Journal of Algebra **239,** 262–271 (2001)
doi:10.1006/jabr.2000.8625, available online at http://www.idealibrary.com on IDEAL®

Source Algebras and Source Modules

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Communicated by Michel Broué

Received April 1, 2000

The aim of this article is to give a self-contained approach in module theoretic terms to two fundamental results in block theory, due to Puig [6, 14.6]: first, there is an embedding of the source algebra of the Brauer correspondent of a block of some finite group into a source algebra of that block, and second, the source algebras of the Brauer correspondence can be described explicitly. Our proof of the first result—Theorem 5 and its Corollary 6 below—is essentially the translation to our terminology of the proof in [1, 4.10].

The second result, describing the source algebras of the Brauer correspondent, follows also from [3], but the proofs in [3] use both the main result on the structure of nilpotent blocks in [5] as well as techniques from [6, Sects. 4, 6], while our approach—from Proposition 9 onwards—requires only some classical results on the structure of blocks with a central defect group.

An account of the results in [6] on source algebras of blocks with a normal defect group can also be found in [7, Sect. 45], except for the explicit description of the central extension $\hat{E}$ (see Section 10 below), as being defined in terms of the multiplicity algebra of the block, which is quoted in [7, (45.10)(b)] without proof, referring back to [6] at the end of [7, Sect. 45].

0021-8693/01 $35.00

1. Let p be a prime, and let $\mathcal{O}$ be a complete local principal ideal domain having a residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p; that is, either $\mathcal{O}$ is a complete discrete valuation ring or $\mathcal{O} = k$.

Throughout the paper, G is a finite group, b is a block of $\mathcal{O}G$, and P is a defect group of b; that is, b is a primitive idempotent of $Z(\mathcal{O}G)$ and P is a p-subgroup of G such that the subgroup $\Delta P = \{(u, u^{-1})\}_{u \in P}$ of $G \times G^0$ is a vertex of $\mathcal{O}Gb$ as an $\mathcal{O}(G \times G^0)$-module, where G^0 is the group opposite to G and $\mathcal{O}G$ is considered as an $\mathcal{O}(G \times G^0)$-module through $(x, y).a = xay$ for any $x, y \in G$ and $a \in \mathcal{O}G$. Note that $\mathcal{O}Gb$ has the trivial $\mathcal{O}\Delta P$-module $\mathcal{O}$ as a source, since we have an isomorphism of $\mathcal{O}(G \times G^0)$-modules $\mathcal{O}G \cong \mathrm{Ind}_{\Delta G}^{G \times G^0}(\mathcal{O})$ mapping $x \in G$ to $(x, 1) \otimes 1_{\mathcal{O}}$.

DEFINITION 2. A *source module of the block b of $\mathcal{O}G$* is an indecomposable direct summand M of $\mathrm{Res}_{G \times P^0}^{G \times G^0}(\mathcal{O}Gb)$ having ΔP as a vertex.

Remark 3. Since $G \times P^0$ contains the vertex ΔP of the $\mathcal{O}(G \times G^0)$-module $\mathcal{O}Gb$, the block b has a source module M. The algebra $\mathrm{End}_{\mathcal{O}(G \times 1)}(M)^0$ is then a *source algebra of the block b* (cf. [4]), which explains our terminology. Indeed, any direct summand of $\mathcal{O}Gb$ as a left $\mathcal{O}G$-module is of the form $\mathcal{O}Gi$ for some idempotent i in $\mathcal{O}Gb$. Since M is a direct summand of $\mathcal{O}Gb$ as an $\mathcal{O}(G \times P^0)$-module, we have in fact $M = \mathcal{O}Gi$ for some P-stable idempotent i in $\mathcal{O}Gb$; moreover, as M is indecomposable, i is primitive in $(\mathcal{O}Gb)^P$, and the condition that ΔP be a vertex of M is then equivalent to requiring $\mathrm{Br}_P(i) \neq 0$, where Br_P: $(\mathcal{O}G)^P \to kC_G(P)$ is the Brauer homomorphism (cf. [7, (37.5)(b)]). The map sending $a \in i\mathcal{O}Gi$ to right multiplication by a on M is an isomorphism of *interior P-algebras* $i\mathcal{O}Gi \cong \mathrm{End}_{\mathcal{O}(G \times 1)}(M)^0$. Note that M is projective as left $\mathcal{O}Gb$-module and as right $\mathcal{O}P$-module. The block b can have more than one isomorphism class of source modules, but they are all conjugate under the action of $N_G(P)$ in the sense that, if M' is another source module of b, there is an automorphism φ of P induced by conjugation with an element of $N_G(P)$ such that $M' \cong \mathrm{Res}_{\mathrm{Id}_G \times \varphi}(M)$. This follows from the interpretation of source modules as actual sources of a module in Remark 6 below.

