# NOTES ON TWO PAPERS BY A. OBERMAN

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Abstract. These are notes on two papers by Adam Oberman: "Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problems" and "Convergent difference schemes for elliptic partial differential equations which are functions of the eigenvalues of the hessian." Definitions, main points and remarks are noted.

## Paper [1].
## Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problem

### Important definitions.

- **Degenerate Elliptic (DE) PDE**: The equation $F[u](x) = F(x, u(x), Du(x), D^2u(x)) = 0$ is *degenerately elliptic* if:
$$F(x, r, p, X) \le F(x, s, p, Y) \quad \text{whenever} \quad r \le s \text{ and } Y \le X.$$
Note that $Y \le X$ means $Y - X$ is non-negative (is this not non-positive?) definite symmetric.
- **Monotonicity for DE PDE**: Let the solution operator $S : g \to u$, where $g$ is a continuous b.c., and $u$, its corresponding unique continuous solution. Given and DE equation $F[u](x) = 0$, $S$ is *monotone* if $\forall$ continuous $g, h$ on $\partial\Omega$,
$$g(x) \le h(x) \forall x \in \partial\Omega \implies S(g)(x) \le S(h)(x) \forall x \in \Omega.$$
- **Non-expansive in the Max norm**: The solution operator $S$ is *non-expansive in the max norm* if,
$$\max_{x\in\Omega} |S(g)(x) - S(h)(x)| \le \max_{x\in\partial\Omega} |g(x) - h(x)|.$$
- **DE schemes**: A scheme, $F = [F^i]_{i=1}^N$ ($i$ index of grid points),
$$F^i[u] \equiv F^i(u_i, u_i - u_j|_{j\in N(i)})$$
is *degenerately elliptic* if each component of $F^i$ for all $i$ is non-decreasing in each variable. In other words, $F^i$ is non-decreasing with respect to $u_i$ and $u_i - u_j|_{j\in N(i)}$.
- **Lipschits Continuity for F.D. schemes**: A F.D. schemes, F, is *Lipschitz continuous* if $\exists K$ such that $\forall i = 1, \ldots, N, x, y \in \Re^{|N(i)|+1}$,
$$|F^i(x) - F^i(y)| \le K||x - y||_\infty$$

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- **Proper schemes**: A F.D. schemes is *proper* if $\exists\delta > 0$ such that $i = 1, \ldots, N$ and $\forall x \in \Re^{|}N(i)|$ and $x_0, y_0 \in \Re$,

$$x_0 \leq y_0 \to F^i(x_0, x) - F(y_0, x) \leq \delta(x_0 - y_0) \quad (< 0)$$

- **Explicit Euler map**: Given $\rho > 0$ and a F.D. scheme $F$, define the *explicit Euler map*, $S_\rho$, by,

$$S_\rho(u) = u - \rho F[u].$$

The $\rho$ is the *time step* for the ODE $du/dt + F[u] = 0$.
- **Nonlinear CFL condition**: Given a Lipschitz continuous scheme $F$ with Lipschitz constant $K$, the *nonlinear CFL condition* for the corresponding Euler map $S_\rho$ is

$$\rho \leq \frac{1}{K}.$$

**Major Results.**

- **(Theorem 1)** Provided mild analytic conditions for well-definedness, *the solution operator of a DE PDE is monotone and non-expansive in the max norm.*
- **(Theorem 2)**Provided mild analytic conditions for well-definedness, *the solution operator of a DE F.D. scheme is monotone and non-expansive in the max norm.*
- **(Theorem 3)** Given a scheme $F$.

$$F \text{ monotone and non-expansive in max norm} \Leftrightarrow F \text{ is DE.}$$

- **(Theorem 5)** Let a F.D. scheme $F$ be proper and DE. Then,

$$F[u] \leq F[v] \implies u \leq v,$$

and solutions are unique, i.e. $F[u] = F[v] = 0 \implies u = v$.
- **(Theorem 6,7,8 - Euler maps)** Let $F$ be Lipschitz and DE, and $S_\rho$ is corresponding Euler map. Then,
  (1) If the nonlinear CFL conditions hold, then $S_\rho$ is monotone.
  (2) If the (strict) inequality of the nonlinear CFL conditions hold, then $S_\rho$ is a (strict) contraction in the max norm in $\Re^N$.
  (3) If the strict inequality of the nonlinear CFL conditions hold, then $S_\rho$ converges to the solution from arbitrary initial condition.
- **(Observations 1,2 - Building elliptic schemes)** Let $F_1, F_2$ be DE functions (or schemes), and $g : \Re^2 \to \Re$ a non-decreasing function. Then,
  (1) $F = g(F_1, F_2)$ is a DE function (or scheme).
  (2) If $F$ is defined as,

$$F[u] = \begin{cases} F_2[u] = 0 & \text{if } G[u] > 0 \\ F_1[u] = 0 & \text{otherwise,} \end{cases}$$

  then it is DE if $G[u]$ is DE.

