STABLE CATEGORIES AND RECONSTRUCTION

JEREMY RICKARD AND RAPHAËL ROUQUIER

Dedicated to the memory of Sandy Green

1. Introduction

The Green correspondence is a fundamental construction in modular representation theory of finite groups. It is expected (Broué's abelian defect group conjecture for example) to be the shadow of a more structural categorical correspondence, yet to be found. In an inductive approach to this, a key case is when the Green correspondence induces a stable equivalence between blocks. This work is an attempt towards a Morita theory for stable equivalences between self-injective algebras. More precisely, given two self-injective algebras A and B and an equivalence between their stable categories, consider the set $\mathcal{S}$ of images of simple B-modules inside the stable category of A. That set satisfies some obvious properties of Hom-spaces and it generates the stable category of A. Keep now only $\mathcal{S}$ and A. Can B be reconstructed ? We show how to reconstruct the graded algebra associated to the radical filtration of (an algebra Morita equivalent to) B. It would be interesting to develop further an obstruction theory for the existence of an algebra B with that given filtration, starting only with $\mathcal{S}$ (this might be studied in terms of localization of A_∞-algebras). Note that a result of Linckelmann [Li] shows that, if we consider only stable equivalence of Morita type, then B is characterized by $\mathcal{S}$ — but this result does not provide a reconstruction of B from $\mathcal{S}$.

We also study a similar problem in the more general setting of a triangulated category $\mathcal{T}$. Given a finite set $\mathcal{S}$ of objects satisfying Hom-properties analogous to those satisfied by the set of simple modules in the derived category of a ring and assuming that the set generates $\mathcal{T}$, we construct a t-structure on $\mathcal{T}$. In the case $\mathcal{T} = D^b(A)$ and A is a symmetric algebra, the first author has shown [Ri] that there is a symmetric algebra B with an equivalence $D^b(B) \xrightarrow{\sim} D^b(A)$ sending the set of simple B-modules to $\mathcal{S}$. The case of a self-injective algebra leads to a slightly more general situation : there is a finite dimensional differential graded algebra B with $H^i(B) = 0$ for $i > 0$ and for $i \ll 0$ with the same property as above.

2. Notations

Let $\mathcal{C}$ be an additive category. Given S a set of objects of $\mathcal{C}$, we denote by $\operatorname{add} S$ the full subcategory of $\mathcal{C}$ of objects isomorphic to finite direct sums of objects of S.

Let k be a field and A a finite dimensional k-algebra. We say that A is split if the endomorphism ring of every simple A-module is k. We denote by A-mod the category of finitely generated left A-modules and by $D^b(A)$ its derived category. For A self-injective, we denote by A-stab the stable category, the quotient of A-mod by projective modules. Given M an A-module, we denote by ΩM the kernel of a projective cover of M and by $\Omega^{-1} M$ the cokernel of an injective hull of M.

3. Simple generators for triangulated categories

3.1. **Category of filtered objects.** Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a full subcategory of $\mathcal{T}$.

We define a category $\mathcal{F}$ as follows.

• Its objects are diagrams

$$M = (\cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{\varepsilon_0} N_0)$$

where M_i is an object of $\mathcal{T}$, $M_i = 0$ for $i \gg 0$, such that

(i) $M_1 \xrightarrow{f_1} M_0 \xrightarrow{\varepsilon_0} N_0$ is the beginning of a distinguished triangle
(ii) for all $i \geq 1$, the cone N_{i-1} of f_i is in $\operatorname{add}\mathcal{S}$
(iii) the canonical map $\operatorname{Hom}(N_0, S) \to \operatorname{Hom}(M_0, S)$ is surjective for all $S \in \mathcal{S}$
(iv) the canonical map $\operatorname{Hom}(N_i, S) \to \operatorname{Hom}(M_i, S)$ is bijective for all $S \in \mathcal{S}$ and $i \geq 1$.

Note that $\varepsilon_i : M_i \to N_i = \operatorname{cone}(f_{i+1})$ is well defined up to unique isomorphism for $i \geq 1$ thanks to property (iv). For $i \geq 0$, we define a new object $M_{\geq i}$ of $\mathcal{F}$ as $\cdots \to M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{\varepsilon_i} N_i$.

• Given another diagram M', we define $\operatorname{Hom}_{\mathcal{F}}(M, M')_0$ as the subspace of $\operatorname{Hom}(N_0, N'_0)$ consisting of those maps g such that there is $h : M_0 \to M'_0$ with $\varepsilon'_0 h = g\varepsilon_0$.

We put $\operatorname{Hom}_{\mathcal{F}}(M, M')_i = \operatorname{Hom}_{\mathcal{F}}(M, M'_{\geq i})_0$ and $\operatorname{Hom}_{\mathcal{F}}(M, M') = \oplus_{i \geq 0} \operatorname{Hom}_{\mathcal{F}}(M, M')_i$.

• Consider now $g_0 \in \operatorname{Hom}_{\mathcal{F}}(M, M')$. By (iv), there are maps $h_0, h_1, \ldots$ and $g_1, g_2, \ldots$ making the following diagrams commutative

$$\begin{array}{ccccccc}
N_i[-1] & \xrightarrow{\rho_i} & M_{i+1} & \xrightarrow{f_{i+1}} & M_i & \xrightarrow{\varepsilon_i} & N_i \\
\downarrow{\scriptstyle g_i[-1]} & & \downarrow{\scriptstyle h_{i+1}} & & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle g_i} \\
N'_i[-1] & \xrightarrow[\rho'_i]{} & M'_{i+1} & \xrightarrow[f'_{i+1}]{} & M'_i & \xrightarrow[\varepsilon'_i]{} & N'_i
\end{array}$$

Here, $\rho_i : N_i[-1] \to M_{i+1}$ and $\rho'_i : N'_i[-1] \to M'_{i+1}$ are the maps making the horizontal rows in the diagram above into distinguished triangles.

Lemma 3.1. *The maps $g_i : N_i \to N'_i$ (for $i \geq 1$) depend only on g_0.*

Proof. We proceed by induction on i. We assume g_{i-1} has been shown to depend only on g_0. Let us consider the lack of unicity of h_i. Consider $h_i, \tilde{h}_i : M_i \to M'_i$ such that $h_i\rho_{i-1} = \rho'_{i-1}g_{i-1}[-1] = \tilde{h}_i\rho_{i-1}$. There is $p : M_{i-1} \to M'_i$ such that $\tilde{h}_i - h_i = pf_i$.

By (iii) and (iv), there exists $q : N_{i-1} \to N_i$ such that $q\varepsilon_{i-1} = \varepsilon'_i p$. We have $\varepsilon'_i p f_i = q\varepsilon_{i-1}f_i = 0$, hence $\varepsilon'_i\tilde{h}_i = \varepsilon'_i h_i$.

By (iv), we deduce that there is a unique map $g_i : N_i \to N'_i$ such that $g_i\varepsilon_i = \varepsilon'_i h_i$ and that map g_i is the unique one such that $g_i\varepsilon_i = \varepsilon'_i\tilde{h}_i$. □

Let $g_0 \in \operatorname{Hom}_{\mathcal{F}}(M, M')_i$ and $g'_0 \in \operatorname{Hom}_{\mathcal{F}}(M', M'')_j$. We define the product $g'_0 g_0$ as the composition $N_0 \xrightarrow{g_0} N'_i \xrightarrow{g'_i} N''_{i+j}$.

Lemma 3.2. *Assume $\operatorname{Hom}(S, T[n]) = 0$ for all $S, T \in \mathcal{S}$ and $n < 0$. Let M be an object of $\mathcal{F}$. Then, the canonical map $\operatorname{Hom}(N_0, S) \to \operatorname{Hom}(M_0, S)$ is an isomorphism.*

Proof. By induction on $-i$, we see that $\mathrm{Hom}(M_i, S[n]) = 0$ for $n < 0$ and $S \in \mathcal{S}$. It follows that $\mathrm{Hom}(M_1[1], S) = 0$, hence the canonical map $\mathrm{Hom}(N_0, S) \to \mathrm{Hom}(M_0, S)$ is injective, as well as being surjective by assumption. □

3.2. ***t*-structures.** Let k be a field and assume $\mathcal{T}$ is a k-linear triangulated category.

We assume from now on the following

Hypothesis 1. (1) $\mathrm{Hom}(S, T) = k^{\delta_{S,T}}$ for $S, T \in \mathcal{S}$
(2) $\mathcal{S}$ generates $\mathcal{T}$ as a triangulated category
(3) $\mathrm{Hom}(S, T[n]) = 0$ for $S, T \in \mathcal{S}$ and $n < 0$.

3.2.1.

