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Self-equivalences of stable module categories

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Abstract. Let P be an abelian p-group, E a cyclic p'-group acting freely on P and k an algebraically closed field of characteristic $p > 0$. In this work, we prove that every self-equivalence of the stable module category of $k[P \rtimes E]$ comes from a self-equivalence of the derived category of $k[P \rtimes E]$. Work of Puig and Rickard allows us to deduce that if a block B with defect group P and inertial quotient E is Rickard equivalent to $k[P \rtimes E]$, then they are splendidly Rickard equivalent. That is, Broué's original conjecture implies Rickard's refinement of the conjecture in this case. All of this follows from a general result concerning the self-equivalences of the thick subcategory generated by the trivial module.

1 Introduction

Let G be a finite group and k a field of characteristic $p > 0$. The study of module categories has come to play a major role in the modular representation theory of finite groups. On the one hand, equivalences of the derived categories of blocks of group algebras are implied in the local representation theory involved with the conjectures of Alperin and Broué (e.g. see [5]). Here the objective is to discern the relationship between the representation theory of a block of kG and that of a corresponding block of some subgroup such as the normalizer of a defect group of the block. On the other hand the structure of the categories plays a large part in the basic homological properties of modules (e. g. see [4]).

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In this paper we study properties of equivalences of derived and stable categories in some cases. Of particular interest is the thick subcategory $\mathcal{T}(G)$ of the stable category generated by the trivial module. In a real sense this subcategory controls cohomology and extensions of modules. The importance of the subcategory has been highlighted in several recent works [4, 7]. Our main theorem in the next section is that any equivalence between $\mathcal{T}(G)$ and $\mathcal{T}(H)$ for groups G and H must take endo-trivial modules to endo-trivial modules. So after tensoring with a suitable endo-trivial module we can assume that any such equivalence takes the trivial kG-module to the trivial kH-module. In addition, the theorem implies that if two p-groups have equivalent stable categories then their group algebras are isomorphic. The same had been proved by Linckelmann for integral group rings of p-groups over the p-adic integers and its extensions.

It is known that if the centralizers of every non-trivial p-element in G is p-nilpotent, then the subcategory $\mathcal{T}(G)$ is equal to the entire stable category of the principal block of kG. This fact was proved for $p > 2$ in [3] and in full generality by Benson in [2]. Also we know from [9] that the endo-trivial modules over abelian p-groups are all Heller translates of the trivial module. Using these facts, we consider the Picard groups of self equivalences of the group algebra of an extension of an abelian p-group P by a cyclic p'-group E that acts freely on P. Our main result is that the Picard group of the stable category of $k[P \rtimes E]$ is generated by the Picard group of the module category and the translation functor. So self-equivalences (of Morita type) of the stable category all lift to equivalences of the derived category.

In the last section we consider further conditions which imply that stable equivalences lift to derived (Rickard) equivalences and also lift to corresponding categories of $\mathcal{O}G$-modules. For the case of $G \simeq P \rtimes E$ as above we are able to show the equivalence of conjectures of Broué and Rickard concerning the existence of derived equivalences between a G-block e and the block of a maximal e-subpair. Also in the case that P is cyclic or elementary abelian of order 4, we prove that some results of Linckelmann, Rickard and the second author on the existence of Rickard equivalence are all equivalent.

2 Stable equivalences and endo-trivial modules

Let k be an algebraically closed field of characteristic $p > 0$ and let G be a finite group. Unless otherwise indicated, the tensor product $M \otimes N$ of two kG-modules M and N will be the tensor product over the base field with diagonal G-action.

Throughout the paper we let $\mathbf{mod(kG)}$ denote the category of all finitely generated left kG-modules and let $\mathbf{stmod(kG)}$ be the corresponding stable category of kG-modules modulo projectives. The stable category is triangulated with triangles in $\mathbf{stmod(kG)}$ corresponding roughly to exact sequences in $\mathbf{mod(kG)}$. The translation functor in the triangulation is the Heller translate Ω^{-1} where for a kG-module M, $\Omega^{-1}(M)$ is isomorphic to the cokernel of an injective hull $M \hookrightarrow Q$. Recall that a subcategory $\mathcal{M}$ of $\mathbf{stmod(kG)}$ is thick provided it is triangulated and it is closed under direct summands. Let $\overline{\mathrm{Hom}}_{kG}(M,N) = Hom_{kG}(M,N)/PHom_{kG}(M,N)$ where $PHom_{kG}(M,N)$ is the set of all homomorphisms which factor through a projective module. By definition $\overline{\mathrm{Hom}}_{kG}$ is the Hom functor for $\mathbf{stmod(kG)}$. For further discussion of the triangulated categories see [10] or [6].