**Other remarks.**

- Schemes that are DE have a stronger property than stability: they are non-expansive in the max norm They are also monotone; but monotonicity alone does not imply stability.
- Experiments on sinusoids show stability alone may not avoid errors (artificial diffusion, in this case) comparable to the data values.

- If $F$ is not a proper scheme, one can define $\bar{F}[u] = F[u] + \varepsilon u$, for $\varepsilon > 0$ small, to make it proper (still need to check this).

## Paper [2].
### Convergent difference schemes for elliptic partial differential equations which are functions of the eigenvalues of the hessian

**Equation of interest.** Let $D^2u$ be the Hessian of a function $u : \Re^2 \to \Re$, and let $\lambda_-[u](x) \leq \lambda_+[u](x)$ be the eigenvalues of $D^2u$. Consider the dirichlet problem on a bounded domain $\Omega \subset \Re^2$,

$$\begin{cases} F[u] \equiv -G(D^2u(x)) + f(x) = 0 & \text{in } \Omega \\ u = g \quad \text{on } \partial\Omega \end{cases}$$

**Important definitions.**

- **Courant-Hilbert characterization of eigenvalues**: Let $v_\theta = (\cos\theta, \sin\theta)$ be the unit vector in the direction $\theta$. Then,

$$\begin{aligned} \lambda_-[u](x) &= \min_\theta \frac{d^2u}{dv_\theta^2} \\ \lambda_+[u](x) &= \max_\theta \frac{d^2u}{dv_\theta^2} \end{aligned}$$

- **Comparison Principle**: For the given PDE, the *comparison principle* is,

$$F[u_1] \leq F[u_2] \text{ in } \Omega, u_1 \leq u_2 \text{ on } \partial\Omega \implies u_1 \leq u_2 \text{ in } \Omega.$$

- **Local spatial and directional resolution**: Let $x_0$ be a point on a grid and $i$ be indices of the neighbours of $x_0$. Then the *local spacial resolution* is,

$$dx(x_0) \equiv \max_i dx_i$$

  and the *local directional resolution* is,

$$d\theta(x_0) \equiv \max_{\theta \in [-\pi,\pi]} \min_i |\theta - \theta_i|$$

- **Schemes for $\lambda_-$ and $\lambda_+$**: Discretize the Courant-Hilbert characterization of the eigenvalues by

$$\begin{aligned} \lambda_-^{dx,d\theta}[u](x_0) &= \min_i \frac{u(x_0 + v_i) - 2u(x_0) + u(x_0 - v_i)}{|v_i|^2} \\ \lambda_+^{dx,d\theta}[u](x_0) &= \max_i \frac{u(x_0 + v_i) - 2u(x_0) + u(x_0 - v_i)}{|v_i|^2} \end{aligned}$$

**Major Results.**

- **(Theorem 1 - Barles-Souganidis)** Suppose the given DE PDE has a unique viscosity solution. Then, for a scheme $F$,

$$F \text{ is consistent, stable and monotone } \Longrightarrow$$
$$F \text{ converges to the viscosity solution.}$$

- **(Lemma 1, Theorem 3)** The schemes defined above for $\lambda_-$ and $\lambda_+$ are degenerately elliptic. Furthermore, for a given $\phi \in C^2$ defined on the neighbourhood of the grid point $x_0$,

$$\lambda_-[\phi](x_0) - \lambda_-^{dx,d\theta}[\phi](x_0) = O(dx^2 + d\theta^2),$$

and likewise for $\lambda_+$.

**Other remarks.**

- Most theoretical results in this paper owe itself to the earlier paper and the Barles-Souganidis result.
- The the directional vector $v_i$ at a point $x_0$ on a grid, often it is assume that $-v_i$ also exists on the grid. In terms of notation, the set of direction vectors is denoted $S = \{v_{-l}, \ldots, v_{-1}, v_1, \ldots, v_l\}$, where $v_{-i} = -v_i$.
- in practice, the number of iterations for convergence within $O(dx^2)$ is proportional to the diameter of the graph, which is of order $1/dx$.
- The "properization" of a scheme $\bar{F}[u] = F[u] + \varepsilon u$, was required only for the Monge-Ampere equation.

## References

[1] A. M. Oberman. 'Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problem", SIAM J. Numer. Anal., Vol. 44, No. 2, pp.879-895 (2006)

[2] A. M. Oberman. "Convergent difference schemes for elliptic partial differential equations which are functions of the eigenvalues of the hessian"

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