Lemma 3.3. *Given $N \in \mathcal{T}$, there is a sequence $0 = M_r \xrightarrow{f_r} \cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$ and $d : \mathbf{Z}_{>0} \to \mathbf{Z}$ non increasing such that $\mathrm{cone}(f_i)[d(i)] \in \mathcal{S}$.*

For such a sequence, the maps $M_{r-1} \to N$ and $N \to \mathrm{cone}(f_1)$ are non zero.

Proof. Since $\mathcal{T}$ is generated by $\mathcal{S}$, there is a sequence $0 = M_r \to \cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$ and $d : \mathbf{Z}_{>0} \to \mathbf{Z}$ such that $\mathrm{cone}(f_i)[d(i)] \in \mathcal{S}$.

We put $N_i = \mathrm{cone}(f_i) = S_i[-d(i)]$ with $S_i \in \mathcal{S}$. Take i such that $d(i) > d(i-1)$. Let T be the cone of $f_{i-1}f_i : M_i \to M_{i-2}$. The octahedral axiom gives a distinguished triangle $S_i[-d(i)] \to T \to S_{i-1}[-d(i-1)] \rightsquigarrow$.

Assume the morphism $S_{i-1}[-d(i-1)] \to S_i[-d(i)+1]$ is non zero. Then it is an isomorphism and $d(i) = d(i-1)+1$. It follows that $T = 0$ and $f_{i-1}f_i$ is an isomorphism. Consequently,

$$0 = M_r \to \cdots \to M_{i+1} \xrightarrow{f_{i-1}f_if_{i+1}} M_{i-2} \to \cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$$

is a new sequence with successive cones being shifts of objects of $\mathcal{S}$.

By induction, we can assume that the morphism $S_{i-1}[-d(i-1)] \to S_i[-d(i)+1]$ is zero. Then, $T \simeq N_i \oplus N_{i-1}$. There is an object M'_{i-1} and distinguished triangles $M_i \to M'_{i-1} \to N_{i-1} \rightsquigarrow$ and $M'_{i-1} \to M_{i-2} \to N_i \rightsquigarrow$. Put $M'_j = M_j$ for $j \neq i-1$. So,

$$0 = M'_r \to \cdots \to M'_2 \to M'_1 \to M'_0 = N$$

is a new sequence with the same cones as in the original sequence except the i and $i-1$ ones which have been swapped. By induction, we can reorder the cones in the sequence so that d is non increasing.

Assume the map $M_{r-1} \to N$ is zero. Let T be its cone. Then $T \simeq N \oplus M_{r-1}[1]$. Note that T is filtered by the $S_i[-d(i)]$ with $-d(i) < -d(r)+1$, hence $\mathrm{Hom}(M_{r-1}[1], T) = 0$. So we have a contradiction. The case of the map $N \to N_1$ is similar. □

Let $\mathcal{T}^{\leq 0}$ (resp. $\mathcal{T}^{>0}$) be the full subcategory of objects N in $\mathcal{T}$ such that there is a sequence $0 = M_r \to \cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$ with $\mathrm{cone}(f_i)$ a direct sum of objects $S[r]$ with $S \in \mathcal{S}$ and $r \geq 0$ (resp. $r < 0$).

Proposition 3.4. *$(\mathcal{T}^{\leq 0}, \mathcal{T}^{>0})$ is a bounded t-structure on $\mathcal{T}$.*

Proof. By induction, we see there is no non-zero map from an object of $\mathcal{T}^{\leq 0}$ to an object of $\mathcal{T}^{>0}$. Furthermore, we have $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{>0} \subseteq \mathcal{T}^{>0}[1]$.

Let $N \in \mathcal{T}$. Pick a sequence as in Lemma 3.3. Take s such that $d(s) > 0$ and $d(s+1) \leq 0$. Let L be the cone of $f_1 \cdots f_s : M_s \to N$. We have a distinguished triangle

$$M_s \to N \to L \rightsquigarrow$$

with $M_s \in \mathcal{T}^{\leq 0}$ and $L \in \mathcal{T}^{>0}$. □

We have a characterization of $\mathcal{T}^{\geq 0}$ and $\mathcal{T}^{\leq 0}$:

Proposition 3.5. *Let $N \in \mathcal{T}$. Then, $N \in \mathcal{T}^{\leq 0}$ if and only if $\mathrm{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$.*

Similarly, $N \in \mathcal{T}^{\geq 0}$ if and only if $\mathrm{Hom}(S[i], N) = 0$ for $S \in \mathcal{S}$ and $i > 0$.

Proof. We have $\mathrm{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$, if $N \in \mathcal{S}[r]$ with $r \geq 0$. By induction, it follows that if $N \in \mathcal{T}^{\leq 0}$, then $\mathrm{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$.

Assume now $\mathrm{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$. Pick a filtration of N as in Lemma 3.3. Then, $d(1) \leq 0$, hence $d(i) \leq 0$ for all i and $N \in \mathcal{T}^{\leq 0}$.

The other case is similar. □

Note that the heart $\mathcal{A}$ of the t-structure is artinian and noetherian. Its set of simple objects is $\mathcal{S}$.

Remark 3.6. Assume $\mathcal{T}$ can be generated by a finite set of objects. Then, there is a finite subcategory $\mathcal{S}'$ of $\mathcal{S}$ generating $\mathcal{T}$. It follows immediately from condition (i) that $\mathcal{S} = \mathcal{S}'$. So, $\mathcal{S}$ has only finitely many objects.

3.2.2. In §3.2.2, we assume $\mathcal{T} = D^b(A)$ where A is a finite dimensional k-algebra. By Remark 3.6, $\mathcal{S}$ is finite (note that $\mathcal{T}$ is generated by the simple A-modules, up to isomorphism).

Proposition 3.7. *Let $S \in \mathcal{S}$. There is a bounded complex of finitely generated injective A-modules $I_{\mathcal{S}}(S) \in \mathcal{T}^{\geq 0}$ such that, given $T \in \mathcal{S}$ and $i \in \mathbf{Z}$, we have*

$$\mathrm{Hom}_{D^b(A)}(T, I_{\mathcal{S}}(S)[i]) = \begin{cases} k & \text{for } i = 0 \text{ and } S = T \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, there is a bounded complex of finitely generated projective A-modules $P_{\mathcal{S}}(S) \in \mathcal{T}^{\leq 0}$ such that, given $T \in \mathcal{S}$ and $i \in \mathbf{Z}$, we have

$$\mathrm{Hom}_{D^b(A)}(P_{\mathcal{S}}(S)[i], T) = \begin{cases} k & \text{for } i = 0 \text{ and } S = T \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The construction of a complex $I_{\mathcal{S}}(S)$ of A-modules with the Hom property is [Ri, §5] (note that the proof of [Ri, Lemma 5.4] is valid for non-symmetric algebras). It is in $\mathcal{T}^{\geq 0}$ by Proposition 3.5. Since $\bigoplus_{i \in \mathbf{Z}} \dim \mathrm{Hom}_{D^b(A)}(V, I_{\mathcal{S}}(S)[i]) = 0$ for all simple A-modules V, we deduce that $I_{\mathcal{S}}(S)$ is isomorphic to a bounded complex of finitely generated injective A-modules.

The second case follows from the first one by passing to A^{opp} and taking the k-duals of elements of $\mathcal{S}$. □

We denote by $\tau^{>0}$, etc... the truncation functors and ${}^tH^0$ the H^0-functor associated to the t-structure constructed in §3.2.1.

Lemma 3.8. *The object ${}^tH^0(I_{\mathcal{S}}(S))$ of $\mathcal{A}$ is an injective hull of S and ${}^tH^0(P_{\mathcal{S}}(S))$ is a projective cover of S.*

Proof. We have a distinguished triangle

$$^tH^0(I_{\mathcal{S}}(S)) \to I_{\mathcal{S}}(S) \to \tau^{>0} I_{\mathcal{S}}(S) \rightsquigarrow .$$

Let $N \in \mathcal{A}$. We have $\mathrm{Hom}(N, \tau^{>0} I_{\mathcal{S}}(S)) = 0$ and $\mathrm{Hom}(N, I_{\mathcal{S}}(S)[1]) = 0$, so we deduce that $\mathrm{Hom}(N, {}^tH^0(I_{\mathcal{S}}(S))[1]) = 0$. It follows that $\mathrm{Ext}^1_{\mathcal{A}}(N, {}^tH^0(I_{\mathcal{S}}(S))) = 0$, hence ${}^tH^0(I_{\mathcal{S}}(S))$ is injective. Since $\mathrm{Hom}(T, (\tau^{>0} I_{\mathcal{S}}(S))[-1]) = 0$, we have $\mathrm{Hom}(T, {}^tH^0(I_{\mathcal{S}}(S))) \xrightarrow{\sim} \mathrm{Hom}(T, I_{\mathcal{S}}(S)) = k^{\delta_{ST}}$ for $T \in \mathcal{S}$. So ${}^tH^0(I_{\mathcal{S}}(S))$ is an injective hull of S. The projective case is similar. □

Let us consider the finite dimensional differential graded algebra

$$B = \mathrm{End}^\bullet_A\Big(\bigoplus_S P_{\mathcal{S}}(S)\Big) = \bigoplus_i \mathrm{Hom}_A\Big(\bigoplus_S P_{\mathcal{S}}(S), \bigoplus_S P_{\mathcal{S}}(S)[i]\Big).$$

Denote by $D^b(B)$ the derived category of finite dimensional differential graded B-modules.

