Solutions to Midterm 2 practice problems.

① C_R is parametrized by $\bar{c}(t) = (R\cos\theta, R\sin\theta)$, $0 \le \theta \le 2\pi$

$\bar{c}'(t) = (-R\sin\theta, R\cos\theta)$ and $\bar{F}(\bar{c}(t)) = \frac{-R\sin\theta}{R^2}\hat{\imath} + \frac{R\cos\theta}{R^2}\hat{\jmath}$

Then

$$I_R = \oint_{C_R} \bar{F}\cdot d\bar{s} = \int_0^{2\pi} \bar{F}(\bar{c}(t))\cdot\bar{c}'(t)\,d\theta = \int_0^{2\pi} \frac{(-R\sin\theta)^2}{R^2} + \frac{(R\cos\theta)^2}{R^2}\,d\theta$$

$$= \int_0^{2\pi} 1\,d\theta = \theta\Big|_0^{2\pi} = 2\pi.$$

② i. $P(x,y) = 9y - y^3$ $\qquad Q(x,y) = e^{\sqrt{y}}(x^2 - 3x)$

$$\frac{\partial P}{\partial y} = 9 - 3y^2 \neq e^{\sqrt{y}}(2x-3) = \frac{\partial Q}{\partial x}$$

$\therefore$ $\bar{F}$ is not conservative

ii. The boundary of $[0,3]\times[0,3]$ consists of 4 segments, 2 of them are horizontal and the other 2 are vertical.

On the horizontal segments $\bar{c}'(t)$ is a scalar multiple of $\hat{\imath} = (1,0)$ independently of what parametrization is used. Also on these segments $\bar{F}$ is a scaler multiple of $\hat{\jmath} = (0,1)$ because on said segments $y = 0$ or $y = 3$ Therefore the integral of F along the horizontal segments is zero.

A similar analysis shows that the integral over the vertical segments is also zero. Therefore $\int_{-\partial([0,3]\times[0,3])} \bar{F}\cdot d\bar{s} = 0$.

(The (-) sign in $-\partial([0,3]\times[0,3])$ indicates that the orientation is clockwise).

(3) $$\frac{d}{dr}\Big(r\big(\cos(h(r)), \sin(h(r))\big)\Big) = r\big(-h'(r)\sin(h(r)), h'(r)\cos(h(r))\big) + 1\big(\cos(h(r)), \sin(h(r))\big)$$

hence

$$\bar{c}'(r) = r h'(r)\big(-\sin(h(r)), \cos(h(r))\big) + \big(\cos(h(r)), \sin(h(r))\big)$$

let
$$\bar{a}_r = \big(-\sin(h(r)), \cos(h(r))\big)$$
$$\bar{b}_r = \big(\cos(h(r)), \sin(h(r))\big)$$

then $a_r \circ b_r = 0$ and $\|\bar{a}_r\| = \|\bar{b}_r\| = \cos^2(h(r)) + \sin^2(h(r)) = 1$

then

$$\|\bar{c}'(r)\|^2 = \bar{c}'(r)\circ\bar{c}'(r) = (r h'(r)\bar{a}_r + \bar{b}_r)\circ(r h'(r)\bar{a}_r + \bar{b}_r)$$
$$= r^2(h'(r))^2\|\bar{a}_r\|^2 + 2r h'(r)\,\bar{a}_r\circ\bar{b}_r + \|\bar{b}_r\|^2$$
$$= r^2(h'(r))^2 + 1$$

and $$\text{Length}(C) = \int_C 1\, ds = \int_{r_1}^{r_2} \|\bar{c}'(r)\|\, dr = \int_{r_1}^{r_2} \sqrt{1 + r^2(h'(r))^2}\, dr.$$

(4) To parametrize S in a way that it is ~~ea~~ not hard to 'read' the parameters, try cylindrical coordinates with y as the variable that plays the role of the 'height'.