Let $\mathcal{T}(G)$ (or $\mathcal{T}$) be the thick subcategory of $\mathbf{stmod(kG)}$ generated by the trivial module k. That is, $\mathcal{T}$ is the smallest full triangulated subcategory of $\mathbf{stmod(kG)}$ containing k and closed under taking direct summands. Let $\mathcal{A}$ be the smallest full subcategory of $\mathbf{mod(kG)}$ containing the syzigies $\Omega^n k$ of the trivial module ($n \in \mathbb{Z}$) and closed under taking extensions and direct summands. Then $\mathcal{A}$ and $\mathcal{T}$ have the same collection of objects. The objects of $\mathcal{A}$ are direct summands of what were called *trivial-homology (TH)* modules in [3]. The canonical functor $\mathbf{mod(kG)} \to \mathbf{stmod(kG)}$ induces an essentially surjective functor $\mathcal{A} \to \mathcal{T}$. Note that $\mathcal{A}$ is a tensor subcategory of $\mathbf{mod(kG)}$ in the sense that it is closed under the taking of tensor products. We know from [3, 2] that $\mathcal{A}$ is the module category of the principal block of kG if and only if the centralizers of non-trivial p-subgroups of G are p-nilpotent.

Let $\bar{K}(\mathcal{T})$ be the image of the Grothendieck group of $\mathcal{T}$ in the Grothendieck group of $\mathbf{stmod(kG)}$ (*i.e.*, the subgroup of the Grothendieck group of $\mathbf{stmod(kG)}$ generated by the classes of the objects of $\mathcal{T}$).

Lemma 2.1. *The group $\bar{K}(\mathcal{T})$ is generated by the class $[k]$ of the trivial module and its order is equal to the order of a Sylow p-subgroup of G.*

Proof. The order of the class $[k]$ of the trivial module in the Grothendieck group of $\mathbf{stmod(kG)}$ is the smallest non zero integer n such that there is a virtually projective character χ equal to n times the Brauer character of k. By [24, Sect. 16.1, Théorème 35 et Exercice 3], this integer is the order of a Sylow p-subgroup of G.

Note that when G is the direct product of a p-group P and a p'-group, then $\mathcal{T}$ is equivalent to the stable category of kP-modules and the result is trivial.

So suppose that M is a module in $\mathcal{A}$ and that ϕ is its Brauer character. In order to show that the class of M in $\bar{K}(\mathcal{T})$ is equal to $\dim M \cdot [k]$, we have to show that the class function f which vanishes on p-singular elements and which is equal to $\phi - \phi(1)1$ on p-regular elements is a generalized character. For then f will be a virtually projective character.

Thanks to Brauer's theorem on characterization of characters [24, Théorème 21], f is a generalized character if and only if its restriction to every nilpotent subgroup of G is a generalized character. If H is a subgroup of G, then restriction induces a functor $\mathcal{T}(G) \to \mathcal{T}(H)$. So, we have to prove that for every nilpotent subgroup H of G, the class in $\bar{K}(\mathcal{T}(H))$ of the restriction of M is $\dim M \cdot [k]$. As we have seen, the lemma holds when G is nilpotent, and hence we are done. □

Remark 1. In general, the canonical map from the Grothendieck group of a thick triangulated subcategory of a triangulated category $\mathcal{D}$ to the Grothendieck group of $\mathcal{D}$ is not injective.

Nevertheless, it may well be that the Grothendieck group of $\mathcal{T}$ itself is generated by the class of k and has order the order of a Sylow p-subgroup of G or equivalently, that the canonical map from the Grothendieck group of $\mathcal{T}$ to the Grothendieck group of $\mathbf{stmod}(\mathbf{kG})$ is injective. This is certainly the case when $\mathcal{A}$ is the module category of the principal block of kG.

Recall that a kG-module M is endo-trivial if $M \otimes M^* \simeq k \oplus P$, where P is a projective module. Thus if M is endo-trivial then $M \otimes -$ and $M^* \otimes -$ induce inverse self-equivalences of the stable module category $\mathbf{stmod}(\mathbf{kG})$.

Let H be another finite group. Suppose that $F : \mathbf{stmod}(\mathbf{kG}) \to \mathbf{stmod}(\mathbf{kH})$ and $F' : \mathbf{stmod}(\mathbf{kH}) \to \mathbf{stmod}(\mathbf{kG})$ are two triangulated functors such that $F(\mathcal{T}(G)) \subseteq \mathcal{T}(H)$, $F'(\mathcal{T}(H)) \subseteq \mathcal{T}(G)$ and that the restrictions of F and F' to functors $\mathcal{T}(G) \to \mathcal{T}(H)$ and $\mathcal{T}(H) \to \mathcal{T}(G)$ are inverse equivalences. Then we can prove the following.