Theorem 3.9. *We have $H^i(B) = 0$ for $i > 0$ and for $i \ll 0$. We have $H^0(B)$-mod $\simeq \mathcal{A}$ and $D^b(B) \simeq D^b(A)$.*

Proof. Let $N \in \mathcal{T}$ and consider a filtration of N as in Lemma 3.3. Take $S \in \mathcal{S}$ such that $S[i]$ is isomorphic to the cone of $M_d \to M_{d-1}$. Then, $\mathrm{Hom}(P_{\mathcal{S}}(S)[i], N) \neq 0$. It follows that the right orthogonal category of $\{P_{\mathcal{S}}(S)[i]\}_{S\in\mathcal{S}, i\in\mathbf{Z}}$ is zero. Since the $P_{\mathcal{S}}(S)$ are perfect, it follows that $\bigoplus_S P_{\mathcal{S}}(S)$ generates the category of perfect complexes of A-modules as a triangulated category closed under taking direct summands [Nee, Lemma 2.2]. The functor $\mathrm{Hom}^\bullet_A(\bigoplus_S P_{\mathcal{S}}(S), -)$ gives an equivalence $D^b(A) \xrightarrow{\sim} D^b(B)$ [Ke, Theorem 4.3].

Let $C = \bigoplus_{S\in\mathcal{S}} P_{\mathcal{S}}(S)$ and $N = {}^tH^0(C)$. We have a distinguished triangle $\tau^{<0}C \to C \to N \rightsquigarrow$. We have $\mathrm{Hom}(\tau^{<0}C, N[i]) = 0$ for $i \le 0$. We deduce that the canonical morphism $\mathrm{Hom}(N, N) \to \mathrm{Hom}(C, N)$ is an isomorphism. We have $\mathrm{Hom}(C, (\tau^{<0}C)[i]) = 0$ for $i \ge 0$ since $\tau^{<0}C$ is filtered by objects in $\mathcal{S}[d]$, $d > 0$ (cf Proposition 3.7). It follows that the canonical morphism $\mathrm{Hom}(C, C) \to \mathrm{Hom}(C, N)$ is an isomorphism.

This shows that the canonical morphism $\mathrm{End}(C) \to \mathrm{End}({}^tH^0(C))$ is an isomorphism. By Lemma 3.8, ${}^tH^0(C)$ is a progenerator for $\mathcal{A}$. So $H^0(B)$-mod $\simeq \mathcal{A}$.

Note that $H^i(B) = 0$ for $i \ll 0$ because $\bigoplus_S P_{\mathcal{S}}(S)$ is bounded. Since $P_{\mathcal{S}}(S)$ is filtered by objects in $\mathcal{S}[d]$ with $d \ge 0$, it follows from Proposition 3.7 that $\mathrm{Hom}(P_{\mathcal{S}}(T), P_{\mathcal{S}}(S)[i]) = 0$ for $i > 0$. So, $H^i(B) = 0$ for $i > 0$. □

The following proposition is clear.

Proposition 3.10. *Let B be a dg-algebra with $H^i(B) = 0$ for $i > 0$ and for $i \ll 0$. Let C be the sub-dg-algebra of B given by $C^i = B^i$ for $i < 0$, $C^0 = \ker d^0$ and $C^i = 0$ for $i > 0$. Then the restriction $D(B) \to D(C)$ is an equivalence.*

Let $\mathcal{S}$ be a complete set of representatives of isomorphism classes of simple $H^0(B)$-modules (viewed as dg-C-modules). Then $\mathcal{S}$ satisfies Hypothesis 1. Furthermore, $\mathcal{A} \simeq H^0(B)$-mod.

So we have a bijection between

- the sets $\mathcal{S}$ (up to isomorphism) satisfying Hypothesis 1
- the equivalences $D^b(B) \xrightarrow{\sim} D^b(A)$ where B is a dg-algebra with $H^i(B) = 0$ for $i > 0$ and for $i \ll 0$ and where B is well-defined up to quasi-isomorphism and the equivalence is taken modulo self-equivalences of $D^b(B)$ that fix the isomorphism classes of simple $H^0(B)$-modules.

We recover a result of Al-Nofayee [Al, Theorem 4] :

Proposition 3.11. *Assume A is self-injective with Nakayama functor ν. The following are equivalent*

- $H^i(B) = 0$ *for* $i \neq 0$
- $\nu(\mathcal{S}) = \mathcal{S}$ *(up to isomorphism).*

Proof. Note that $\mathcal{S}$ is stable under ν if and only if $\{P_{\mathcal{S}}(S)\}_{S\in\mathcal{S}}$ is stable under ν (up to isomorphism). Given $S, T \in \mathcal{S}$ and $i \in \mathbf{Z}$, we have

$$\mathrm{Hom}_{D^b(A)}(P_{\mathcal{S}}(S), P_{\mathcal{S}}(T)[i])^* \simeq \mathrm{Hom}_{D^b(A)}(P_{\mathcal{S}}(T), \nu(P_{\mathcal{S}}(S))[-i]).$$

If $\mathcal{S}$ is stable under ν, then $\mathrm{Hom}_{D^b(A)}(P_{\mathcal{S}}(T), \nu(P_{\mathcal{S}}(S))[-i]) = 0$ for $i > 0$, hence $H^{<0}(B) = 0$.

Assume now $H^{<0}(B) = 0$. Then, viewed as an object of $D^b(B)$, $\nu(P_{\mathcal{S}}(S))$ is concentrated in degree 0. Since it is perfect, it is isomorphic to a projective indecomposable module, hence to $P_{\mathcal{S}}(S')$ for some $S' \in \mathcal{S}$. So, $\mathcal{S}$ is stable under ν. □

We recover now the main result of [AlRi]:

Corollary 3.12. *Let A be a self-injective algebra and B an algebra derived equivalent to A. Then B is self-injective.*

From Proposition 3.11, we recover [Ri, Theorem 5.1] :

Theorem 3.13. *If A is symmetric then $H^i(B) = 0$ for $i \neq 0$,* i.e., *there is an equivalence $D^b(\mathcal{A}) \xrightarrow{\sim} D^b(A)$ where $\mathcal{S}$ is the set of images of the simple objects of $\mathcal{A}$.*

Remark 3.14. Theorem 3.13 does not hold in general for a self-injective algebra. Take $A = k[\varepsilon]/(\varepsilon^2) \rtimes \mu_2$, where $\mu_2 = \{\pm 1\}$ acts on $k[\varepsilon]/(\varepsilon^2)$ by multiplication on ε. Assume k does not have characteristic 2. This is a self-injective algebra which is not symmetric. The Nakayama functor swaps the two simple A-modules U and V.

Let P_U (resp. P_V) be a projective cover of U (resp. V). Take $S = U$ and $T = P_U[1]$. Then, the set $\mathcal{S} = \{S, T\}$ satisfies Hypothesis 1. We have $I_{\mathcal{S}}(T) \simeq T$ and $I_{\mathcal{S}}(S) \simeq 0 \to P_U \to P_V \to 0$, a complex with homology V in degree 0 and -1.

The dg-algebra B has homology $H^0(B)$ isomorphic to the path algebra of the quiver $\bullet \longrightarrow \bullet$, $H^{-1}(B) = k$ and $H^i(B) = 0$ for $i \neq 0, -1$.

The derived category of the hereditary algebra $H^0(B)$ is not equivalent to $D^b(A)$.

3.3. **Graded of an abelian category.** Let $\mathcal{A}$ be an abelian k-linear artinian and noetherian category with finitely many simple objects up to isomorphism and $\mathcal{S}$ a complete set of representatives of isomorphism classes of simple objects. We assume $\mathcal{A}$ is split, *i.e.*, endomorphism rings of simple objects are isomorphic to k. Let $\mathcal{T} = D^b(\mathcal{A})$.