4. There is a block e of $\mathcal{O}C_G(P)$ having $Z(P)$ as a defect group such that $\mathcal{O}C_G(P)e$ is isomorphic to a direct summand of $\mathrm{Res}_{C_G(P) \times C_G(P)^0}^{G \times G^0}(\mathcal{O}Gb)$; any other block of $\mathcal{O}C_G(P)$ with this property is then conjugate to e by some element of $N_G(P)$. Equivalently, e is a block of $\mathcal{O}C_G(P)$ satisfying $\mathrm{Br}_P(be) \neq 0$; that is, (P, e) is a maximal b-Brauer pair.

We set $H = N_G(P, e)$, the subgroup of all elements in $N_G(P)$ which stabilize e, and $E = H/PC_G(P)$, the *inertial quotient of the block b* (associated with e). Moreover, e is still a block of both $\mathcal{O}PC_G(P)$ and $\mathcal{O}H$ having

P as a defect group. If k is large enough for $\mathcal{O}C_G(P)e$ then E is a p'-group (see, e.g., [7, (37.12)]).

Observe that $N_{G\times P^0}(\Delta P) = \Delta P(C_G(P) \times 1) \subset H \times P^0$. Therefore, if we take a source module N of $\mathcal{O}He$, it follows from the Green correspondence [7, (20.8)] that $\mathrm{Ind}_{H\times P^0}^{G\times P^0}(N)$ has, up to isomorphism, a unique indecomposable direct summand M with vertex ΔP and trivial source, and then M is a source module for $\mathcal{O}Gb$. The next result makes this more precise:

THEOREM 5. (i) *Up to isomorphism, there is a unique indecomposable direct summand X of* $\mathrm{Res}_{G\times H^0}^{G\times G^0}(\mathcal{O}Gb)$ *with vertex ΔP which is isomorphic to a direct summand of* $\mathrm{Ind}_{H\times H^0}^{G\times H^0}(\mathcal{O}He)$.

(ii) *If N is a source module for the block e of $\mathcal{O}H$, then $M = X \otimes_{\mathcal{O}H} N$ is a source module for the block b of $\mathcal{O}G$.*

Proof. The existence and uniqueness of X follows from the Green correspondence; similarly, since M is a direct summand of $\mathrm{Ind}_{H\times P^0}^{G\times P^0}(N)$, the Green correspondence implies that $M = M' \oplus M''$, where M' is indecomposable with vertex ΔP and all indecomposable direct summands of M'' have vertices strictly smaller than ΔP.

But X is also isomorphic to a direct summand of $\mathrm{Ind}_{G\times P^0}^{G\times H^0}(Z)$ for some indecomposable $\mathcal{O}(G \times P^0)$-module Z with vertex ΔP. Thus, using Mackey's formula, the $\mathcal{O}(G \times P^0)$-module $M = X \otimes_{\mathcal{O}H} N$ is isomorphic to a direct summand of

$$\mathrm{Res}_{G\times P^0}^{G\times H^0} \mathrm{Ind}_{G\times P^0}^{G\times H^0}(Z) = \bigoplus_{x\in[H^0\backslash P^0]} {}^{(1,x)}Z.$$

Since H normalizes P, the module ${}^{(1,x)}Z$ has vertex ${}^{(1,x)}\Delta P$ for any $x \in H^0$. Together with the fact that all indecomposable direct summands of M'' have vertex strictly smaller than ΔP follows that $M = M'$, which concludes the proof. ∎

Remark 6. The Green correspondence does not require the ground ring to be commutative; one might therefore as well consider the group algebra AG^0 of G^0 over the ring $A = \mathcal{O}G$. We have an obvious isomorphism $AG^0 \cong \mathcal{O}(G \times G^0)$, which allows us to consider $\mathcal{O}Gb$ as an AG^0-module. With the notation of Theorem 5, as an AG^0-module, $\mathcal{O}Gb$ has vertex P^0; the source module M, considered as an AP^0-module, is a source of $\mathcal{O}Gb$ and the AH^0-module X is the Green correspondent of $\mathcal{O}Gb$ having M as source.

Theorem 5 is a particular case of [1, 4.10], just translated to a module theoretic language: with the notation of the theorem, let f be a primitive idempotent in $(\mathcal{O}Gb)^H$ such that $\mathrm{Br}_P(ef) \neq 0$ and let j be a primitive idempotent in $(\mathcal{O}He)^P$ such that $\mathrm{Br}_P(j) \neq 0$. Then $X \cong \mathcal{O}Gf$ and $N \cong \mathcal{O}H_j$. Thus $M \cong \mathcal{O}Gjf$, and the assertion that M is indecomposable is equivalent to the assertion that the idempotent jf is primitive in $(\mathcal{O}Gb)^P$. The referee pointed out that one can choose for j any primitive idempotent in $\mathcal{O}C_G(P)e$. This follows from the obvious equality $(\mathcal{O}H)^P = \mathcal{O}C_G(P) + \ker(\mathrm{Br}_P^{\mathcal{O}H})$ together with the fact that $\ker(\mathrm{Br}_P^{\mathcal{O}H})$ is contained in the kernel of the canonical homomorphism $\mathcal{O}H \to kH/P$; hence it is contained in $J((\mathcal{O}H)^P)$.