Proposition 2.2. *If M is an endo-trivial module in $\mathcal{T}(G)$, then $F(M)$ is endo-trivial. So F and F' induce inverse isomorphisms between the groups of isomorphisms classes of endo-trivial modules in $\mathcal{T}(G)$ and $\mathcal{T}(H)$.*

Proof. We may assume that p divides the order of H. By composing the self-equivalence of $\mathbf{stmod}(\mathbf{kG})$ given by $M \otimes -$ with F, we may also assume that M is the trivial module k.

The functors F and F' induces inverse isomorphisms $\phi : \bar{K}(\mathcal{T}(G)) \to \bar{K}(\mathcal{T}(H))$ and $\phi' : \bar{K}(\mathcal{T}(H)) \to \bar{K}(\mathcal{T}(G))$. Let $V = F(k)$. Then, the class χ of V in $\bar{K}(\mathcal{T}(H))$ is a ($\mathbb{Z}$-module) generator, by Lemma 2.1. So, there is a non-zero integer r prime to p such that $\chi = r[k]$. So, $\dim V \equiv r[p^a]$, where p^a is the order of a Sylow p-subgroup of H. Hence, $\dim V$ is prime to p.

It follows that there is a kH-module X such that $V \otimes V^* \simeq k \oplus X$ as kH-modules [1, Theorem 3.1.9]. Now, for every integer i, we have

$$\begin{aligned}\overline{\mathrm{Hom}}_{kH}(\Omega^i k, V \otimes V^*) &\simeq \overline{\mathrm{Hom}}_{kH}(\Omega^i V, V) \simeq \overline{\mathrm{Hom}}_{kG}(\Omega^i k, k) \\ &\simeq \hat{H}^i(G, k).\end{aligned}$$

So, $\dim \hat{H}^i(G,k) \geq \dim \hat{H}^i(H,k)$ for all i, where

$$\hat{H}^i(G,k) \simeq \overline{\mathrm{Hom}}_{kG}(\Omega^i k, k)$$

is the Tate cohomology. Swapping the roles of G and H in the discussion above shows that $\dim \hat{H}^i(H,k) \geq \dim \hat{H}^i(G,k)$ for all i. Hence, $\dim \hat{H}^i(G,k) = \dim \hat{H}^i(H,k)$ for all i, and $\overline{\mathrm{Hom}}_{kH}(\Omega^i k, X) = 0$ for all i. Since X is in $\mathcal{T}(H)$, this implies that X is projective [3, Corollary 3.8]. □

Corollary 2.3. *The functor $\tilde{F} = (F(k)^* \otimes -) \circ F$ gives an equivalence $\mathcal{T}(G) \to \mathcal{T}(H)$ such that $\tilde{F}(k) \simeq k$ (in $\mathcal{T}(H)$).*

Note that Linckelmann's analogous result [14, Theorem 3.1] for p-groups H, G and a discrete valuation ring $\mathcal{O}$ of characteristic zero with residue field k follows immediately. That is, if $\tilde{F} : \mathbf{stmod}(\mathcal{O}\mathbf{G}) \to \mathbf{stmod}(\mathcal{O}\mathbf{H})$ is an equivalence, then $\tilde{F} \otimes k : \mathbf{stmod}(\mathbf{kG}) \to \mathbf{stmod}(\mathbf{kH})$ is an equivalence, hence $\tilde{F}(\mathcal{O}) \otimes k$ is endo-trivial, so $\tilde{F}(\mathcal{O})$ is endo-trivial.

The following result answers a question of Linckelmann [14, p.93] (cf Sect. 3 for the definition of stable equivalences of Morita type) :

Corollary 2.4. *Let G and H be two p-groups such that there is a stable equivalence of Morita type between kG and kH. Then, kG and kH are isomorphic.*

Proof. By Corollary 2.3, there is a functor $F' : \mathbf{mod}(\mathbf{kG}) \to \mathbf{mod}(\mathbf{kH})$ inducing a stable equivalence such that $F'(k) \simeq k \oplus$ projective module. By [14, Theorem 2.1], there is a functor $F : \mathbf{mod}(\mathbf{kG}) \to \mathbf{mod}(\mathbf{kH})$ (a direct summand of F') inducing a stable equivalence, with $F(k) \simeq k$. Now, by [14, Theorem 2.1], such a functor gives a Morita equivalence between kG and kH. Therefore we are done, because every Morita equivalence between kG and kH comes from an isomorphism $kG \xrightarrow{\simeq} kH$, because both algebras are basic algebras. □

We should point out that it is a open question as to whether the existence of an isomorphism between kG and kH implies that the p-groups G and H are isomorphic.

3 Self stable equivalences for some frobenius groups

In this section we show how the results of the previous section can be used to characterize the self-equivalences of the stable category in some specific cases. Throughout the section we assume that P is an abelian p-group, E is a cyclic p'-group that acts freely on P and $G = P \rtimes E$. Because the centralizer of every p-element is p-nilpotent, the category $\mathcal{T}$ is the full stable category of kG [3,2,4]. Let $\hat{E} = \mathrm{Hom}(E, k^\times)$ be the character group. For notational convenience we shall identify the elements of $\hat{E}$ with their underlying modules.