That is set $\quad y = y$ and $\begin{array}{l} x = r\cos\theta \\ z = r\sin\theta . \end{array}$

The choice of the parameters y and θ ~~aa~~ is natural, namely $0 \le y \le 1$ and $0 \le \theta \le 2\pi$. The only thing that is left to do, is to express the surface in terms of y and θ.

To do this, notice that $x^2+y^2+z^2=4$ leads to

$y^2+r^2=4$ and $r=\sqrt{4-y^2}$.

The parametrization is then $G(y,\theta) = \left(\sqrt{4-y^2}\cos\theta,\ y,\ \sqrt{4-y^2}\sin\theta\right)$

$$0 \le y \le 1, \quad 0 \le \theta \le 2\pi.$$

$$T_y = \left(\frac{-y\cos\theta}{\sqrt{4-y^2}},\ 1,\ \frac{-y\sin\theta}{\sqrt{4-y^2}} \right) = \frac{1}{\sqrt{4-y^2}} \left(-y\cos\theta,\ \sqrt{4-y^2},\ -y\sin\theta\right)$$

$$T_\theta = \left(-\sqrt{4-y^2}\sin\theta,\ 0,\ \sqrt{4-y^2}\cos\theta\right) = \sqrt{4-y^2}\left(-\sin\theta,\ 0,\ \cos\theta\right)$$

$$T_y \times T_\theta = \frac{1}{\sqrt{4-y^2}} \cdot \sqrt{4-y^2}\left(\sqrt{4-y^2}\cos\theta,\ y,\ \sqrt{4-y^2}\sin\theta\right)$$

Notice that since $0 \le y \le 1$ $T_y \times T_\theta$ is the outward normal.
(we should keep this in mind when integrating vector fields, for scalar functions it makes no difference)

We then have

$$\| T_y \times T_\theta \|^2 = (4-y^2)\cos^2\theta + y^2 + (4-y^2)\sin^2\theta = 4 - y^2 + y^2 = 4$$

$$\therefore \quad \| T_y \times T_\theta \| = 2$$

Finally,

$$\iint_S y \, dS = \int_0^1 \int_0^{2\pi} y \, (2) \, d\theta \, dy = y^2 \Big|_0^1 \theta \Big|_0^{2\pi} = 2\pi .$$

(5) The parametrization in polar coordinates y, θ is

$$G(y,\theta) = (y\cos\theta, \, y\sin\theta, \, g(y))$$

$$T_y = (\cos\theta, \, \sin\theta, \, g'(y))$$

$$T_\theta = (-y\sin\theta, \, y\cos\theta, \, 0) = y(-\sin\theta, \, \cos\theta, \, 0)$$

$$T_y \times T_\theta = y\left(-(\cos\theta)\, g'(y), \, -(\sin\theta)\, g'(y), \, 1 \right)$$

$$\| T_y \times T_\theta \|^2 = y^2\left((g'(y))^2 + 1 \right) \Rightarrow \| T_y \times T_\theta \| = y\sqrt{1 + (g'(y))^2} \quad b/c \ y > 0$$

$$\therefore \quad Area(S) = \int_S 1 \, dS = \int_0^{2\pi}\int_c^d \| T_y \times T_\theta \| \, dy \, d\theta = 2\pi \int_c^d y\sqrt{1 + (g'(y))} \, dy .$$

(6) For $\iint_S z^2\,dS$ use the parametrization (See problem 4)

$$G(z,\theta) = \left(\sqrt{R^2-z^2}\cos\theta,\ \sqrt{R^2-z^2}\sin\theta,\ z\right)$$

$$0 \le \theta \le 2\pi \qquad -R \le z \le R$$

then

$$T_z = \frac{1}{\sqrt{R^2-z^2}}\left(-z\cos\theta,\ -z\sin\theta,\ 1\right)$$

$$T_\theta = \sqrt{R^2-z^2}\left(-\sin\theta,\ \cos\theta,\ 0\right)$$

$$T_z \times T_\theta = \left(-\sqrt{R^2-z^2}\cos\theta,\ -\sqrt{R^2-z^2}\sin\theta,\ -z\right)$$

and $\|T_z \times T_\theta\| = R$. Then $\iint_S z^2\,dS = \int_{-R}^{R}\int_0^{2\pi} z^2 (R)\,d\theta\,dz$