Note that, again by the Green correspondence, the unique indecomposable direct summand of $\mathrm{Res}^{G\times H^0}_{H\times H^0}(X)$ with vertex ΔP is isomorphic to $\mathcal{O}He$. From this and Theorem 5 we obtain the embedding of source algebras, due to Puig:

COROLLARY 7 [6, 14.6; 1, 4.10]. *Set* $A = \mathrm{End}_{\mathcal{O}(G\times 1)}(M)^0$ *and* $B = \mathrm{End}_{\mathcal{O}(H\times 1)}(N)^0$. *The functor* $X \otimes_{\mathcal{O}H}$ *induces an injective homomorphism of interior P-algebras*

$$B \to A$$

which is split as homomorphism of $B-B$*-bimodules*; *in particular*, *we have*

$$\mathrm{rk}_{\mathcal{O}}(A)/|P| \equiv \mathrm{rk}_{\mathcal{O}}(B)/|P| \pmod{p}.$$

Proof. Since $M = X \otimes_{\mathcal{O}H} N$, tensoring by X induces a homomorphism of interior P-algebras from B to A. This homomorphism is split injective as homomorphism of $B-B$-bimodules, because N is isomorphic to a direct summand of $\mathrm{Res}^{G\times P^0}_{H\times P^0}(M)$ by the usual Green correspondence. Since, moreover, any other direct summand has a vertex of order strictly smaller than $|P|$, we get the statement on the $\mathcal{O}$-ranks. ∎

If k is large enough, it is possible to be much more precise about source modules of $\mathcal{O}He$ and the corresponding source algebras. For this, we recall some standard facts from block theory (see, e.g., [7, Sect. 39]).

8. Assume that k is large enough for $\mathcal{O}C_G(P)e$; that is, the semi-simple quotient of $\mathcal{O}C_G(P)e$ is isomorphic to a matrix algebra over k. Set $\overline{C} = C_G(P)/Z(P)$ and denote by $\bar{e}$ the image of e in $\mathcal{O}\overline{C}$ under the canonical surjection $\mathcal{O}C_G(P) \to \mathcal{O}\overline{C}$.

8.1. The idempotent $\bar{e}$ is a block of $\mathcal{O}\overline{C}$ having the trivial subgroup as defect group, and the block algebra $\mathcal{O}\overline{C}e$ is isomorphic to a matrix algebra $M_n(\mathcal{O})$ over $\mathcal{O}$ for some positive integer n;

8.2. as a block of $\mathcal{O}C_G(P)$, e has $Z(P)$ as defect group and the block algebra $\mathcal{O}C_G(P)e$ is isomorphic to the matrix algebra $M_n(\mathcal{O}Z(P))$ over $\mathcal{O}Z(P)$;

8.3. as block of $\mathcal{O}PC_G(P)$, e has P as defect group and the block algebra $\mathcal{O}PC_G(P)e$ is isomorphic to the matrix algebra $M_n(\mathcal{O}P)$ over $\mathcal{O}P$.

The next proposition is a translation of the statements in 8 to our terminology involving source modules, from which we obtain the source algebras in each of the cases in 8 (as originally determined by Puig in [6, 14.5, 14.6]):

PROPOSITION 9. *Let $\bar{V}$ be a projective indecomposable $\mathcal{O}\bar{C}\bar{e}$-module, and let V be the $\mathcal{O}(\Delta P(C_G(P) \times 1))$-module obtained by taking a projective cover of $\bar{V}$ as a $\mathcal{O}C_G(P)e$-module and restricting it through the group homomorphism $\Delta P(C_G(P) \times 1) \to C_G(P)$ which maps (uz, u^{-1}) to z for any $u \in P$ and any $z \in C_G(P)$. Set*

$$U = \mathrm{Ind}_{\Delta P(C_G(P)\times 1)}^{PC_G(P)\times P^0}(V).$$

(i) *Up to isomorphism, $\bar{V}$ is the unique source module of $\mathcal{O}\bar{C}\bar{e}$, and we have an algebra isomorphism $\mathrm{End}_{\mathcal{O}(\bar{C}\times 1)}(\bar{V})^0 \cong \mathcal{O}$.*

(ii) *Up to isomorphism, $\mathrm{Res}_{C_G(P)\times Z(P)^0}^{\Delta P(C_G(P)\times 1))}(V)$ is the unique source module of $\mathcal{O}C_G(P)e$, and the natural map*

$$\mathcal{O}Z(P) \to \mathrm{End}_{\mathcal{O}(C_G(P)\times 1)}(V)^0$$

is an algebra isomorphism.

