Math 32B **MIDTERM 2 (PRACTICE PROBLEMS)** Spring 2013

CALCULUS OF SEVERAL VARIABLES

1. **Circulation around a vortex**
   Let $\vec{F}$ be the vortex vector field
   $$\vec{F}(x,y) = \frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\imath}}.$$
   Calculate
   $$I_R = \oint_{\mathcal{C}_R} \vec{F} \cdot \, \mathrm{d}\vec{s}$$
   where $\mathcal{C}_R$ is the circle $x^2 + y^2 = R^2$ oriented counterclockwise.

2. Let $\vec{F}(x,y) = (9y - y^3)\hat{\boldsymbol{\imath}} + \mathrm{e}^{\sqrt{y}}(x^2 - 3x)\hat{\boldsymbol{\imath}}$ and let $\mathcal{C}$ be the boundary of the box $[0,3]\times[0,3]$ oriented clockwise.
   i. Show that $\vec{F}(x,y)$ is not conservative.
   ii. Show that
   $$\oint_C \vec{F} \cdot \, \mathrm{d}\vec{s} = 0$$
   .

3. Note that a curve $\mathcal{C}$ in polar form $\theta = h(r)$ for $r_1 \le r \le r_2$ can be parametrized by $\vec{c}(r) = r\big(\cos h(r), \sin h(r)\big)$ because the x- and y-coordinates are given by $x = r\cos\theta$ and $y = r\sin\theta$.
   Show that $||\vec{c}\,'(r)||^2 = 1 + r^2\big(h'(r)\big)^2$ and conclude that the length of $\mathcal{C}$ is
   $$\mathrm{Length}(\mathcal{C}) := \int_{\mathcal{C}} 1 \, \mathrm{d}s = \int_{r_1}^{r_2} \sqrt{1 + r^2\big(h'(r)\big)^2} \, \mathrm{d}r.$$

4. Calculate $\iint_{\mathcal{S}} y \, \mathrm{d}S$ over the portion of the sphere $x^2 + y^2 + z^2 = 4$ where $0 \le y \le 1$.

5. **Surfaces of Revolution: The method of Cylindrical Shells**
   Let $\mathcal{S}$ be a surface formed by rotating the region under the graph $z = g(y)$ in the yz-plane for $c \le y \le d$ about the z-axis, where $c \ge 0$.
   Prove the formula
   $$\mathrm{Area}(\mathcal{S}) := \int_{\mathcal{S}} 1 \, \mathrm{d}S = 2\pi \int_c^d y\sqrt{1 + \big(g'(y)\big)^2} \, \mathrm{d}y$$
   .
   Hint: Use polar coordinates y and θ to parametrize $\mathcal{S}$.

6. Let $\mathcal{S}$ be the sphere of radius R centered at the origin. By choosing appropriate parametrizations of $\mathcal{S}$ show that
   $$\iint_{\mathcal{S}} x^2 \, \mathrm{d}S = \iint_{\mathcal{S}} y^2 \, \mathrm{d}S = \iint_{\mathcal{S}} z^2 \, \mathrm{d}S,$$

then show that

$$\iint_{\mathcal{S}} x^2 \, \mathrm{d}S = \frac{4}{3}\pi R^4$$

by adding the integrals.

Hint: The Surface Area of a sphere of radius R is $4\pi R^2$.

7. Compute $\iint_{\mathcal{S}} \langle x, y, z\rangle \cdot \, \mathrm{d}S$, for the part of the spere $x^2 + y^2 + z^2 = 1$ where $\frac{1}{2} \leq z \leq \frac{3}{2}$, oriented by the inward-pointing normal.

8. Prove that if $\vec{F}(x, y, z) = P(x, y, z)\hat{\boldsymbol{\imath}} + Q(x, y, z)\hat{\boldsymbol{\imath}} + R(x, y, z)\hat{\boldsymbol{k}}$, and $\mathcal{S}$ is the part of a graph $z = g(x, y)$ lying over a domain $\mathcal{D}$ in the xy-plane, where the orientation is determined by the upwards pointing normal, then

$$\iint_{\mathcal{S}} \vec{F} \cdot \, \mathrm{d}S = \iint_{\mathcal{D}} \Big(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\Big) \, \mathrm{d}x \, \mathrm{d}y.$$