Let $\mathrm{gr}\mathcal{A}$ be the category with objects the objects of $\mathcal{A}$ and where $\mathrm{Hom}_{\mathrm{gr}\mathcal{A}}(M, N)$ is the graded vector space associated to the filtration of $\mathrm{Hom}_{\mathcal{A}}(M, N)$ given by $\mathrm{Hom}_{\mathcal{A}}(M, N)^i = \{f \,|\, \mathrm{im}\, f \subseteq \mathrm{rad}^i N\}$.

Given M in $\mathcal{A}$, let $M_i = \mathrm{rad}^i M$, $f_i : M_i \to M_{i-1}$ the inclusion, $N_0 = M/M_1$ and $\varepsilon_0 : M \to M/M_1$ the projection. This defines an object of $\mathcal{F}$.

We obtain a functor $\mathrm{gr}\mathcal{A} \to \mathcal{F}$.

Proposition 3.15. *The canonical functor* $\mathrm{gr}\mathcal{A} \to \mathcal{F}$ *is an equivalence.*

Proof. The image of $\mathrm{Hom}_{\mathcal{A}}(N, N')$ in $\mathrm{Hom}_{\mathcal{A}}(N, N'_0)$ is isomorphic to the quotient of $\mathrm{Hom}_{\mathcal{A}}(N, N')$ by $\mathrm{Hom}_{\mathcal{A}}(N, \mathrm{rad}\, N')$ and it follows that the functor is fully faithful.

Let us show that it is essentially surjective. Let $M \in \mathcal{F}$. Let $r \geq 0$ such that $M_{r+1} = 0$. Then, $M_r \xrightarrow{\sim} N_r$ has homology concentrated in degree 0 and is semi-simple. By induction on $-i$, it follows from the distinguished triangle $M_{i+1} \to M_i \to N_i \rightsquigarrow$ that M_i has homology concentrated in degree 0.

Note that we have an exact sequence $0 \to H^0 M_{i+1} \to H^0 M_i \to H^0 N_i \to 0$. Since the canonical map $\mathrm{Hom}(H^0 N_i, S) \to \mathrm{Hom}(H^0 M_i, S)$ is bijective for any simple S, it follows that $H^0 N_i$ is the largest semi-simple quotient of $H^0 M_i$. So, $M_i \xrightarrow{\sim} \mathrm{rad}^i M_0$ and M comes from an object of $\mathcal{A}$. □

4. Simple generators for stable categories

4.1. **From equivalences.** Let k be a field and A a split self-injective k-algebra with no projective simple module.

Let B be another split self-injective k-algebra with no projective simple module, and let $F : B\text{-stab} \xrightarrow{\sim} A\text{-stab}$ be an equivalence of triangulated categories. Let $\mathcal{S}'$ be a complete set of representatives of isomorphism classes of simple B-modules. For $L \in \mathcal{S}'$, let L' be an indecomposable A-module isomorphic to $F(L)$ in A-stab. Let $\mathcal{S} = \{L'\}_{L \in \mathcal{S}'}$. Then,

(i) $\mathrm{Hom}_{A\text{-stab}}(S, T) = k^{\delta_{S,T}}$ for $S, T \in \mathcal{S}$
(ii) Every object M of A-stab has a filtration $0 = M_r \to M_{r-1} \to \cdots \to M_1 \to M_0 = M$ such that the cone of $M_i \to M_{i-1}$ is isomorphic to an object of $\mathcal{S}$.

Note that (ii) is equivalent to

(ii') Given M in A-mod, there is a projective module P such that $M \oplus P$ has a filtration $0 = N_r \subset N_{r-1} \subset \cdots \subset N_1 \subset N_0 = M \oplus P$ with the property that N_i/N_{i-1} is isomorphic (in A-mod) to an object of $\mathcal{S}$.

Linckelmann has shown the following [Li, Theorem 2.1 (iii)] :

Proposition 4.1. *Assume that F is induced by an exact functor B-mod $\to$ A-mod. If $\mathcal{S}$ consists of simple modules, then there is a direct summand of F that is an equivalence B-mod $\xrightarrow{\sim}$ A-mod.*

We deduce :

Corollary 4.2. *Let B_1, B_2 be split self-injective algebras with no projective simple modules and $G_i : B_i$-mod $\to$ A-mod exact functors inducing stable equivalences. Assume $\mathcal{S}_1 = \mathcal{S}_2$ (up to isomorphism). Then, B_1 and B_2 are Morita equivalent.*

So, if we assume in addition that F comes from an exact functor G between module categories, then B is determined by $\mathcal{S}$, up to Morita equivalence.

The functor G is isomorphic to $X \otimes_B -$ where X is an (A, B)-bimodule. We can (and will) choose G so that X has no non-zero projective direct summand. Then, $G(L)$ is indecomposable for L simple [Li, Theorem 2.1 (ii)], so $\mathcal{S} = \{G(L)\}_{L \in \mathcal{S}'}$, up to isomorphism.

Proposition 4.3. *An A-module M is in the image of G if and only if there is a filtration $0 = M_r \subset M_{r-1} \subset \cdots \subset M_1 \subset M_0 = M$ such that M_i/M_{i-1} is isomorphic to an object of $\mathcal{S}$.*

Proof. Take L a B-module. Then the image by G of a filtration of L whose successive quotients are simple provides a filtration as required.

Conversely, we proceed by induction on r. We have an exact sequence $0 \to G(N) \to M \to G(L) \to 0$ and a corresponding element $\zeta \in \operatorname{Ext}^1_A(G(L), G(N))$. We have an isomorphism $\operatorname{Ext}^1_B(L, N) \xrightarrow{\sim} \operatorname{Ext}^1_A(G(L), G(N))$ and we take ζ' to be the inverse image of ζ under this isomorphism. This gives an exact sequence $0 \to N \to M' \to L \to 0$, and hence an exact sequence $0 \to G(N) \to G(M') \to G(L) \to 0$ with class ζ. It follows that $M \simeq G(M')$ and we are done. ☐

4.2. **Filtrable objects.**

4.2.1. Given two A-modules M and N, we write $M \sim N$ to denote the existence of an isomorphism between M and N in A-stab. Given $f, g \in \operatorname{Hom}_A(M, N)$, we write $f \sim g$ if $f - g$ is a projective map.

Lemma 4.4. *Let $f, f' : M \to N$ be two surjective maps with $f \sim g$. Then there is $\sigma \in \operatorname{Aut}_A(M)$ with $f' = f\sigma$ and $\sigma \sim \mathrm{id}_M$.*

Similarly, let $f, f' : N \to M$ be two injective maps with $f \sim g$. Then there is $\sigma \in \operatorname{Aut}_A(M)$ with $f' = \sigma f$ and $\sigma \sim \mathrm{id}_M$.

Proof. Let $L = \ker f$ and $L' = \ker f'$. Let $L = L_0 \oplus P$ and $L' = L'_0 \oplus P'$ with P, P' projective and L_0, L'_0 without non-zero projective direct summands. We have an isomorphism $\bar\alpha_0 \in \operatorname{Hom}_{A\text{-stab}}(L_0, L'_0)$ in A-stab giving rise to an isomorphism of distinguished triangles in A-stab

$$\begin{array}{ccccccc} L_0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \Omega^{-1}L_0 \\ {\scriptstyle \bar\alpha_0}\downarrow{\scriptstyle \sim} & & \| & & \| & & {\scriptstyle \Omega^{-1}(\bar\alpha_0)}\downarrow{\scriptstyle \sim} \\ L'_0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \Omega^{-1}L'_0 \end{array}$$

Let $\alpha_0 \in \operatorname{Hom}_A(L_0, L'_0)$ lifting $\bar\alpha_0$. This is an isomorphism. There is now a commutative diagram of A-modules, where the exact rows come from the elements of $\operatorname{Ext}^1_A(N, L_0)$ and $\operatorname{Ext}^1_A(N, L'_0)$ defined above :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_0 & \longrightarrow & M_0 & \longrightarrow & N & \longrightarrow & 0 \\ & & {\scriptstyle \alpha_0}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sigma_0}\downarrow{\scriptstyle \sim} & & \| & & \\ 0 & \longrightarrow & L'_0 & \longrightarrow & M'_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

We have $M \simeq M_0 \oplus P \simeq M'_0 \oplus P'$, hence $P \simeq P'$. Let $\alpha : L \xrightarrow{\sim} L'$ extending α_0. Then there is $\sigma : M \xrightarrow{\sim} M$ making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sigma}\downarrow{\scriptstyle \sim} & & \| & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

and we are done.

The second part of the lemma has a similar proof — it can also be deduced from the first part by duality. ☐

4.2.2.

Hypothesis 2. Let $\mathcal{S}$ be a finite set of indecomposable finitely generated A-modules such that $\mathrm{Hom}_{A\text{-stab}}(S,T) = k^{\delta_{S,T}}$ for $S, T \in \mathcal{S}$.