Lemma 3.1. *Let $F : \mathbf{stmod(kG)} \to \mathbf{stmod(kG)}$ be an equivalence. Then, there is a permutation σ of $\hat{E}$ and an integer n such that $F(V)$ is isomorphic to $\Omega^n\sigma(V)$ in $\mathbf{stmod(kG)}$, for all $V \in \hat{E}$.*

Proof. Let M be an indecomposable endo-trivial module for kG. Then, the restriction N of M to kP is endo-trivial, hence isomorphic to $\Omega^n_{kP}k \oplus (kP)^r$ for some integers n, r by [9, Theorem 10.1]. It follows that M is a direct summand of the induced module

$$\mathrm{Ind}_P^G N \simeq \bigoplus_{V \in \hat{E}} \Omega^n_{kG} V \oplus \text{ projective module.}$$

Hence, $M \simeq \Omega^n_{kG}V$ for some $V \in \hat{E}$.

By Proposition 2.2, we know that $F(V)$ is endo-trivial for $V \in \hat{E}$. Hence, there is a function $n : \hat{E} \to \mathbb{Z}$ and a function $\sigma : \hat{E} \to \hat{E}$ such that $F(V)$ is isomorphic to $\Omega^{n(V)}\sigma(V)$ in $\mathbf{stmod(kG)}$ for $V \in \hat{E}$.

When P is cyclic, we have $\hat{E} = \{\Omega^{2i}k\}_{0 \le i \le |E|-1}$ and $\Omega^{2|E|}V \simeq V$ for $V \in \hat{E}$. Hence in this case we assume that n takes values in $\{0, 1\}$. Note that we are already done when P has order 2. So, we assume P has order greater than 2.

Suppose that there are elements $V, V' \in \hat{E}$ with $\sigma(V) = \sigma(V')$. Then, $V' \simeq \Omega^{n(V')-n(V)}_{kG}V$, hence, restricting to P, we get $k \simeq \Omega^{n(V')-n(V)}_{kP}k$. Assume that $V \neq V'$, so that $n(V) \neq n(V')$. If P is not cyclic, then it has non-periodic cohomology, hence this is impossible. If P is cyclic, then $\Omega_{kP}k = k$, which again is not possible, because P does not have order 2. Consequently σ is a permutation as asserted.

Assume that P is cyclic Let V be an element of $\hat{E}$ such that $n(V) = 0$. Let $V' = \sigma^{-1}(\Omega^{-2}\sigma(V))$. We have that

$$\overline{\mathrm{Hom}}(V', V) \simeq \overline{\mathrm{Hom}}(\Omega^{n(V')}\sigma(V'), \sigma(V)) \simeq$$

$$\overline{\mathrm{Hom}}(\Omega^{n(V')-2}\sigma(V), \sigma(V)) \simeq \hat{H}^{n(V')-2}(G, k).$$

Since V' and V are distinct simple modules, we have that $\overline{\mathrm{Hom}}(V', V) = 0$. Now, $\hat{H}^{-1}(G, k) \neq 0$, and hence $n(V') = 0$. Thus (when P is a cyclic group) n is constant and we are done.

From now, we assume that P is not cyclic. Let $W \in \hat{E}$ with $n(W)$ minimal. Let $\tau : \hat{E} \to \hat{E}$ defined by $\tau(V) = \sigma(VW)\sigma(W)^{-1}$. Then τ is bijective. Let $h : \hat{E} \to \mathbb{Z}[[t, t,^{-1}]]$ given by

$$h(V) = \sum_{n \in \mathbb{Z}} \dim \mathrm{Ext}^n(k, V)t^n.$$

We have that

$$\operatorname{Ext}^i(k, VW^{-1}) \simeq \operatorname{Ext}^i(W,V)$$
$$\simeq \operatorname{Ext}^i(F(W),F(V)) \simeq \operatorname{Ext}^{i+n(W)-n(V)}(k, \sigma(V)\sigma(W)^{-1}).$$

Hence

$$h(VW^{-1}) = t^{n(V)-n(W)} h(\sigma(V)\sigma(W)^{-1}) \text{ for } V \in \hat{E},$$

and

$$h(V) = t^{n(VW)-n(W)} h(\tau(V)) \text{ for } V \in \hat{E}.$$

Let $V \in \hat{E}$. There exists an integer r such that $\tau^r(V) = V$. Then,

$$h(V) = t^{(n(VW)-n(W))+(n(\tau(V)W)-n(W))+\cdots(n(\tau^{r-1}(V)W)-n(W))} h(V).$$

Now $h(V)$ is not periodic, since P is not cyclic [7]. Hence, we have $(n(VW) - n(W)) + (n(\tau(V)W) - n(W)) + \cdots (n(\tau^{r-1}(V)W) - n(W)) = 0$. Because $n(W)$ is minimal, it follows that $n(\tau^i(V)W) - n(W) \geq 0$ for all i, hence $n(VW) = n(W)$. So, n is constant. □

For P cyclic, this lemma is due to Linckelmann [15, (proof of) Proposition 5.1].