(iii) *Up to isomorphism, U is the unique source module of $\mathcal{O}PC_G(P)e$, and the natural map*

$$\mathcal{O}P \to \mathrm{End}_{\mathcal{O}(PC_G(P)\times 1)}(U)^0$$

is an algebra isomorphism.

Proof. Statement (i) is a trivial consequence of the fact that by 8.1 the block algebra $\mathcal{O}\bar{C}\bar{e}$ is a matrix algebra over $\mathcal{O}$ and has defect group {1}.

The restriction to $C_G(P)$ of V is indecomposable, since it is a projective cover of the simple module $k \otimes_{\mathcal{O}} \bar{V}$. Thus V and its restriction to $C_G(P) \times Z(P)^0$ are indecomposable. Moreover, ΔP acts trivially on V. Thus ΔP and $\Delta Z(P)$ are vertices of V and its restriction to $C_G(P) \times Z(P)^0$, respectively. This shows already that $\mathrm{Res}_{C_G(P)\times Z(P)^0}^{\Delta P(C_G(P)\times 1}(V)$ is a source module of $\mathcal{O}C_G(P)e$. As the normalizer of the defect group $Z(P)$ acts transitively on the isomorphism classes of source modules (cf. Remark 3), this shows also the uniqueness, up to isomorphism in this situation. The natural algebra homomorphism in (ii) is injective, since V is projective as an $\mathcal{O}Z(P)$-module. Since $\mathcal{O} \otimes_{\mathcal{O}Z(P)} \mathrm{End}_{\mathcal{O}(C_G(P)\times 1)}(V) \cong \mathrm{End}_{\mathcal{O}(\bar{C}\times 1)}(\bar{V}) \cong \mathcal{O}$, it follows from Nakayama's lemma that the algebra homomorphism is (ii) is also surjective. This shows (ii).

The natural map $\mathcal{O}P \to \mathrm{End}_{\mathcal{O}(PC_G(P)\times 1)}(U)^0$ is $\mathcal{O}$-split injective, since U is projective as an $\mathcal{O}(1 \times P^0)$-module. In order to show the surjectivity, it suffices to show that both sides have the same $\mathcal{O}$-rank. By Mackey's formula, we have $\mathrm{Res}^{PC_G(P)\times P^0}_{PC_G(P)\times 1}(U) \cong \mathrm{Ind}^{PC_G(P)\times 1}_{C_G(P)\times 1}(V)$. Thus Frobenius' reciprocity implies that $\mathrm{End}_{\mathcal{O}(PC_G(P)\times 1)}(U) \cong \mathrm{Hom}_{\mathcal{O}(C_G(P)\times 1)}(V, \oplus_x {}^{(x,1)}V)$, with x running over a right transversal of $C_G(P)$ in $PC_G(P)$. Now ${}^{(x,1)}V \cong V$ as left $\mathcal{O}C_G(P)$-modules for any $x \in PC_G(P)$, and whence the rank of $\mathrm{End}_{\mathcal{O}(PC_G(P)\times 1)}(U)$ is $[PC_G(P) : C_G(P)]\mathrm{rk}_{\mathcal{O}}(\mathrm{End}_{\mathcal{O}(C_G(P)\times 1)}(V)) = [P : Z(P)]|Z(P)| = |P|$. This shows that U is indecomposable, even when restricted to $PC_G(P) \times 1$. Moreover, ΔP is a vertex of V and thus of U, too. Therefore U is a source module for $\mathcal{O}PC_G(P)e$. Again, by Remark 3, it is unique up to isomorphism. ∎

Before we go further, we recall some facts about central extensions of a finite group by the group $\mathcal{O}^\times$ of invertible elements in $\mathcal{O}$. The reason for central extensions to occur in this context is the following: the module $\bar{V}$, viewed as module for $PC_G(P)$ via the canonical homomorphism $PC_G(P) \to PC_G(P)/P \cong C_G(P)/Z(P)$, cannot, in general, be extended to H. It can, however, be extended to a *twisted group algebra* $\mathcal{O}_* \hat{H}$ for some central $\mathcal{O}^\times$-extension $\hat{H}$ of H which arises naturally from the action of H on the matrix algebra $\mathcal{O}\bar{C}\bar{e}$.

