Answer the questions in the spaces provided. If you run out of room use scratch paper and attach it to end of the exam. Show your work. Correct answers not accompanied by sufficient explanations will receive little or no credit. Please ask any of the proctors if you have any questions about a problem. Calculators are allowed but definitely not needed. No books, PDAs, cell phones or other devices (other than calculators) will be permitted.

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Student Name: SOLUTIONS

Student Signature: ____________________________________________

Section: _______ Meets on: ___________ TA Name: ________________
1. A) The Flux of a vector field $\vec{F}(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j}$ across an oriented curve $C$ is given by

$$\text{Flux}_C(\vec{F}) = \int_C P(x, y) \, dy - Q(x, y) \, dx.$$ 

Calculate the flux of $\vec{F}(x, y) = e^y\hat{i} + (2x - 1)\hat{j}$ across the parabola $y = x^2$, $0 \leq x \leq 1$, oriented from left to right.

The parabola can be parametrized by $\vec{c}(t) = (t, t^2)$, $0 \leq t \leq 1$. Then $dx = x'(t) \, dt = (1) \, dt$, $dy = y'(t) \, dt = (2t) \, dt$ and the flux is

$$\text{Flux}_C(\vec{F}) = \int_C P \, dy - Q \, dx = \int_0^1 e^{t^2} \, 2t \, dt - \int_0^1 (2t - 1) \, dt$$

$$= \left. e^{t^2} \right|_0^1 - \left. t^2 \right|_0^1 + \left. t \right|_0^1$$

$$= e - 1 - 1^2 + 1 - 0^2 + 0$$

$$= e - 1 - 1 + 1 = e - 1.$$
B) I. Which of the following statements are true for all vector fields, and which are true only for conservative vector fields?

i) The line integral along a path from $P$ to $Q$ does not depend on which path is chosen. \hspace{2cm} i) \text{conservative}

ii) The line integral around a closed curve is zero. \hspace{2cm} ii) \text{conservative}

iii) The cross-partials of the components are equal. \hspace{2cm} iii) \text{conservative}

iv) The line integral over an oriented curve $C$ does not depend on how $C$ is parametrized. \hspace{2cm} iv) \text{all}

v) The line integral is equal to the difference of a potential function at the two endpoints. \hspace{2cm} v) \text{conservative}

vi) The line integral is equal to the integral of the tangential component along the curve. \hspace{2cm} vi) \text{all}

vii) The line integral changes signs if the orientation is reversed. \hspace{2cm} vii) \text{all}

II. Let $\vec{F}$ be a vector field on an open, connected domain $\mathcal{D}$. Which of the following statements are always true, and which are true under additional hypothesis on $\mathcal{D}$?

i) If $\vec{F}$ has a potential function, then $\vec{F}$ is conservative. \hspace{2cm} i) \text{always}

ii) If $\vec{F}$ is conservative, then the cross-partials of $\vec{F}$ are equal. \hspace{2cm} ii) \text{always}

iii) If the cross-partials of $\vec{F}$ are equal, then $\vec{F}$ is conservative. \hspace{2cm} iii) \text{additional hyp.}
(20 pts) 2. Calculate the integral of $f(x, y, z) = e^z$ over the portion of the plane $x + 2y + 2z = 3$, where $x, y, z \geq 0$.

**HINT:**

i) An appropriate choice of the order of integration might help you avoid having to integrate by parts.

The answer is $3(e^{3/2} - \frac{5}{2})$.

Solving for $x$ in $x + 2y + 2z = 3$ leads to $x = 3 - 2y - 2z$. Since $x$ must be non-negative we have $3 - 2y - 2z \geq 0$.

