Mathematics 110AH

HW #4

1. Let $\phi : G \to G'$ be a group homomorphism. Show
   a. $\ker(\phi) \subset G$ and $\mathrm{im}(\phi) \subset G'$ are subgroups.
   b. ϕ is an isomorphism if and only if ϕ is bijective.

2. To what group is $\mathrm{ST}_2(\mathbf{R})$ isomorphic (besides itself)? Prove your answer.

3. Prove Wilson's Theorem which states: Let $p > 1$ be an integer. Then p is a prime if and only if $(p-1)! \equiv -1 \pmod{p}$. [This is not a practical test for primality.] [Hint: Let $1 \leq j \leq p-1$. If p is a prime when can $j^2 \equiv 1 \pmod{p}$?]

4.(*) Let G be a group and H and K subgroups of G. Show all of the following:
   (a) If $K \subset H \subset G$ and K has finite index in G then $[G : K] = [G : H][H : K]$. (This is even true when nothing is finite if read correctly.)
   (b) Let $HK := \{hk \mid h \in H,\, k \in K\}$. Then $HK/K = \{hK \mid h \in H\}$. Then (clearly) $H/(H \cap K)$ is a subset of $G/(H \cap K)$ and HK/K is a subset of G/K. Show that $f : H/(H \cap K) \to HK/K$ by $h(H \cap K) \mapsto hK$ is a well-defined bijection.
   (c) (Poincaré) Suppose both H and K have finite index in G. Then $H \cap K$ has finite index in G.

5.(*) Show that $Aut(G)$ is a group and $Inn(G) \triangleleft Aut(G)$. Let G be a cyclic group. Determine $Aut(G)$ and $Inn(G)$ up to isomorphism as groups that we know.

6. Show that a subgroup $H \subset G$ is normal if and only if $gH = Hg$ for all $g \in G$. If H is not normal is it still true that for each $g \in G$ there is an $a \in G$ such that $gH = Ha$?

7. Find all subgroups of S_3 and determine which ones are normal.

8. Let G be a group of order p^n where p is a prime and $n \geq 1$. Prove that there exists an element of order p in G.

9. Prove that a group of order 30 can have at most 7 groups of order 5.