10. Assume that k is large enough for the block e of $\mathcal{O}C_G(P)$. The group H acts on the matrix algebra $S = \mathcal{O}\bar{C}\bar{e}$, since H normalizes P and stabilizes e. Let $\hat{H}$ be the subgroup of $H \times S^\times$ of all pairs $(x, s) \in H \times S^\times$ satisfying ${}^x t = sts^{-1}$ for all $t \in S$; that is, such that x and s induce the same automorphism of S. In other words, $\hat{H}$ is the pullback of the obvious maps from H and $S^\times$ to $\mathrm{Aut}(S)$.

By the Skolem–Noether theorem, any automorphism of S is inner; in particular, the projection to the first component $\hat{H} \to H$ mapping (x, s) to x is surjective. The kernel of this map is $\{(1_H, \lambda 1_S)\}_{\lambda \in \mathcal{O}^\times} \cong \mathcal{O}^\times$; thus $\hat{H}$ is a central extension of H by $\mathcal{O}^\times$. We consider $\hat{H}$ as an $\mathcal{O}^\times$-group endowed with the map $\mathcal{O}^\times \to \hat{H}$ sending $\lambda \in \mathcal{O}^\times$ to $\hat{\lambda} = (1_H, \lambda 1_S)$ and call it the *$\mathcal{O}^\times$-group defined by the action of H on S*. Its *opposite $\mathcal{O}^\times$-group* is the group $\check{H}$, which is equal to $\hat{H}$ as an abstract group, endowed with the group homomorphism $\mathcal{O}^\times \to \check{H}$ sending $\lambda \in \mathcal{O}^\times$ to $\check{\lambda} = (1_H, \lambda^{-1} 1_S)$.

The twisted group algebra $\mathcal{O}_* \hat{H}$ is the quotient of the group algebra $\mathcal{O}\hat{H}$ by the ideal generated by the set of elements $\lambda.\hat{x} - 1_{\mathcal{O}}.(\hat{\lambda}\hat{x})$ with λ running over $\mathcal{O}^\times$ and $\hat{x}$ running over $\hat{H}$. This is an $\mathcal{O}$-free algebra of $\mathcal{O}$-rank $|H|$; indeed, if for any $x \in H$ we denote by $\hat{x}$ an element of $\hat{H}$ lifting x, then the set $\{\hat{x}\}_{x \in H}$ is an $\mathcal{O}$-basis of $\mathcal{O}_* \hat{H}$, and multiplication is given by $\hat{x}\hat{y} = \lambda_{x,y}\widehat{xy}$ for any two $x, y \in H$ and uniquely determined coefficients $\lambda_{x,y} \in \mathcal{O}^\times$ (this is a two-cocycle determining the central extension $\hat{H}$ of H). If the extension $\hat{H}$ of H splits; that is, $\hat{H} \cong H \times \mathcal{O}^\times$, any such isomorphism induces an isomorphism of $\mathcal{O}$-algebras $\mathcal{O}_* \hat{H} \cong \mathcal{O}H$.

We identify $PC_G(P)$ with its canonical image in $\hat{H}$ (via the inclusion $PC_G(P) \subset H$ and the obvious homomorphism $PC_G(P) \to PC_G(P)/P \cong C_G(P)/Z(P) \to S^\times$; see [6, 6.5]) and set $\hat{E} = \hat{H}/PC_G(P)$. Similarly, we set $\check{E} = \check{H}/PC_G(P)$. The $\mathcal{O}^\times$-groups $\hat{E}$ and $\check{E}$ are central $\mathcal{O}^\times$-extensions of E.

The projection to the second component $\hat{H} \to S^\times$ mapping (x, s) to s induces a surjective algebra homomorphism $\mathcal{O}_* \hat{H} \to \mathcal{O}\overline{C}\bar{e}$ which extends the canonical homomorphism $\mathcal{O}PC_G(P)e \to \mathcal{O}\overline{C}\bar{e}$. Note that this homomorphism maps the elements of P to the unit element of $\mathcal{O}\overline{C}\bar{e}$. In particular, any $\mathcal{O}\overline{C}\bar{e}$-module can be viewed as an $\mathcal{O}_* \hat{H}/P$-module through this homomorphism. Note that $k \otimes_{\mathcal{O}} \overline{V}$ is, up to isomorphism, the unique simple $\mathcal{O}\overline{C}\bar{e}$-module, which necessarily remains simple when considered as an $\mathcal{O}_* \hat{H}/P$-module.

Since E is a p'-group, the inverse image of E in $\mathrm{Aut}(P)$ has $\mathrm{Int}(P)$ as a normal Sylow-p-subgroup; thus it has a complement isomorphic to E; unique up to conjugacy (cf. [7, (45.1)]), which we are going to denote again by E. We denote by $P \rtimes E$ the corresponding semi-direct product. The $\mathcal{O}^\times$-group $\check{E}$ acts still on P via the canonical surjection $\check{E} \to E$, and we denote again by $P \rtimes \check{E}$ the corresponding semi-direct product, viewed as an $\mathcal{O}^\times$-group in the obvious way, that is, with the $\mathcal{O}^\times$-group structure coming from $\check{E}$. We use the sign $\circ$ to denote the opposite product of a group.

