HW #1

1. Let a and n be positive integers. Show if $a^n - 1$ is prime and $n > 1$ then $a = 2$ and n is prime. If $2^n + 1$ is prime, what can you say about n?

2.(*) Let x, y, z, n, a, b be integers. Show
   a) If $x|y$ and $x|z$ then $x|ay + bz$.
   b) If $x|y$ then $x|yn$.
   c) If $x|y$ and $y \neq 0$ then $|y| \geq |x| \geq x$.
   d) If $xy = 0$ then $x = 0$ or $y = 0$.
   e) If $xa = xb$ then $x = 0$ or $a = b$.

3.(*) Let a, b, n be positive integers with $n > 1$. Determine when $n^{\frac{a}{b}}$ is rational. Prove. [You can use the Fundamental Theorem of Arithmetic.]

4. Prove the cartesian product of finitely many countable sets is countable.

5. Prove any two (finite) line segments have the same cardinality.

6. Let $F = \mathbf{R}, \mathbf{C}$ or $\mathbf{Q}$ [or any FIELD]. Let $F[t]$ be the set of polynomials with coefficients in F with the usual addition and multiplication. State and prove the analog of the Division Algorithm for Integers. (Use your knowledge of such division. Use degrees of polynomials as a substitute for statement (ii) in the Division Algorithm.) What can you do if you take polynomials with coefficients in $\mathbf{Z}$?

7. Prove that the number of subsets of a set with n elements is 2^n.

8. The first nine Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, 21, 34$. What is the nth Fibonacci number F_n. Show that $F_n < 2^n$.

9. Note that Euclid's proof of the infinitude of primes clearly shows that if p_n is the nth prime then $p_{n+1} \leq p_n^n + 1$. Be more careful and show that $p_{n+1} \leq 2^{2^{n+1}}$. Using this, can you then show $\pi(x) \geq \log\log(x)$ where $\pi(x)$ is the number of primes less than x if $x \geq 2$. [This is a bad estimate.]

10. When Gauss was ten years old he almost instantly recognized that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. [Actually, what he did was a bit harder.] What is a formula for the sum of the first n cubes? Prove your result?