1. Let $H_i, i \in I$, be a collection of subgroups of $G$. Prove that $\bigcap H_i$ is a subgroup of $G$. Is $\bigcup H_i$ a subgroup of $G$?

2. Let $G$ be a group and $W$ a subset of $G$. Show that $\langle W \rangle = \{ g \in G | \exists w_1, \ldots, w_r \in W, \text{not necessarily distinct}, n_1, \ldots, n_r \in \mathbb{Z} \ni g = w_1^{n_1} \cdots w_r^{n_r} \}$.

3. Show if $G$ is a group in which $(ab)^2 = a^2b^2$ for all $a, b \in G$ then $G$ is abelian.

4. Determine all groups up to order 6.

5. Let $p$ be a prime. Show that $F = \mathbb{Z}/p\mathbb{Z}$ is a field [i.e., in the commutative ring $\mathbb{Z}/p\mathbb{Z}$ every non-zero element has a multiplicative inverse]. Compute $|G|$ if $G = GL_n(F), SL_n(F), ST_n(F),$ or $D_n(F)$.

[Hint: First show that $F$ is a domain, i.e., a commutative ring satisfying: if $ab = 0$ in $F$ then $a = 0$ or $b = 0$. Then show that any domain with finitely many elements is a field.]

6.(*) Let $m_i > 1, 1 \leq i \leq n$, be pairwise relatively prime integers. Let $m = m_1 \cdots m_n$. Let $\varphi(m)$ denote the order of the group $(\mathbb{Z}/m\mathbb{Z})^\times$. The function $\varphi : \mathbb{Z}^+ \to \mathbb{Z}^+$ is called the Euler phi function. [We let $\varphi(1) = 1.$] Show that there exists an isomorphism

$$(\mathbb{Z}/m\mathbb{Z})^\times \to (\mathbb{Z}/m_1\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/m_n\mathbb{Z})^\times.$$ 

In particular $\varphi(m) = \varphi(m_1) \cdots \varphi(m_n)$. Compute $\varphi(p^r)$ when $p$ is a prime and $r$ is a positive integer.

7.(*) Prove the Cyclic Subgroup Theorem which states: Let $H$ be a subgroup of the cyclic group $G = \langle g \rangle$. Let $e$ be the identity of $G$. Let $n$ be a positive integer. Then

(i) $H = \{ e \}$ or $H = \langle g^m \rangle$ where $m \geq 1$ is the least integer such that $g^m \in H$. If $G$ is infinite then $H$ is infinite or $\{ e \}$. If $G$ is finite of order $n$ then $m | n$.

(ii) If $|G| = n$ and $m | n$ then $\langle g^m \rangle$ is the unique subgroup of $G$ of order $n/|m|$.

(iii) If $|G| = n$ and $m \not | n$ then $G$ does not have a subgroup of order $m$.

(iv) If $|G| = n$ then the number of subgroups of $G$ is equal to the number of divisors of $|G|$.

(v) If $G$ has prime order then the only subgroups of $G$ are $\{ e \}$ and $G$.

8.(*) Let $G$ be an abelian group. Let $a, b \in G$ have finite order $m, n$, respectively. Suppose that $m$ and $n$ are relatively prime. Show that $ab$ has order $mn$. Is this true if $G$ is not abelian? Prove or give a counterexample.