We can now parametrize the plane using $y$ and $z$ as parameters.

$$g(y, z) = (3 - 2y - 2z, y, z),$$

with $(y, z) \in D = \{ y \geq 0, z \geq 0, 2y + 2z \leq 3 \}$

Then

$$T_y = \frac{\partial g}{\partial y} = (-2, 1, 0),$$

$$T_z = \frac{\partial g}{\partial z} = (-2, 0, 1),$$

$$T_y \times T_z = (1, -2, 2)$$

and $$|T_y \times T_z| = \sqrt{1 + 4 + 4} = 3$$

Therefore

$$\int\int\int_D e^z \, dV = \int\int\int_D e^z (3) \, dA = 3 \int\int_D e^{\frac{3}{2} - y} \, dy \, dz = 3 \int_0^{\frac{3}{2}} 3 \left( e^{\frac{3}{2} - y} - 1 \right) \, dy = \left( -3 e^{\frac{3}{2} - y} \right) \Bigg|_y^0 = \left( -3 e^{3/2} - 3 \right)$$

$$= -3e^{3/2} + 3e^{3/2} - 3\left( \frac{3}{2} \right) + 3(0) = 3e^{3/2} - 3 - \frac{9}{2} = 3e^{3/2} - \frac{15}{2} = 3\left(e^{3/2} - \frac{5}{2}\right).$$
(30 pts) 3. The Flux of a vector field \( \vec{F}(x, y, z) \) across a closed surface \( S \) is given by the surface integral of \( \vec{F} \) over \( S \). In symbols

\[
\text{Flux}_S(\vec{F}) = \iint_S \vec{F} \cdot d\vec{S},
\]

where the normal unit vector \( \hat{n} \) that determines the orientation of \( S \) is the one that points towards the outside of the volume enclosed by \( S \).

Let \( \vec{F} = y\hat{i} + yz\hat{j} + (x^2 - 5z)\hat{k} \), compute \( \text{Flux}_S(\vec{F}) \) where \( S \) is the boundary of the cylinder shown in Figure 1. Keep in mind that \( S \) also includes the top and the bottom of the cylinder.

HINTS:

i) Figuring out the normal component of the vector field on the top and the bottom of the cylinder might help you avoid unnecessary computations.

ii) The following integrals might come in handy:

\[
\int_0^{2\pi} \sin t \cos t \, dt = \int_0^{2\pi} \sin mt \, dt = \int_0^{2\pi} \cos mt \, dt = 0, \text{ for } m = 1, 2, \ldots
\]

\[
\int_0^{2\pi} \sin^2 t \, dt = \int_0^{2\pi} \cos^2 t \, dt = \pi.
\]

**Top:** \( \hat{n} = \langle 0, 1, 0 \rangle \) and \( \vec{F} \big|_{\text{top}} = \vec{F}(y, y, 2) \bigg|_{z=5} = \langle y, 5y, 0 \rangle \)

\[
(\vec{F} \cdot \hat{n}) \bigg|_{\text{top}} = \langle y, 5y, 0 \rangle \cdot \langle 0, 1, 0 \rangle = 0 + 0 + 0 = 0.
\]

**Bottom:** \( \hat{n} = \langle 0, 0, -1 \rangle \) and \( \vec{F} \big|_{\text{bottom}} = \vec{F}(y, y, 2) \bigg|_{z=0} = \langle y, 0, 0 \rangle \)

\[
(\vec{F} \cdot \hat{n}) \bigg|_{\text{bottom}} = -\langle y, 0, 0 \rangle \cdot \langle 0, 0, 1 \rangle = 0 + 0 + 0 = 0.
\]

Therefore

\[
\iint_{\text{Top}} \vec{F} \cdot d\vec{S} = \iint_{\text{Bottom}} \vec{F} \cdot d\vec{S} = 0 \quad \text{and}
\]

\[
\text{Flux}_S(\vec{F}) = \iint_{S_1} \vec{F} \cdot d\vec{S} \quad \text{where } S_1 \text{ is the surface where}
\]

\[
\begin{align*}
x^2 + y^2 &= 4 \\
0 &\leq z \leq 5
\end{align*}
\]
$S_1$ can be parametrized by

$G_1(\theta, z) = \left( 2 \cos \theta, 2 \sin \theta, z \right)$ \hspace{1cm} $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 5$

Then

$T_\theta = \left( -2 \sin \theta, 2 \cos \theta, 0 \right)$

$T_z = \left( 0, 0, 1 \right)$

$T_\theta \times T_z = \left( 2 \cos \theta - 0 \right)\hat{i} - \left( -2 \sin \theta - 0 \right)\hat{j} + (0 - 0)\hat{k}$

$= \left( 2 \cos \theta, 2 \sin \theta, 0 \right)$

Observe that $T_\theta \times T_z$ has the right orientation as it points towards the outside of the cylinder.