For $\iint_S y^2\,dS$ use $H(y,\theta) = \left(\sqrt{R^2-z^2}\cos\theta,\ y,\ \sqrt{R^2-z^2}\sin\theta\right)$

$$0 \le \theta \le 2\pi \qquad -R \le y \le R$$

and just as before.

$$\iint_S y^2\,dS = \int_{-R}^{R}\int_0^{2\pi} y^2 (R)\,d\theta\,dy = \int_{-R}^{R}\int_0^{2\pi} z^2 (R)\,d\theta\,dz.$$

The same goes for $\iint_S x^2\,dS$.

It follows that

$$\iint_S x^2\,dS = \frac{1}{3}\left[\iint_S x^2\,dS + \iint_S y^2\,dS + \iint_S z^2\,dS\right] = \frac{1}{3}\iint_S \overbrace{x^2+y^2+z^2}^{=R^2 \text{ on } S}\,dS$$

$$= \frac{R^2}{3}\iint_S 1\,dS = \frac{R^2}{3}\left(4\pi R^2\right) = \frac{4\pi R^4}{3}.$$

(7) This problem can be solved using the parametrizations described in problems 4 and 6, but to have a different point of view let's try spherical coordinates.

$$G(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$T_\theta = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$$

$$T_\varphi = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$$

$$T_\theta \times T_\varphi = (-\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi)$$

Since $\frac{1}{2} \le z \le \frac{\sqrt{3}}{2}$ we get from $z = \cos\varphi$ that then angle φ ranges from $\frac{\pi}{6}$ to $\frac{\pi}{3}$. $\left(\begin{array}{l}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \cos\pi/3 = \frac{1}{2}\end{array}\right)$

Then

$$\iint_S \langle x,y,z\rangle \cdot d\vec{S} = \int_{\pi/6}^{\pi/3}\int_0^{2\pi} \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle \cdot \langle -\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi\rangle \, d\theta \, d\varphi$$

$$= \cdots = \int_{\pi/6}^{\pi/3}\int_0^{2\pi} -\sin\varphi \, d\theta\, d\varphi = \cos\varphi\Big|_{\pi/6}^{\pi/3} (2\pi)$$

Lots of room for mistakes here!

$$= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)(2\pi)$$

$$= \pi(1-\sqrt{3}) < 0$$

consistent with the fact that $\langle x,y,z\rangle$ and the normal to S point in opposite directions.

(8) Parametrize S by x, y-coordinates.

$$G(x,y) = (x, y, g(x,y))$$

$$T_x = \left(1, 0, \frac{\partial g}{\partial x}(x,y)\right)$$

$$T_y = \left(0, 1, \frac{\partial g}{\partial y}(x,y)\right)$$

$$T_x \times T_y = \left(-\frac{\partial g}{\partial x}(x,y), -\frac{\partial g}{\partial y}(x,y), 1\right)$$ points upwards

$$\iint_S \bar{F} \cdot d\bar{S} = \iint_D \bar{F}(G(x,y)) \cdot (T_x \times T_y)\, dx\, dy$$

$$= \iint_D -P(x,y,g(x,y))\frac{\partial g}{\partial x}(x,y) - Q(x,y,g(x,y))\frac{\partial g}{\partial y}(x,y) + R(x,y,g(x,y))\, dx\, dy$$

or dropping the dependence on (x,y)

$$\iint_S \bar{F} \cdot d\bar{S} = \iint_D -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\, dx\, dy.$$