An $\mathcal{S}$-*filtration* for an A-module M is a filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ such that $\bar{M}_i = M_i/M_{i+1}$ is in $\mathrm{add}(\mathcal{S})$ for $0 \le i \le r-1$.

We say that M is *filtrable* if it admits an $\mathcal{S}$-filtration.

Lemma 4.5. *Let M be a non-projective filtrable A-module. Then there is $S \in \mathcal{S}$ such that* $\mathrm{Hom}_{A\text{-stab}}(M,S) \neq 0$ *(resp. such that* $\mathrm{Hom}_{A\text{-stab}}(S,M) \neq 0$*).*

Proof. Assume $\mathrm{Hom}_{A\text{-stab}}(M,S) = 0$ for all $S \in \mathcal{S}$. Since M is filtrable, it follows that $\mathrm{End}_{A\text{-stab}}(M) = 0$, and hence M is projective, which is not true. The second case is similar. ☐

Lemma 4.6. *Let M be a filtrable module and $S \in \mathcal{S}$. Given $f : M \to S$ non-projective, there is $g : M \to S$ surjective with filtrable kernel such that $f \sim g$. Similarly, given $f : S \to M$ non-projective, there is $g : S \to M$ injective with filtrable cokernel such that $f \sim g$.*

Proof. We proceed by induction on the number of terms in a filtration of M. The result is clear if $M \in \mathcal{S}$.

Let $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} T \to 0$ be an exact sequence with $T \in \mathcal{S}$ and N filtrable.

Assume first $f\alpha : N \to S$ is projective. Then there is $p : M \to S$ projective and $g : T \to S$ with $f - p = g\beta$. Since g is not projective, it is an isomorphism. Consequently, $f - p$ is surjective and its kernel is isomorphic to N by Lemma 4.4, so we are done.

Assume now $f\alpha : N \to S$ is not projective. By induction, there is $q : N \to S$ projective such that $f\alpha + q$ is surjective with filtrable kernel N'. Since $\alpha : N \to M$ is injective, there is a projective map $p : M \to S$ with $q = p\alpha$. Now, we have an exact sequence $0 \to N/N' \xrightarrow{\bar{\alpha}} M/\alpha(N') \to T \to 0$ and a non-projective surjection $f + p : M/\alpha(N') \to S$. Since $(f+p)\bar{\alpha} : N/N' \xrightarrow{\sim} S$ is an isomorphism, it follows that the kernel of the map $M/\alpha(N') \to S$ is isomorphic to T. Since N' is filtrable, it follows that $\ker(f+p)$ is filtrable and we are done. The second assertion follows by duality. ☐

From Lemmas 4.4 and 4.6, we deduce :

Lemma 4.7. *Let $S \in \mathcal{S}$ and let M be a filtrable module.*

If $f : M \to S$ be a surjective and non-projective map, then $\ker f$ *is filtrable.*

Similarly, if $g : S \to M$ is injective and non-projective, then $\mathrm{coker}\, g$ *is filtrable.*

From Lemmas 4.5 and 4.6, we deduce :

Lemma 4.8. *Let M be filtrable non-projective. Then there is a submodule S of M, with $S \in \mathcal{S}$, such that M/S is filtrable and the inclusion $S \to M$ is not projective. Similarly, there is a filtrable submodule N of M such that $M/N \in \mathcal{S}$ and $M \to M/N$ is not projective.*

Proposition 4.9. *Let M be an A-module with a decomposition $M \sim M_1' \oplus M_2'$ in the stable category. If M is filtrable then there is a decomposition $M = M_1 \oplus M_2$ such that M_i is filtrable and $M_i \sim M_i'$*

Proof. We can assume M is not projective, for otherwise the proposition is trivial. We prove the proposition by induction on the dimension of M.

Let $M = T_1 \oplus T_2 \oplus P$ with P projective, T_i without non-zero projective direct summand and $T_i \sim M'_i$. Denote by $\pi : M \to T_1$ the projection.

By Lemma 4.5, there is $S \in \mathcal{S}$ such that $\mathrm{Hom}_{A\text{-stab}}(M, S) \neq 0$. Hence, $\mathrm{Hom}_{A\text{-stab}}(T_i, S) \neq 0$ for $i = 1$ or $i = 2$. Assume for instance $i = 1$. Pick a non-projective map $\alpha : T_1 \to S$. So, $\alpha\pi : M \to S$ is not projective. By Lemma 4.6, there is a surjective map $\beta : M \to S$ with $\beta \sim \alpha\pi$ and $N = \ker \beta$ filtrable. Then $N \sim L \oplus T_2$, where L is the kernel of $\alpha + p : T_1 \oplus P_S \to S$ and $p : P_S \to S$ is a projective cover of S. By induction, we have $N = N_1 \oplus N_2$ with N_i filtrable and $N_1 \sim L$, $N_2 \sim T_2$. Now, the map $S \to L[1]$ gives a map $S \to N_1[1]$ (in A-stab). Let M_1 be the extension of S by N_1 corresponding to that map. Then $M \simeq M_1 \oplus N_2$, the modules M_1 and N_2 are filtrable, $M_1 \sim M'_1$, and $N_2 \sim M'_2$. □

Let M be a filtrable module. We say that M *has no projective remainder* if there is no direct sum decomposition $M = N \oplus P$ with $P \neq 0$ projective and N filtrable.

Lemma 4.10. *Let M be a filtrable module with no projective remainder and let $S \in \mathcal{S}$.*

For $f : M \to S$ surjective, $\ker f$ *is filtrable if and only if f is non-projective.*

For $f : S \to M$ injective, $\operatorname{coker} f$ *is filtrable if and only if f is non-projective.*

Proof. Assume f is projective. Then there is a decomposition $M = N \oplus P$ and $f = (0, g)$ with P projective. Now, $\ker f = N \oplus \ker g$. If $\ker f$ is filtrable, then it follows from Lemma 4.9 that M has a non-zero projective submodule whose quotient is filtrable.

The converse is given by Lemma 4.7. The second part of the Lemma has a similar proof. □

Lemma 4.11. *Let $M = M_0 \oplus M_1$ with M and M_0 filtrable and such that M_0 has no projective remainder. Then M_1 is filtrable.*

Proof. We proceed by induction on $\dim M_0$ — the result is clear for $M_0 = 0$. Assume $M_0 \neq 0$. Let $f : M_0 \to S$ be a surjection with $S \in \mathcal{S}$ and $\ker f$ filtrable. By Lemma 4.10, f is not projective. Then $f' : M \xrightarrow{\text{can}} M_0 \xrightarrow{f} S$ is a non-projective surjection. By Lemma 4.7, $\ker f'$ is filtrable. We have $\ker f' = \ker f \oplus M_1$ and we are done. □

4.2.3. We now turn to filtrations by objects in $\mathrm{add}(\mathcal{S})$.

Lemma 4.12. *Let M be a filtrable module and N a filtrable submodule of M such that $M/N \in \mathrm{add}\,\mathcal{S}$. Then, N is minimal with these properties if and only if N has no projective remainder and the canonical map* $\mathrm{Hom}_{A\text{-stab}}(M/N, S) \to \mathrm{Hom}_{A\text{-stab}}(M, S)$ *is surjective for every $S \in \mathcal{S}$.*

Proof. Let N be a minimal filtrable submodule of M such that $M/N \in \mathrm{add}\,\mathcal{S}$. Denote by $i : N \to M$ the injection and $p : M \to M/N$ the quotient map.

Let $S \in \mathcal{S}$. Fix $f_1, \ldots, f_r : M/N \to S$ such that $\sum_i f_i : M/N \to S^r$ is surjective and $\ker \sum_i f_i$ has no direct summand isomorphic to S. Let T be the subspace of $\mathrm{Hom}_{A\text{-stab}}(M, S)$ generated by $f_1 p, \ldots, f_r p$. Assume this is a proper subspace, so there is $f' : M \to S$ whose image in $\mathrm{Hom}_{A\text{-stab}}(M, S)$ is not in T. Then $f' i : N \to S$ is not projective, hence there is a projective map $q : N \to S$ such that $f' i + q$ is surjective and has filtrable kernel N' (Lemma 4.6). There is $q' : M \to S$ projective such that $q = q' i$. Now, $M/N' \simeq M/N \oplus S$ and this contradicts the minimality of N. It follows that the canonical map $\mathrm{Hom}_{A\text{-stab}}(M/N, S) \to \mathrm{Hom}_{A\text{-stab}}(M, S)$ is surjective. Assume $N = N' \oplus P$ with N' filtrable with no projective remainder and P projective.