Let $\mathcal{O}$ be a commutative ring and let A and B be two blocks of finite groups over $\mathcal{O}$. Let C be a bounded complex of (A,B)-bimodules, each of which is projective as a left A-module and as a right B-module. Assume

$$C \otimes_B C^* \simeq A \oplus X \text{ as complexes of } (A,A)-\text{bimodules and}$$
$$C^* \otimes_A C \simeq B \oplus Y \text{ as complexes of } (B,B)-\text{bimodules.}$$

Then, we say that

- C induces a *Rickard equivalence* (between A and B) if X and Y are homotopy equivalent to 0.
- C induces a *stable equivalence of Morita type* if C is concentrated in degree 0 and if X and Y are projective.
- C induces a *Morita equivalence* if C is concentrated in degree 0 and if X and Y are zero.

We denote by $\operatorname{Pic}(A)$ the Picard group of A, *i.e.*, the group of isomorphism classes of (A,A)-bimodules M inducing self Morita equivalences of A, where the product of the classes of M and N is the class of $M \otimes_A N$.

Similarly, we denote by $\operatorname{StPic}(A)$ the group of isomorphism classes in the stable category of (A,A)-bimodules of objects inducing self stable equivalences of A, where the product of the classes of M and N is the class of $M \otimes_A N$.

If M is a bimodule inducing a self Morita equivalence of A, then it induces a self stable equivalence of A. This gives rise to a morphism $\mathrm{Pic}(A) \to \mathrm{StPic}(A)$. If M is isomorphic to A in the stable category of (A, A)-bimodules, then M is isomorphic (as an (A, A)-bimodule) to the direct sum of A with a projective (A, A)-bimodule. Since M is indecomposable, this shows that M and A are isomorphic as (A, A)-bimodules. So, the canonical map $\mathrm{Pic}(A) \to \mathrm{StPic}(A)$ is injective. In what follows we identify the image of $\mathrm{Pic}(A))$ with its image in $\mathrm{StPic}(A)$.

Finally, we denote by $\mathrm{TrPic}(A)$ the group of isomorphism classes in the homotopy category of (A, A)-bicomplexes of objects which induce self Rickard equivalences of A, where the product of the classes of M and N is the class of $M \otimes_A N$. Recall that a Rickard equivalence from A to B induces an equivalence of the derived categories of A and B. See [23] for general properties about StPic and TrPic.

Assume C induces a Rickard equivalence between A and B. In the derived category of (A, B)-bimodules, C is isomorphic to a bounded complex of bimodules which are all projective, except the degree n term M, for some n. Then, $\Omega^n_{A\otimes B^\circ} M$ induces a stable equivalence between A and B [19, proof of Corollary 5.5] and the isomorphism class of $\Omega^n_{A\otimes B^\circ} M$ in the stable category of (A, B)-bimodules depends only on C. In particular, we have defined a morphism

$$\rho : \mathrm{TrPic}(A) \to \mathrm{StPic}(A).$$

Our main result in this section is that every self equivalence of the stable category of kG-modules is, up to Heller translation, induced by a self Morita equivalence of kG.

Theorem 3.2. *Suppose that $G = P \rtimes E$ as above. Then*

$$\mathrm{StPic}(kG) = \mathrm{Pic}(kG) \cdot S(kG),$$

where $S(kG)$ is the cyclic subgroup generated by $\Omega_{k(G\times G^\circ)} kG$. (Here G° is the opposite group of G.)

Proof. Let M be a (kG, kG)-bimodule in $\mathrm{StPic}(kG)$. By Lemma 3.1, there is an integer n such that $M' = \Omega^n_{kG\otimes kG^\circ} M$ has the property that $M' \otimes_{kG} V$ is the direct sum of a simple module and a projective module, for every $V \in \hat{E}$. By [14, Theorem 2.1], M' has an indecomposable direct summand M'' such that M'' is in $\mathrm{StPic}(kG)$ and then $M'' \otimes_{kG} V$ is indecomposable, for every $V \in \hat{E}$. So, $M'' \otimes_{kG} V$ is actually simple, for V simple. Then it is a consequence of [14, Theorem 2.1] that M'' actually induces a Morita equivalence, *i.e.*, M'' is in $\mathrm{Pic}(kG)$. □

Note that the group $S(kG)$ is finite precisely when P is cyclic. Furthermore, $S(kG) \cap \mathrm{Pic}(kG) \neq 1$ implies that P is cyclic.