PROPOSITION 11. *Let $\tau: \hat{H} \to H$ and $\pi: \hat{H} \to \hat{E}$ be the canonical surjections.*

(i) *The product map (τ, π): $\hat{H} \to H \times \hat{E}$ is injective, and its image consists of all pairs $(x, \hat{e}) \in H \times \hat{E}$ with the property that the natural images of x and $\hat{e}$ in* $\mathrm{Aut}(P)/\mathrm{Int}(P)$ *coincide; equivalently, the triple $(\hat{H}, \tau, \pi)$ is the pullback of the natural maps from H and $\hat{E}$ to* $\mathrm{Aut}(P)/\mathrm{Int}(P)$.

(ii) *Set $\hat{L} = N_{H \times (P \rtimes \check{E})^0}(\Delta P)$. For any $(x, u \circ \check{e}) \in \hat{L}$, where $x \in H$, $u \in P$, $\check{e} \in \check{E}$, there is a unique $\hat{x} = \alpha(x, u \circ \check{e}) \in \hat{H}$ such that $\tau(\hat{x}) = x$ and $\pi(\hat{x})^{-1} = \check{e}$, and this map is a surjective homomorphism of $\mathcal{O}^\times$-groups*

$$\alpha : \hat{L} \to \hat{H}$$

which maps ΔP to P and whose kernel is $1 \times Z(P)^0$.

Proof. (i) Since $\ker(\tau) \cong \mathcal{O}^\times$ and $\ker(\pi) = PC_G(P)$, the product map (τ, π) is injective, and for any $\hat{x} \in \hat{H}$, the images of $\tau(\hat{x})$ and $\pi(\hat{x})$ in $\mathrm{Aut}(P)/\mathrm{Int}(P)$ coincide. Let $x \in H$ and $\hat{e} \in \hat{E}$ such that the images of x and $\hat{e}$ in $\mathrm{Aut}(P)/\mathrm{Int}(P)$ coincide. This means that the class $\hat{e} \in \hat{E} = \hat{H}/PC_G(P)$ can be represented by an element of the form $\hat{x} = (x, s)$ of $\hat{H}$ for some $s \in S^\times$, which proves (i).

(ii) An element $(x, u \circ \check{e}) \in H \times (P \rtimes \check{E})^0$ normalizes ΔP if x and $u^{-1}\check{e}^{-1}$, the latter now viewed as an element of $P \rtimes \hat{E}$, induce the same automorphism of P. In particular, the images of x and $\check{e}^{-1}$ in $\mathrm{Aut}(P)/\mathrm{Int}(P)$ coincide, which by (i) implies that there is a unique $\hat{x} \in \hat{H}$ satisfying $\tau(\hat{x}) = x$ and $\pi(\hat{x})^{-1} = \check{e}$. From this uniqueness it follows also that the map α thus defined is a group homomorphism. Conversely, given $\hat{x} \in \hat{H}$, the elements $x = \tau(\hat{x}) \in H$ and $\hat{e} = \pi(\hat{x}) \in \hat{E}$ have the same image in $\mathrm{Aut}(P)/\mathrm{Int}(P)$, which means that there is $u \in P$ such that $(x, u \circ \hat{e}^{-1})$ normalizes ΔP. This shows the surjectivity of α. Moreover, if in this case both $x = 1$ and $\hat{e} = 1$, then $u \in Z(P)$, which implies that $\ker(\alpha) = 1 \times Z(P)^0$. Finally, α maps $(u, u^{-1}) \in \Delta P$ to the unique element $\hat{u}$ of $\hat{H}$ satisfying $\tau(\hat{u}) = u$ and $\pi(\hat{u}) = 1$ which forces $\hat{u} = u$. ∎

The group homomorphism α in the previous proposition allows us to consider any $\mathscr{O}_* \hat{H}/P$-module as an $\mathscr{O}_* \hat{L}/\Delta P$-module. In particular, the source module $\overline{V}$ for $\mathscr{O}\overline{C}\overline{e}$ from Proposition 9 becomes an $\mathscr{O}_* \hat{L}/\Delta P$-module in this way. The next proposition shows that the $\mathscr{O}\Delta(C_G(P) \times 1))$-module V from Proposition 9 extends to $\hat{L}$.