By Lemma 4.11, P is filtrable. We have $M/N' \simeq M/N \oplus P$. Since M/N is a maximal quotient of M in $\operatorname{add}(\mathcal{S})$ and P is filtrable, it follows that $P = 0$.

Conversely, take $f : N \to S$ surjective with filtrable kernel such that the extension of M/N by S splits. Then f lifts to $M \to S$ and it is not projective by Lemma 4.10. This contradicts the surjectivity of $\operatorname{Hom}_{A\text{-stab}}(M/N, S) \to \operatorname{Hom}_{A\text{-stab}}(M, S)$. Consequently, N is minimal. □

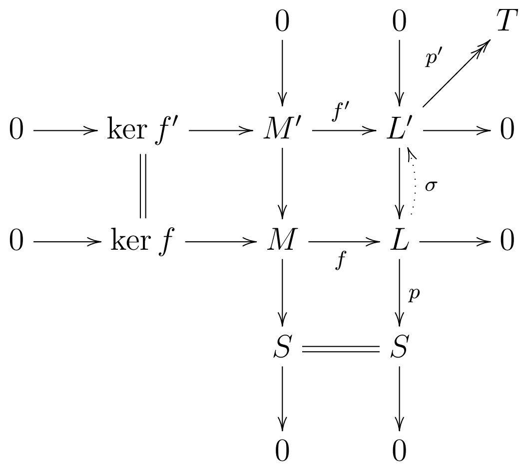
Lemma 4.13. *Let M be a filtrable A-module with no projective remainder.*

Let $f : M \to L$ be a surjection with $L \in \operatorname{add}\mathcal{S}$. Then $\ker f$ is filtrable if and only if the canonical map $\operatorname{Hom}_{A\text{-stab}}(L, S) \to \operatorname{Hom}_{A\text{-stab}}(M, S)$ is injective for all $S \in \mathcal{S}$.

Proof. Note that the canonical map $\operatorname{Hom}_{A\text{-stab}}(L, S) \to \operatorname{Hom}_{A\text{-stab}}(M, S)$ is injective if and only if, given $p : L \to S$ surjective with $S \in \mathcal{S}$, pf is not projective.

Assume $\ker f$ is filtrable. Let $p : L \to S$ be a surjective map with $S \in \mathcal{S}$. Then $\ker pf$ is filtrable, hence pf is not projective (Lemma 4.10).

Let us now prove the converse by induction on the dimension of M. Assume that given $p : L \to S$ surjective with $S \in \mathcal{S}$, then pf is not projective. Pick $p : L \to S$ surjective and let $L' = \ker p$. Let $M' = \ker pf$. Then f induces a surjection $f' : M' \to L'$ and we have $L' \in \operatorname{add}\mathcal{S}$ (since p is split). Let $p' : L' \to T$ be a surjective map with $T \in \mathcal{S}$. Fix a left inverse $\sigma : L \to L'$ to the inclusion $L' \to L$.



If $S \neq T$, then $\operatorname{Hom}_{A\text{-stab}}(S, T) = 0$, and hence $p'\sigma f$ doesn't factor through S in the stable category. On the other hand, if $S = T$ then pf and $p'\sigma f$ define linearly independent elements of $\operatorname{Hom}_{A\text{-stab}}(M, S)$. Consequently, $p'\sigma f$ doesn't factor through S in the stable category. It follows that $p'f'$ is not projective. By Lemma 4.7, M' is filtrable. By induction, it follows that $\ker f'$ is filtrable and we are done. □

Proposition 4.14. *Let M be a filtrable A-module with no projective remainder.*

Let N be a minimal filtrable submodule of M such that $M/N \in \operatorname{add}\mathcal{S}$. Then there is an isomorphism

$$M/N \xrightarrow{\sim} \bigoplus_{S \in \mathcal{S}} S \otimes \operatorname{Hom}_{A\text{-stab}}(M, S)$$

that induces the canonical map $M \to \bigoplus_{S \in \mathcal{S}} S \otimes \operatorname{Hom}_{A\text{-stab}}(M, S)$ in the stable category.

Given $\tau \in \operatorname{Aut}(N)$ such that $\tau \sim \operatorname{id}_N$, there is $\sigma \in \operatorname{Aut}(M)$ with $\sigma \sim \operatorname{id}_M$ and $\sigma_{|N} = \tau$.

Let N' be a minimal filtrable submodule of M such that $M/N' \in \operatorname{add}\mathcal{S}$. Then there is $\sigma \in \operatorname{Aut}(M)$ such that $N' = \sigma(N)$ and $\sigma \sim \operatorname{id}_M$.

Proof. The first part of the proposition follows from Lemmas 4.12 and 4.13.

Let $\tau \in \mathrm{Aut}(N)$ such that $\tau = \mathrm{id}_N + p$ with $p : N \to N$ projective. Then there is a projective map $q : M \to N$ with $p = qi$. Let $\sigma = \mathrm{id}_M + q$. Then $\sigma_{|N} = \tau$. Now, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\ & & {\scriptstyle \tau}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \sigma}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \end{array}$$

and hence σ is an automorphism of M.

Let N' be a minimal filtrable submodule of M such that $M/N' \in \operatorname{add}\mathcal{S}$. Then we have shown that $M/N \xrightarrow{\sim} M/N'$ and that via such an isomorphism, the maps $M \to M/N$ and $M \to M/N'$ are stably equal. Now, Lemma 4.4 shows there is $\sigma \in \mathrm{Aut}(M)$ with $N' = \sigma(N)$ and $\sigma \sim \mathrm{id}_M$. □

Let M be filtrable. An *$\mathcal{S}$-radical filtration* of M is a filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ such that M_i is a minimal filtrable submodule of M_{i-1} with $M_{i-1}/M_i \in \operatorname{add}\mathcal{S}$.

Proposition 4.15. *Let M be a filtrable A-module with no projective remainder. Let $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ and $0 = M'_{r'} \subseteq M'_{r'-1} \subseteq \cdots \subseteq M'_0 = M$ be two $\mathcal{S}$-radical filtrations of M. Then, $r = r'$ and there is an automorphism of M that swaps the two filtrations and that is stably the identity.*

Proof. We prove this lemma by induction on the dimension of M. By Proposition 4.14, there is $\sigma \in \mathrm{Aut}(M)$ such that $\sigma(M'_1) = M_1$ and $\sigma \sim \mathrm{id}_M$. Now, by induction, we have $r = r'$ and there is $\tau \in \mathrm{Aut}(M_1)$ such that $\tau\sigma(M'_i) = M_i$ for $i > 0$ and $\tau \sim \mathrm{id}_{M_1}$. By Proposition 4.14, there is $\tau' \in \mathrm{Aut}(M)$ such that $\tau'_{|M_1} = \tau$ and $\tau' \sim \mathrm{id}_M$. Now, $\tau'\sigma$ sends M'_i onto M_i. □

Remark 4.16. A filtrable projective module can have two $\mathcal{S}$-radical filtrations with non-isomorphic layers.

Consider $A = k\mathfrak{A}_4$, the group algebra of the alternating group of degree 4 and assume k has characteristic 2 and contains a cubic root of 1. Let B be the principal block of $k\mathfrak{A}_5$. Then, the restriction functor is a stable equivalence between B and A. Let $\mathcal{S}$ be the set of images of the simple B-modules. Denote by k the trivial A-module and by k_+, k_- the non-trivial simple A-modules. Then $\mathcal{S} = \{k, S_+, S_-\}$ where S_ε is a non-trivial extension of k_ε by $k_{-\varepsilon}$. Let P and P' be the two projective indecomposable B-modules that don't have k as a quotient. Then $\mathrm{Res}_{\mathfrak{A}_4} P \simeq \mathrm{Res}_{\mathfrak{A}_4} P'$. This projective module has two $\mathcal{S}$-radical filtrations with non-isomorphic layers : one coming from the radical filtration of P and one coming from the radical filtration of P'.

While $\mathcal{S}$-radical filtrations are not unique in general for filtrable modules with a projective remainder, there are some cases where uniqueness still holds :

Proposition 4.17. *Assume A is a symmetric algebra. Let $0 \to S \to M \to T \to 0$ and $0 \to S' \to M \to T' \to 0$ be two exact sequences with $S, S', T, T' \in \mathcal{S}$. Assume that the sequences don't both split. Then there is an automorphism of M swapping the two exact sequences.*

Proof. If M is non-projective, then this is a consequence of Proposition 4.14.