Corollary 3.3. *For G as in the theorem, the canonical map $\rho : \mathrm{TrPic}(kG) \to \mathrm{StPic}(kG)$ is surjective. Moreover if $\mathcal{O}$ be a commutative complete local ring with residue field k, then,*

$$\mathrm{StPic}(\mathcal{O}G) = \mathrm{Pic}(\mathcal{O}G) \cdot S(\mathcal{O}G)$$

and the canonical map $\rho : \mathrm{TrPic}(\mathcal{O}G) \to \mathrm{StPic}(\mathcal{O}G)$ is surjective.

Proof. The image of the class of the complex $kG[1]$, consisting of the one nonzero term kG concentrated in degree 1, in $\mathrm{StPic}(kG)$ is the class of $\Omega_{k(G\times G^\circ)}kG$. By Theorem 3.2, this gives the required surjectivity.

Let M be an indecomposable $(\mathcal{O}G, \mathcal{O}G)$-bimodule in $\mathrm{StPic}(\mathcal{O}G)$ which becomes trivial in $\mathrm{StPic}(kG)$. Then, $M\otimes k \simeq kG \oplus L$ where L is a projective (kG, kG)-bimodule. Hence, there is a projective $(\mathcal{O}G, \mathcal{O}G)$-bimodule L' such that $L' \otimes k \simeq L$ and L' is a direct summand of M. Since M is indecomposable, we get $L = 0$. So, $M\otimes k \simeq kG$, hence $(M\otimes_{\mathcal{O}G} M^*)\otimes k \simeq kG$. Since $M\otimes_{\mathcal{O}G} M^* \simeq \mathcal{O}G \oplus X$ for some projective $(\mathcal{O}G, \mathcal{O}G)$-bimodule X, we get $X = 0$. This shows that M is in $\mathrm{Pic}(\mathcal{O}G)$.

So, the kernel of the canonical map $\mathrm{StPic}(\mathcal{O}G) \to \mathrm{StPic}(kG)$ is contained in $\mathrm{Pic}(\mathcal{O}G)$, hence the results over $\mathcal{O}$ follow from those over k. □

4 Splendid Rickard equivalences

Let $\mathcal{O}$ be a complete discrete valuation ring with residue field k. For V a complex of $\mathcal{O}$-modules, we put $kV = k \otimes_{\mathcal{O}} V$. In this section we demonstrate some conditions under which the existence of a Rickard equivalence between blocks implies also the existence of a splendid Rickard equivalence. Moreover under the proper circumstances the existence of a stable equivalence of the blocks with $\mathcal{O}$ coefficients implies an equivalence with k coefficients and vice versa.

Let G be a finite group, P an abelian p-subgroup of G and let H be a finite group having P as a normal subgroup. Let $\Delta P = \{(x, x^{-1}) | x \in P\} \leq G \times H^\circ$.

Lemma 4.1. *Let M be a $k(G \times H^\circ)$-module with vertex ΔP and source V where V is an H-stable endo-permutation module. Then, there is a complex $\tilde{C}$ of $\mathcal{O}(G\times H^\circ)$-modules with trivial sources and vertices contained in ΔP, such that $k\tilde{C}$ has homology only in degree 0, $H^0(\tilde{C}) \otimes k \simeq H^0(k\tilde{C}) \simeq M$ and $\tilde{C} \otimes_{\mathcal{O}H} \tilde{C}^*$ is homotopy equivalent to $H^0(\tilde{C}) \otimes_{\mathcal{O}H} H^0(\tilde{C})^*$.*

Proof. By [20, Theorem 7.2], there is a bounded complex X of p-permutation kP-modules having the following properties. First X has no terms in positive degrees, and its homology is nonzero only in degree 0, where it is isomorphic to V. In addition, $\mathrm{Res}^{P\times P^\circ}_{\Delta P}(X \otimes X^*)$ is homotopy equivalent to its 0-homology. Since Rickard's construction gives a complex that is unique up to homotopy equivalence (cf [9, Theorem 12.5, (12.8)]), we can assume that X is H-stable by removing all of its indecomposable direct summands that are homotopy equivalent to 0.

Let $X' = \mathrm{Ind}_{\Delta P}^{G\times H^\circ} X$ and let Y be the restricted complex

$$Y = \mathrm{Res}^{(G\times H^\circ)\times(G\times H^\circ)^\circ}_{((G\times 1)\times(G\times 1)^\circ)\Delta' H} X' \otimes X'^*$$

where $\Delta' H = \{(1,h^{-1})\times(1,h) | h\in H)\}$. Then by the Mackey Theorem

$$Y \simeq \bigoplus_{h\in H/P} z^{-1}\otimes \mathrm{Ind}^{((G\times 1)\times(G\times 1)^\circ)(\Delta' H)^z}_{\Delta^h P} \mathrm{Res}^{\Delta P\times(\Delta P)^\circ}_{\Delta^h P} X\otimes X^*$$

