1. The area of a circle

(2 pts) A) Let \( r \) be a positive real number and let \( O = (0, 0) \) and \( P = (r, 0) \) be points in the coordinate plane. Let \( \ell_\theta \) be the (half)-line that emanates from \( O \) and makes an angle \( \theta \) with respect to the positive \( x \)-axis (see figure).

Let \( Q \) be the point on \( \ell_\theta \) whose distance to \( O \) is \( r \) and let \( Q' \) be the point on \( \ell_\theta \) whose perpendicular projection onto the positive \( x \)-axis is \( P \) (see figure 1).

Find the \( x \) and \( y \) coordinates of \( Q \) and \( Q' \).

B) Partition the angle \( 2\pi \) into \( N \) equal parts of size \( \Delta \phi \) and let
\[
\mathcal{C}_r = \text{area of a circle with radius } r, \\
\mathcal{A} = \text{area of the triangle } \triangle OPQ, \\
\mathcal{A}' = \text{area of the triangle } \triangle OPQ',
\]
where \( O, P, Q, Q' \) are as in the above figure with angle \( \theta = \Delta \varphi \).

(2 pts) i. Show that
\[
\mathcal{A} = \frac{r^2 \sin(\Delta \varphi)}{2} \quad \text{and} \quad \mathcal{A}' = \frac{r^2 \tan(\Delta \varphi)}{2}.
\]

(3 pts) ii. Explain why the inequalities
\[
\frac{Nr^2 \sin(\Delta \varphi)}{2} \leq \mathcal{C}_r \leq \frac{Nr^2 \tan(\Delta \varphi)}{2} \tag{2}
\]
hold for any \( r > 0 \) and any natural number \( N \geq 5 \).

(3 pts) iii. Find an expression for \( N \) in terms of \( \Delta \varphi \) and take the limit as \( \Delta \varphi \to 0 \) in (2) to find \( \mathcal{C}_r \).