Assume M is projective. Since A is symmetric, we have a non-projective map $T \simeq \Omega^{-1}S \to S$. It follows that $S = T$. Similarly, $T' = S'$. We have exact sequences

$$0 \to \mathrm{Hom}(S', S) \to \mathrm{Hom}(S', M) \to \mathrm{Hom}(S', S) \to \mathrm{Ext}^1(S', S) \to 0$$

$$0 \to \mathrm{Hom}(S', S') \to \mathrm{Hom}(S', M) \to \mathrm{Hom}(S', S') \to \mathrm{Ext}^1(S', S') \to 0$$

We have $\Omega^{-1}S' \simeq S'$, and hence $\dim \mathrm{Ext}^1(S', S') = 1$. Consequently, $\dim \mathrm{Hom}(S', M)$ is an odd integer. It follows that $\mathrm{Ext}^1(S', S) \neq 0$, hence $\mathrm{Hom}_{A\text{-stab}}(S', S) \neq 0$, so $S' = S$ and we are done by Lemma 4.4. □

Lemma 4.18. *Let $0 = M_r \subset M_{r-1} \subset \cdots \subset M_0 = M$ be a filtration of M with $M_{i-1}/M_i \in \mathrm{add}\,\mathcal{S}$.*

(i) *If M has no projective remainder, then M_i has no projective remainder, for all i.*
(ii) *If the filtration is an $\mathcal{S}$-radical filtration, then M_i has no projective remainder for $i \geq 1$.*

Proof. Consider an exact sequence $0 \to N \oplus P \to M \to L \to 0$ of filtrable modules with P projective and N filtrable. Then there is an extension M' of L by N such that $M = M' \oplus P$ and M' is filtrable. The first part of the lemma follows.

Assume now the filtration is an $\mathcal{S}$-radical filtration. Assume for some $i \geq 1$, we have $M_i = N \oplus P$ with N filtrable with no projective remainder and P projective and filtrable (Lemma 4.11). Then, $M = M' \oplus P$ with P filtrable by (i). There is an exact sequence $0 \to L \to P \to S \to 0$ with $S \in \mathcal{S}$ and L filtrable. Now, the canonical surjection $M' \oplus P \to M/M_1 \oplus S$ has filtrable kernel and this contradicts the minimality of M_1. □

Proposition 4.19. *Let M_1 and M_2 be two filtrable A-modules with no projective remainder. If $M_1 \sim M_2$, then $M_1 \simeq M_2$.*

Proof. We prove the proposition by induction on $\min(\dim M_1, \dim M_2)$. Fix an isomorphism ϕ from M_2 to M_1 in the stable category. Let $X = \bigoplus_{S \in \mathcal{S}} S \otimes \mathrm{Hom}_{A\text{-stab}}(M_1, S)$ and $g_1 \in \mathrm{Hom}_{A\text{-stab}}(M_1, X)$ be the canonical map. Let $g_2 = g_1\phi$. By Propositions 4.14 and 4.15, there are exact sequences

$$0 \to N_1 \to M_1 \xrightarrow{f_1} X \to 0 \quad \text{and} \quad 0 \to N_2 \to M_2 \xrightarrow{f_2} X \to 0$$

with the image of f_i in the stable category equal to g_i. So, there is an isomorphism from N_2 to N_1 in the stable category compatible with ϕ. By Lemma 4.18, N_1 and N_2 have no projective remainder. By induction, we deduce that there is an isomorphism $N_2 \xrightarrow{\sim} N_1$ lifting the stable isomorphism. So, M_1 and M_2 are extensions of isomorphic modules, with the same class in Ext^1, hence are isomorphic. □

4.3. **Generators and reconstruction.**

4.3.1. We assume from now on that

Hypothesis 3. $\mathcal{S}$ satisfies Hypothesis 2 and given $M \in A$-mod, there is a projective A-module P such that $M \oplus P$ is filtrable.

Proposition 4.20. *Let $S \in \mathcal{S}$. Let $P_S \to S$ be a projective cover of S and P minimal projective such that $\Omega S \oplus P$ is filtrable. Let $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = \Omega S \oplus P$ be an $\mathcal{S}$-radical filtration.*

Then $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 \subseteq P_S \oplus P$ is an $\mathcal{S}$-radical filtration.

If A is symmetric, then $M_{r-1} \simeq S$.

Proof. Let $f_1 : P_S \to S$ be a surjective map and $f = (f_1, 0) : P_S \oplus P \to S$. Let $T \in \mathcal{S}$ and $g : P_S \oplus P \to T$ such that we have an exact sequence $0 \to L \to P_S \oplus P \xrightarrow{f+g} S \oplus T \to 0$ with L filtrable.

We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & P_S \oplus P & \xrightarrow{f+g} & S \oplus T & \longrightarrow & 0 \\ & & \| & & \uparrow & & \downarrow{\scriptstyle (0,\mathrm{id})} & & \\ 0 & \longrightarrow & L & \longrightarrow & \Omega S \oplus P & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

The surjection $\Omega S \oplus P \to T$ is projective and has filtrable kernel. From Lemma 4.10, we get a contradiction to the minimality of P. It follows that $\Omega S \oplus P$ is a minimal submodule of $P_S \oplus P$ such that the quotient is in $\operatorname{add} \mathcal{S}$.

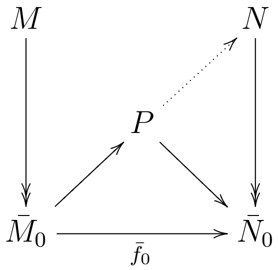
We have $\operatorname{Hom}_{A\text{-stab}}(T, \Omega S) \simeq \operatorname{Hom}_{A\text{-stab}}(S, T)^*$, since A is symmetric. Now, $\operatorname{Hom}_{A\text{-stab}}(M_{r-1}, \Omega S \oplus P) \neq 0$ by Lemma 4.10. The second part of the proposition follows. □

Let M and N be two A-modules with filtrations $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ $0 = N_s \subseteq N_{s-1} \subseteq \cdots \subseteq N_0 = N$. Let $\operatorname{Hom}^f_A(M, N)$ be the subspace of $\operatorname{Hom}_A(M, N)$ of filtered maps (*i.e.*, those g such that $g(M_i) \subseteq N_i$). We put $\bar{M}_i = M_i/M_{i+1}$. We denote by ϕ_i the composition of canonical maps $\phi_i : \operatorname{Hom}^f_A(M, N) \to \operatorname{Hom}_A(\bar{M}_i, \bar{N}_i) \to \operatorname{Hom}_{A\text{-stab}}(\bar{M}_i, \bar{N}_i)$.

We view $N' = N_i$ as a filtered module with the induced filtration $0 = N'_{s-i} \subseteq N'_{s-i-1} = N_{s-1} \subseteq \cdots \subseteq N'_1 = N_{i+1} \subseteq N'_0 = N'$.

Lemma 4.21. *Let M be a filtrable A-module with an $\mathcal{S}$-radical filtration and N be a filtrable A-module with an $\mathcal{S}$-filtration. Let $f \in \operatorname{Hom}^f_A(M, N)$ with $\phi_0(f) = 0$. Then $\phi_i(f) = 0$ for all i.*

Proof. The map $\bar{f}_0 : \bar{M}_0 \to \bar{N}_0$ induced by f is projective. So there is a projective module P and a commutative diagram



Let p be the composition $p : M \to \bar{M}_0 \to P \to N$. Then $f - p \sim f$, $f - p$ and f have the same restriction to M_1, and $\overline{(f-p)}_0 = 0$. Consequently it is enough to prove the lemma in the case where $\bar{f}_0 = 0$.

From now on, we assume $\bar{f}_0 = 0$. Assume the map $\bar{f}_1 : \bar{M}_1 \to \bar{N}_1$ is not projective. So there is $S \in \mathcal{S}$ and a (split) surjection $g : \bar{N}_1 \to S$ such that $g\bar{f}_1 : \bar{M}_1 \to S$ is not projective. Let $s : S \to \bar{M}_1$ be a right inverse to g, and let L be the kernel of $g\bar{f}_1$.

We have an exact sequence $0 \to L \to M/M_2 \xrightarrow{(\mathrm{can}, gf)} \bar{M}_0 \oplus S \to 0$. So the inverse image of L in M_1 is a filtrable submodule of M with quotient isomorphic to $\bar{M}_0 \oplus S$. This contradicts the fact that M_1 is a minimal filtrable submodule of M such that $M/M_1 \in \operatorname{add} \mathcal{S}$. So $\bar{f}_1$ is projective; *i.e.*, $\phi_1(f) = 0$.