We then have

$\iint_{S_1} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^5 \left( 2 \sin \theta, (2 \sin \theta)z, z^2 - 5z \right) \cdot \left( 2 \cos \theta, 2 \sin \theta, 0 \right) \, dz \, d\theta$

$= \int_0^{2\pi} \int_0^5 \left( 4 \sin \theta \cos \theta + 4 \sin^2 \theta \right) \, dz \, d\theta = \int_0^{2\pi} \left( 20 \sin \theta \cos \theta + 5 \sin^2 \theta \right) \, d\theta$

$= 20 \int_0^{2\pi} \sin \theta \cos \theta \, d\theta + 5 \int_0^{2\pi} \sin^2 \theta \, d\theta = 50 \pi$
4. Let $C$ be a curve in polar form $r = f(\theta)$ for $\theta_1 \leq \theta \leq \theta_2$ (see Figure 2), parametrized by 

$$c(\theta) = (r \cos \theta, r \sin \theta) = (f(\theta) \cos \theta, f(\theta) \sin \theta).$$

i) Show that $\dddot{c}(\theta) = f'(\theta)\hat{e}_r + f(\theta)\hat{e}_\theta$, where $\hat{e}_r = (\cos \theta, \sin \theta)$ and $\hat{e}_\theta = (-\sin \theta, \cos \theta)$.

ii) Show that if $L(C) = \int_C 1 \, ds$, represents the length of $C$, then

$$L(C) = \int_{\theta_1}^{\theta_2} \sqrt{(f'(\theta))^2 + (f''(\theta))^2} \, d\theta$$

**Hints:** $\hat{e}_r \perp \hat{e}_\theta$, $||\hat{e}_r|| = ||\hat{e}_\theta|| = 1$, and $||\dddot{c}(\theta)|| = \dddot{c}(\theta) \cdot \dddot{c}(\theta)$.

iii) Use polar coordinates to show that along $C$, the vortex vector field 

$$\mathbf{F}(x, y) = \frac{-y}{x^2 + y^2} \hat{i} + \frac{x}{x^2 + y^2} \hat{j}$$

is given by $\mathbf{F}(c(\theta)) = \frac{1}{f(\theta)} \hat{e}_\theta$ and conclude that

**Observe that**

$$\ddot{c}(\theta) = \Psi(\theta) \hat{e}_r \quad \text{and} \quad \int_C \mathbf{F} \cdot d\mathbf{s} = \theta_2 - \theta_1.$$
iii) In general the vortex vector field in polar coordinates is

\[ \mathbf{F} = -\frac{r \sin \theta}{r^2} \mathbf{e}_r + \frac{r \cos \theta}{r^2} \mathbf{e}_\theta = -\frac{\sin \theta}{r} \mathbf{e}_r + \frac{\cos \theta}{r} \mathbf{e}_\theta \]

\[ = \frac{1}{r} \mathbf{e}_\theta \]

Along the curve \( \mathbf{c}(t) \) we have \( r = \psi(t) \) and \( \mathbf{F} \bigg|_{\mathbf{c}(t)} = \frac{1}{\psi(t)} \mathbf{e}_\theta \)

Therefore

\[ \int_{\psi(t)}^{\psi(t_2)} \mathbf{F} \cdot d\mathbf{c} = \int_{\theta_1}^{\theta_2} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \, d\theta = \int_{\theta_1}^{\theta_2} \frac{1}{\psi(t)} \mathbf{e}_\theta \cdot \left( \psi(t) \mathbf{e}_\theta + \psi'(t) \mathbf{e}_r \right) \, d\theta \]

\[ = \int_{\theta_1}^{\theta_2} \mathbf{e}_\theta \cdot \mathbf{e}_\theta + \frac{\psi'(t)}{\psi(t)} \mathbf{e}_\theta \cdot \mathbf{e}_r \, d\theta = \int_{\theta_1}^{\theta_2} 1 \, d\theta = \theta \bigg|_{\theta_1}^{\theta_2} \]

\[ = \theta_2 - \theta_1 \]