where $z = (1,h^{-1})\times(1,1)\in(G\times H^\circ)\times(G\times H^\circ)$ and

$$\Delta^h P = \{(x,x)\times((x^h)^{-1},(x^h)^{-1}) | x\in P\}.$$

Because X is H-stable,

$$\mathrm{Res}^{\Delta P\times(\Delta P)^\circ}_{\Delta^h P} X\otimes X^* \simeq \mathrm{Res}^{\Delta P\times(\Delta P)^\circ}_{\Delta^1 P} X\otimes X^*$$

is homotopy equivalent to its 0-homology. Note that we regard these modules as kP-modules through the isomorphisms $P\to\Delta^h P$ given by $x\mapsto(x,x)\times((x^h)^{-1},(x^h)^{-1})$. Hence, Y is homotopy equivalent to its 0-homology, and $X'\otimes_{kH} X'^*$ is homotopy equivalent to its 0-homology.

Now by [20, Lemma 7.5], there is a direct summand C of X' whose degree 0 homology is isomorphic to M. Then C is a complex of $k(G\times H^\circ)$-modules with trivial sources and vertices contained in ΔP such that $C\otimes_{kH} C^*$ is homotopy equivalent to $M\otimes_{kH} M^*$. By [20, Lemma 5.1], we can lift C to a complex $\tilde{C}$ of $\mathcal{O}(G\times H^\circ)$-modules with trivial sources and vertices contained in ΔP such that $\tilde{C}$ has no terms in positive degrees and C is isomorphic to $k\tilde{C}$.

Note that $\tilde{C}\otimes_{\mathcal{O}H}\tilde{C}^*$ is a complex of $\mathcal{O}(G\times G^\circ)$-modules with trivial sources which becomes isomorphic to $C\otimes_{kH} C^*$ after tensoring by k. This last complex is isomorphic to the direct sum of $M\otimes_{kH} M^*$ with a complex homotopy equivalent to zero. The uniqueness of liftings [20, Lemma 5.1] shows that $\tilde{C}\otimes_{\mathcal{O}H}\tilde{C}^*$ is homotopy equivalent to $H^0(\tilde{C})\otimes_{\mathcal{O}H} H^0(\tilde{C})^*$. □

A bimodule induces a stable equivalence if it does so over the residue field, as shown by the following well known result.

Lemma 4.2. *Let A and B be two blocks of finite groups over $\mathcal{O}$ and let M an $(A\otimes B^\circ)$-module that is projective both as an A-module and as a B°-module. If kM induces a stable equivalence between kA and kB, then M induces a stable equivalance between A and B.*

Proof. Let $\varepsilon : M^* \otimes_A M \to B$ and $\eta : B \to M \otimes_A M^*$ be the units and counits given by the biadjoint pair $(M^* \otimes_A -, M \otimes_B -)$. Then, $1_k \otimes \varepsilon$ and $1_k \otimes \eta$ are the units and counits given by the biadjoint pair $(kM^* \otimes_{kA} -, kM \otimes_{kB} -)$. So, $(1_k \otimes \varepsilon)(1_k \otimes \eta)$ is an automorphism of kB, hence $\varepsilon\eta$ is an automorphism of B. It follows that there is a (B, B)-bimodule X such that $M^* \otimes_A M \simeq B \oplus X$. We know that X is projective because kX is projective. Similarly, one proves that $M \otimes_B M^* \simeq A \oplus$ projective module, and hence M induces a stable equivalence between A and B. □

Let now A be a block of $\mathcal{O}G$, B a block of $\mathcal{O}H$, both with defect groups P. A bounded complex C of $A \otimes B^\circ$ inducing a Rickard equivalence between A and B is splendid if all its terms have trivial sources and vertices contained in ΔP [20].

Theorem 4.3. *Assume there is a $(kA \otimes (kB)^\circ)$-module M whose vertex is ΔP and whose source is V. Assume that V is an H-stable endo-permutation module and M induces a stable equivalence between A and B. Assume furthermore that the canonical maps $\mathrm{TrPic}(B) \to \mathrm{StPic}(B)$ and $\mathrm{TrPic}(kB) \to \mathrm{StPic}(kB)$ are surjective. Then, the following assertions are equivalent.*

(i) *A and B are splendidly Rickard equivalent.*
(ii) *A and B are Rickard equivalent.*
(iii) *Every stable equivalence of Morita type between A and B lifts to a Rickard equivalence.*
(iv) *kA and kB are splendidly Rickard equivalent.*
(v) *kA and kB are Rickard equivalent.*
(vi) *Every stable equivalence of Morita type between kA and kB lifts to a Rickard equivalence.*
(vii) *The stable equivalence induced by M lifts to a splendid Rickard equivalence.*

Proof. Clearly (i) implies (ii), (ii) implies (v), (iv) implies (v) and (vii) implies (iv).