PROPOSITION 12. *Let W be a projective cover of $\overline{V}$ viewed as an $\mathscr{O}_* \hat{L}/\Delta P$-module, and consider then W as an $\mathscr{O}_* \hat{L}$-module through the canonical map $\hat{L} \to \hat{L}/\Delta P$. Let V be the $\mathscr{O}\Delta P(C_G(P) \times 1)$-module as in Proposition* 9. *We have*

$$\mathrm{Res}^{\hat{L}}_{\Delta P(C_G(P)\times 1)}(W) \cong V.$$

Proof. As a left $\mathscr{O}C_G(P)$-module, V is a projective cover of the simple module $k \otimes_{\mathscr{O}} \overline{V}$; in particular, V has a unique maximal submodule $\mathrm{rad}(V)$. Since $k \otimes_{\mathscr{O}} \overline{V}$ is still simple as an $\mathscr{O}_* \hat{L}/\Delta P$-module, W has a unique maximal submodule $\mathrm{rad}(W)$, too, $\mathrm{Res}^{\hat{L}}_{\Delta P(C_G(P)\otimes 1)}(W)$ has a direct summand isomorphic to V, and we have

$$\mathrm{Res}^{\hat{L}}_{\Delta P(C_G(P)\times 1)}(W/\mathrm{rad}(W)) \cong V/\mathrm{rad}(V) \cong k \bigotimes_{\mathscr{O}} \overline{V}.$$

This isomorphism implies that $\mathrm{rad}(W)$ is also a maximal submodule of $\mathrm{Res}^{\hat{L}}_{\Delta P(C_G(P)\times 1)}(W)$.

The group homomorphism $\alpha\colon \hat{L} \to \hat{H}$ from Proposition 11(ii) induces an isomorphism $\hat{L}/\Delta P(C_G(P) \times 1) \cong \hat{H}/PC_G(P) \cong \hat{E}$. As E is a p'-group, we have therefore $J(\mathscr{O}_* \hat{L}) = \mathscr{O}_* \hat{L}.J(\mathscr{O}\Delta P(C_G(P) \times 1))$. This means that $\mathrm{rad}(W)$ is also the radical and thus the unique maximal submodule of $\mathrm{Res}^{\hat{L}}_{\Delta P(C_G(P)\times 1)}(W)$; in particular, the latter is indecomposable and whence isomorphic to V. ∎

We determine now the source modules of $\mathscr{O}He$ and obtain from this description the structure of the source algebras of $\mathscr{O}He$ as given by Puig in

[6, 14.6] (extending earlier work of Külshammer on blocks with normal defect groups [2]). We keep the notation introduced above.

THEOREM 13. *Set* $N = \mathrm{Ind}_{PC_G(P)\times P^0}^{H\times P^0}(U)$, *where* U *is a source module for* $\mathcal{O}PC_G(P)e$.

(i) *Up to isomorphism,* N *is the unique source module for* $\mathcal{O}He$.

(ii) *We have* $N \cong \mathrm{Res}_{H\times P^0}^{H\times(P\rtimes\check{E})^0}\mathrm{Ind}_{\hat{L}}^{H\times(P\rtimes\check{E})^0}(W)$.

(iii) *As a right* $\mathcal{O}_*(P\rtimes\check{E})$*-module,* $\mathrm{Ind}_{\hat{L}}^{H\times(P\rtimes\check{E})^0}(W)$ *is free of rank* n.

(iv) (*Puig* [6, 14.6]). *We have an isomorphism of interior* P*-algebras*

$$\mathrm{End}_{\mathcal{O}(H\times 1)}(N)^0 \cong \mathcal{O}_*(P\rtimes\check{E});$$

in particular, as an $\mathcal{O}$*-algebra,* $\mathcal{O}He$ *is isomorphic to the matrix algebra* $M_n(\mathcal{O}_*(P\rtimes\check{E}))$.

Proof. Since U is an indecomposable direct summand of $\mathcal{O}PC_G(P)e$ as an $\mathcal{O}(PC_G(P)\times P^0)$-module having ΔP as a vertex, the module N is isomorphic to a direct summand of $\mathcal{O}He$ as an $\mathcal{O}(H\times P^0)$-module having any source module of $\mathcal{O}He$ as a direct summand.

