Mathematics 110AH

HW #7

1. Let G be a finite p-group. Show if $p^n \mid |G|$, then G has a normal subgroup of order p^n.
2. An element $\sigma \in S_n$ is called regular if it is the identity or it has no fixed points and it is a product of disjoint cycles of the same length. Prove that σ is regular if and only if σ is a power of an n-cycle.
3. Let G be a finite group of order greater than two. Prove both of the following:
   i. If G is abelian, then $\mathrm{Aut}(G)$ never has odd order.
   ii. If G is not abelian, then $\mathrm{Aut}(G)$ is never cyclic.
4. Prove that every finite group of odd order is solvable if and only if every nonabelian simple group is of even order.
5. Show that every group of non-prime order less than 60 has a proper normal subgroup.