We now prove by induction that $\phi_i(f) = 0$ for all i. Assume $\phi_d(f) = 0$. Then, we apply the result above to the filtered modules M_d and N_d to get $\phi_{d+1}(f) = 0$. □

4.3.2. We define a category $\mathcal{G}$ as follows.

- Its objects are A-modules together with a fixed $\mathcal{S}$-radical filtration.
- We define $\mathrm{Hom}_{\mathcal{G}}(M,N)_i$ as the image of $\mathrm{Hom}^f_A(M,N_i)$ in $\mathrm{Hom}_{A\text{-stab}}(\bar{M}_0, \bar{N}_i)$. We put $\mathrm{Hom}_{\mathcal{G}}(M,N) = \oplus_i \mathrm{Hom}_{\mathcal{G}}(M,N)_i$.
- Let $f \in \mathrm{Hom}_{\mathcal{G}}(M,N)_i$ and $g \in \mathrm{Hom}_{\mathcal{G}}(L,M)_j$. Let $\tilde{f} : M \to N_i$ be a filtered map lifting f. It induces a map $\phi_j(\tilde{f}) \in \mathrm{Hom}_{A\text{-stab}}(\bar{M}_j, \bar{N}_{i+j})$ independent of the choice of $\tilde{f}$ (Lemma 4.21). We define the product fg to be $\phi_j(\tilde{f}) \circ \phi_0(g)$.

Given $S \in \mathcal{S}$, let $P_S \to S$ be a projective cover of S and Q_S projective minimal such that $\Omega S \oplus Q_S$ is filtrable. Fix a radical filtration of $P_S \oplus Q_S$ with first term $\Omega S \oplus Q_S$.

Let $M = \oplus_{S\in\mathcal{S}}(P_S \oplus Q_S)$. This comes with an $\mathcal{S}$-radical filtration. We have constructed a $\mathbf{Z}_{\geq 0}$-graded k-algebra $\mathrm{End}_{\mathcal{G}}(M)$.

The following Lemma is clear.

Lemma 4.22. *Let $\mathcal{S}$ be a complete set of representatives of isomorphism classes of simple A-modules. Then we have an equivalence $\mathrm{gr}(A\text{-mod}) \xrightarrow{\sim} \mathcal{G}$. If A is basic, then $\mathrm{End}_{\mathcal{G}}(M)$ is isomorphic to the graded algebra associated with the radical filtration of A.*

We have now obtained our partial reconstruction result :

Theorem 4.23. *Let B be a selfinjective algebra with no simple projective module. Let M be an (A,B)-bimodule inducing a stable equivalence and having no projective direct summand. Let $\mathcal{S} = \{M \otimes_B L\}$ where L runs over a complete set of representatives of isomorphism classes of simple B-modules.*

Then, there is an equivalence $\mathrm{gr}(B\text{-mod}) \xrightarrow{\sim} \mathcal{G}$. If B is basic, there is an isomorphism between the graded algebra associated with the radical filtration of B and $\mathrm{End}_{\mathcal{G}}(M)$.

4.3.3. The category $\mathcal{G}$ can be constructed directly as in §3.1, using only the stable category with its triangulated structure.

Proposition 4.24. *Let M be a module with an $\mathcal{S}$-filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$. This is an $\mathcal{S}$-radical filtration if and only if*

- $\mathrm{Hom}_{A\text{-stab}}(M_i/M_{i+1}, S) \to \mathrm{Hom}_{A\text{-stab}}(M_i, S)$ *is an isomorphism for all $S \in \mathcal{S}$ and $i > 0$,*
- $\mathrm{Hom}_{A\text{-stab}}(M_0/M_1, S) \to \mathrm{Hom}_{A\text{-stab}}(M_0, S)$ *is surjective for all $S \in \mathcal{S}$, and*
- *M_i has no projective remainder for $i > 0$.*

Assume the filtration is an $\mathcal{S}$-radical filtration. Then M has no projective remainder if and only if $\mathrm{Hom}_{A\text{-stab}}(M_0/M_1, S) \to \mathrm{Hom}_{A\text{-stab}}(M_0, S)$ is an isomorphism.

Proof. Let M be a module with an $\mathcal{S}$-radical filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$. The canonical map $\mathrm{Hom}_{A\text{-stab}}(M_i/M_{i+1}, S) \to \mathrm{Hom}_{A\text{-stab}}(M_i, S)$ is surjective for all $S \in \mathcal{S}$, by Lemma 4.12. Note that M_i has no projective remainder for $i > 0$, by Lemma 4.18. It follows that the canonical map $\mathrm{Hom}_{A\text{-stab}}(M_i/M_{i+1}, S) \to \mathrm{Hom}_{A\text{-stab}}(M_i, S)$ is an isomorphism for all $S \in \mathcal{S}$ (Lemma 4.13).

Let us now prove the other implication. Since M_i has no projective remainder for $i > 0$, it follows from Lemma 4.12 that $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1$ is an $\mathcal{S}$-radical filtration of M_1.

Assume the filtration is an $\mathcal{S}$-radical filtration. If M has no projective remainder, then $\mathrm{Hom}_{A\text{-stab}}(M_0/M_1, S) \to \mathrm{Hom}_{A\text{-stab}}(M_0, S)$ is injective by Lemma 4.13.

Assume now that $\mathrm{Hom}_{A\text{-stab}}(M/M_1, S) \to \mathrm{Hom}_{A\text{-stab}}(M, S)$ is bijective. Assume $M = M' \oplus P$ with M' filtrable and P projective. We have $\mathrm{Hom}_{A\text{-stab}}(M/M_1, S) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-stab}}(M, S) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-stab}}(M', S)$. There is a surjective map $g : M' \to M/M_1$ with filtrable kernel such that the composition $M \xrightarrow{\mathrm{can}} M' \xrightarrow{g} M/M_1$ is equal to the canonical map $M \to M/M_1$ in the stable category, by Proposition 4.14. By Lemma 4.4, we have $M_1 \simeq \ker g \oplus P$. Since M_1 has no projective remainder by the first part of the proposition, we get $P = 0$, hence M has no projective remainder. $\square$

Let $\mathcal{T} = A$-stab. Note that $\mathcal{S}$ is determined by its image in $\mathcal{T}$ and it satisfies Hypothesis 3 if and only if $\mathrm{Hom}_{\mathcal{T}}(S, T) = k^{\delta_{ST}}$ for all $S, T \in \mathcal{S}$ and every object of $\mathcal{T}$ is an iterated extension of objects of $\mathcal{S}$.

We have a functor $\mathcal{G} \to \mathcal{F}$: it sends a module M with an $\mathcal{S}$-radical filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ to $\cdots \to 0 \to M_{r-1} \to \cdots \to M_1 \to M \to M/M_1$ (cf Proposition 4.24).

Proposition 4.25. *The canonical functor $\mathcal{G} \xrightarrow{\sim} \mathcal{F}$ is an equivalence.*

Proof. The functor is clearly fully faithful.

Start with $0 = N_r \xrightarrow{f_r} N_{r-1} \to \cdots \to N_1 \xrightarrow{f_1} N_0 \xrightarrow{\varepsilon_0} M_0$. Adding a projective direct summand to the N_i's, we can lift the maps f_i to maps that are injective in the module category and such that the successive quotients have no projective direct summands. So we have a filtration $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_1 \subseteq M'_0$ such that M'_i/M'_{i+1} is stably isomorphic to a direct sum of objects of $\mathcal{S}$. Since it has no projective summand, it is actually isomorphic to a sum of objects of $\mathcal{S}$; *i.e.*, we have an $\mathcal{S}$-filtration. Consider i maximal such that M'_i has a projective remainder. Then $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_i$ is an $\mathcal{S}$-radical filtration by Proposition 4.24 (first part). The second part of Proposition 4.24 shows now that M'_i has no projective remainder, a contradiction. So the filtration is an $\mathcal{S}$-filtration. $\square$

References

[Al] S. Al-Nofayee, *Equivalences of derived categories for selfinjective algebras*, J. Alg. **313** (2007), 897–904.

[AlRi] S. Al-Nofayee and J. Rickard, *Rigidity of tilting complexes and derived equivalences for self-injective algebras*, preprint arXiv:1311.0504.

[Ke] B. Keller, *Deriving DG Categories*, Ann. Sci. Ec. Norm. Sup. **27** (1994), 63–102.

[Li] M. Linckelmann, *Stable equivalences of Morita type for self-injective algebras and p-groups*, Math. Z. **223** (1996), 87-100.

[Nee] A. Neeman, *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. Éc. Norm. Sup. **25** (1992), 546–566.

[Ri] J. Rickard, *Equivalences of derived categories for symmetric algebras*, J. of Algebra **257** (2002), 460–481.

Jeremy Rickard : University of Bristol, School of Mathematics, University Walk, Bristol BS8 1TW, UK

E-mail address: J.Rickard@bristol.ac.uk

Raphaël Rouquier : UCLA Mathematics Department, Los Angeles, CA 90095-1555, USA

E-mail address: rouquier@math.ucla.edu