By Proposition 9 we have $N \cong \mathrm{Ind}_{\Delta P(C_G(P)\times 1)}^{H\times P^0}(V)$, and by Proposition 12 together with Mackey's formula, we get

$$N \cong \mathrm{Ind}_{\Delta P(C_G(P)\times 1)}^{H\times P^0}\mathrm{Res}_{\Delta P(C_G(P)\times 1)}^{\hat{L}}(W) \cong \mathrm{Res}_{H\times P^0}^{H\times(P\rtimes\check{E})^0}\mathrm{Ind}_{\hat{L}}^{H\times(P\rtimes\check{E})^0}(W).$$

In particular, any such isomorphism shows that N has a right $\mathcal{O}_*(P\rtimes\check{E})$-module structure, or equivalently, any such isomorphism induces a homomorphism of interior P-algebras

13.1. $\mathcal{O}_*(P\rtimes\check{E}) \to \mathrm{End}_{\mathcal{O}(H\times 1)}(N)^0$

mapping $a \in \mathcal{O}_*(P\rtimes\check{E})$ to the endomorphism of N given by the action of a on N. Another application of Mackey's formula shows that

$$\mathrm{Res}_{1\times(P\rtimes\check{E})^0}^{H\times(P\rtimes\check{E})^0}\mathrm{Ind}_{\hat{L}}^{H\times(P\rtimes\check{E})^0}(W) \cong \mathrm{Ind}_{1\times Z(P)^0}^{1\times(P\rtimes\check{E})^0}\mathrm{Res}_{1\times Z(P)^0}^{\hat{L}}(W)$$

is free as a right $\mathcal{O}_*(P\rtimes\check{E})$-module by 12 and has therefore rank n by 8.2; in particular, the homomorphism 13.1 is $\mathcal{O}$-split injective. In order to show that 13.1 is an isomorphism it suffices to show that both sides have the same $\mathcal{O}$-rank.

By an appropriate version of Frobenius' reciprocity, we have an $\mathcal{O}$-linear isomorphism $\mathrm{End}_{\mathcal{O}(H\times 1)}(N) \cong \mathrm{Hom}_{\mathcal{O}(PC_G(P)\times 1}(U, \mathrm{Res}_{PC_G(P)\times 1}^{H\times P^0}(N))$, and Mackey's formula implies that $\mathrm{Res}_{PC_G(P)\times 1}^{H\times P^0}(N) \cong \oplus_x {}^{(x,1)}U$, where x runs over a set of representatives in H of $H/PC_G(P)$. The direct summands ${}^{(x,1)}U$ in this sum are all isomorphic to U as left $\mathcal{O}PC_G(P)$-modules since the elements of H stabilize e and since $\mathcal{O}PC_G(P)e$ has only one isomorphism class of projective indecomposable modules (cf. 8.3). This shows that $\mathrm{rk}_{\mathcal{O}}(\mathrm{End}_{\mathcal{O}(H\times 1)}(N)) = |E|\mathrm{rk}_{\mathcal{O}}(\mathrm{End}_{\mathcal{O}(PC_G(P)\times 1)}(U)) = |E|\,|P|$, where the last equality comes from 9(iii). Therefore 13.1 is an isomorphism. Since we know already that every source module of $\mathcal{O}He$ occurs as a direct summand, up to isomorphism, of N, all we have to show is that N is indecomposable as an $\mathcal{O}(H \times P^0)$-module. This follows, for instance, from the referee's observation at the end of Remark 6. Equivalently, by 13.1 it suffices to show that the unit element of $\mathcal{O}_*(P \times \check{E})$ is primitive in $(\mathcal{O}_*(P \rtimes \check{E}))^P$. This is well known, and there are various possibilities to see this: one way is to observe that the rank of any source algebra of $\mathcal{O}He$ is at least $|P|\,|E|$ because for any $x \in H$, any source algebra has a direct summand isomorphic to $\mathcal{O}[Px]$ as an $\mathcal{O}(P \times P^0)$-module. A more direct proof goes as follows, involving the Brauer construction [7, Sect. 11]: first, $(\mathcal{O}_*(P \rtimes \check{E}))(P) \cong kZ(P)$ and so $\mathcal{O}_*(P \rtimes \check{E})$ has a unique block and a unique local point of P whose multiplicity is then necessarily 1; second, E is a p'-group acting on the semi-simple quotient of $\mathcal{O}_*(P \rtimes \check{E})^P$ permuting transitively the indecomposable direct algebra summands (because any E-orbit yields an E-stable idempotent which lifts to a block of $\mathcal{O}_*(P \rtimes \check{E})$, whence it must be the unit element). Combining these observations shows that the unit element is primitive in $\mathcal{O}_*(P \rtimes \check{E})^P$. Yet another proof of this fact follows from decomposing $\mathcal{O}_*(P \rtimes \check{E}) \cong \bigoplus_{e \in E} \mathcal{O}Pe$ as an $\mathcal{O}(P \times P^0)$-module; the direct summands in this decomposition are indecomposable, mutually nonisomorphic, and transitively permitted by the action of E. Thus $\mathcal{O}_*(P \rtimes \check{E})$ is indecomposable as an $\mathcal{O}_*((P \rtimes \check{E}) \times P^0)$-module, showing again that the unit element is primitive in $(\mathcal{O}_*(P \rtimes \check{E}))^P$. This concludes the proof of Theorem 13. ∎

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