H)$. Show the following:
   ($a$) If $K \triangleleft\triangleleft H$ and $H \triangleleft G$ then $K \triangleleft G$.
   ($b$) $Z(G) \triangleleft\triangleleft G$.
   ($c$) $G' \triangleleft\triangleleft G$.
   ($d$) Inductively define $G^{(n)}$ as follows: $G^{(1)} = G'$. Having defined $G^{(n)}$ define $G^{(n+1)} := (G^{(n)})'$. [So $G^{(n+1)}$ is generated by commutators $xyx^{-1}y^{-1}$ with $x, y \in G^{(n)}$ and the inverses of such commutators.] Then $G^{(n+1)} \triangleleft\triangleleft G$.

5. A group $G$ is called *solvable* if there exist subgroups $N_i \subset G$, $i = 1, \ldots, r$ some $r$, satisfying all of the following:
   - $(i)$ $N_i \triangleleft N_{i+1}$ for $i = 1, \ldots, r-1$.
   - $(ii)$ $1 = N_1$ and $G = N_r$.
   - $(iii)$ $N_{i+1}/N_i$ is abelian for $i = 1, \ldots, r-1$.

   Prove all of the following **using the Isomorphism Theorems**:
   - $(a)$ A subgroup of a solvable group is solvable.
   - $(b)$ The homomorphic image of a solvable group is solvable, i.e., if $G$ is a solvable group and $\varphi : G \to \tilde{G}$ a group homomorphism, then $\mathrm{im}(\varphi) = \varphi(G)$ is solvable.
   - $(c)$ If $N \triangleleft G$ and both $N$ and $G/N$ are solvable, then so is $G$.

6. Let $p$ and $q$ be positive primes. Show each of the following: (You may assume the Sylow theorems without proving them but not the result that says a group of order $p^n q$ is solvable for any $n$.)
   - $(a)$ Any $p$-group is solvable.
   - $(b)$ If $|G| = pq$, $p^2q$, $p^2q^2$, or $pqr$ with $p, q, r$ primes, then $G$ is solvable.

7. Let $G$ be a finite group, $H < G$ a subgroup. Suppose that $H$ is a $p$-group, $p$ a prime, and $p \mid [G : H]$. Show that $p \mid [N_G(H) : H]$ and $H < N_G(H)$. In particular, if $G$ is a $p$-group and $H < G$, then $H < N_G(H)$.

8. Suppose that $H$ is a proper subgroup of a finite group $G$. Show that $G \neq \bigcup_{g \in G} gHg^{-1}$.

9. Let $S$ be a $G$-set. Suppose that both $S$ and $G$ are finite. If $x \in G$, let
$$F_S(x) := \{s \in S \mid x \cdot s = s\}.$$
   Show the number of orbits $N$ of this action satisfies
$$N = \frac{1}{|G|} \sum_G |F_S(x)|.$$

10. Let $p$ be an odd prime. Classify all groups $G$ of order $2p$ up to isomorphism.