We apply Lemma 4.1 and take $\tilde{M} = H^0(\tilde{C})$. Then, $M \simeq k\tilde{M}$ and $\tilde{M}$ is projective as an A-module and as a B°-module. By Lemma 4.2, $\tilde{M}$ induces a stable equivalence of Morita type, hence (iii) implies (ii).

Now assume (ii). Let C' be a Rickard complex for A and B and M' an $(A \otimes B^\circ)$-module associated to C', inducing a stable equivalence. Let N be an $(A \otimes B^\circ)$-module inducing a stable equivalence. Then, $M'^* \otimes_A N$ defines an element of $\mathrm{StPic}(B)$, which lifts to a complex Y in $\mathrm{TrPic}(B)$ by the hypothesis. Then, $C' \otimes_B Y$ is a Rickard complex lifting N. So, (ii) implies (iii). Similarly, (v) implies (vi).

Every splendid Rickard equivalence between kA and kB lifts to a splendid Rickard equivalence between A and B by [20, Theorem 5.2], hence (iv) implies (i).

Finally let us assume (vi). Let C' be a Rickard complex lifting M. Replacing M by $\Omega^n M$ if necessary, we may choose C' to be the cone of a map $f : M \to X$, with X a bounded complex of projective $(A \otimes B^\circ)$-modules. Let $\tilde{C}$ be a complex associated to M, as in Lemma 4.1, $C = k\tilde{C}$ and let g be the canonical map $C \to H^0(C) = M$ (this is a quasi-isomorphism). Let D be the cone of $fg : C \to X$. Then, D is quasi-isomorphic to C' and D is a complex of modules with trivial sources and vertices contained in ΔP.

Since $C \otimes_{kB} C^*$ is homotopy equivalent to a complex concentrated in degree 0, it follows that $D \otimes_{kB} C^*$ is homotopy equivalent to a bounded complex all of whose terms are projective except the term in degree 0. Hence, $E = D \otimes_{kB} D^*$ is homotopy equivalent to a bounded complex all of whose terms are projective except for the degree 0 term. Since C' is a Rickard complex and D is quasi-isomorphic to C', it follows that E has homology only in degree 0, isomorphic to kA. As the positive degree terms of E are projective, we have that E is homotopy equivalent to a bounded complex with no terms in positive degrees. Similarly, the negative degree terms of E are injective, hence E is homotopy equivalent to a complex concentrated in degree 0, and with $H^0(E) \simeq kA$. Therefore from [20, Theorem 2.1] we have that D induces a splendid Rickard equivalence between kA and kB. So, (vii) is a consequence of (vi). □

Let e a block idempotent of $\mathcal{O}G$. Let (P, e_P) be a maximal e-subpair and N its normalizer. We assume that P is abelian and that $E = N/C_G(P)$ is a cyclic p'-group acting freely on P. Let $H = P \rtimes E$.

Corollary 4.4. *The following assertion are equivalent.*

(i) $\mathcal{O}Ge$ *and* $\mathcal{O}H$ *are splendidly Rickard equivalent.*
(ii) $\mathcal{O}Ge$ *and* $\mathcal{O}H$ *are Rickard equivalent.*
(iii) Every stable equivalence of Morita type between $\mathcal{O}Ge$ *and* $\mathcal{O}H$ *lifts to a Rickard equivalence.*
(iv) kGe *and* kH *are splendidly Rickard equivalent.*
(v) kGe *and* kH *are Rickard equivalent.*
(vi) Every stable equivalence of Morita type between kGe *and* kH *lifts to a Rickard equivalence.*
(vii) The stable equivalence induced by M *lifts to a splendid Rickard equivalence.*

Proof. A theorem of Puig [16, Remarque 6.8] asserts that there is an E-stable endo-permutation kP-module V with vertex P and a direct summand M of $\mathrm{Ind}_{\Delta P}^{G\times H^\circ} V$ that induces a stable equivalence between kGe and kH. So, the corollary follows from Corollary 3.3 and the Theorem. □

Broué [5, 6.2] has conjectured that if e is a block of a finite group G and if (D, f) is a maximal e-subpair such that D is abelian, then the derived categories of $\mathcal{O}Ge$ and $\mathcal{O}N_G(D, f)f$ are derived equivalent. Rickard [20] further conjectured that they are splendidly equivalent. Thus we have verified that the conjectures are equivalent in this special case.

A few other references should be noted. When P is cyclic, Rickard has proven in [17] that kGe and $k[P \rtimes E]$ are Rickard equivalent. Linckelmann [11] proved also that $\mathcal{O}Ge$ and $\mathcal{O}[P \rtimes E]$ are Rickard equivalent. The Corollary shows directly the equivalence of the two statements and shows further that we have splendid equivalences as proven in [21, 22]. Here, the approach goes back to [18].

Similarly, when P is a Klein four group, Linckelmann [12] has proven that kGe and $k[P \rtimes E]$ are Rickard equivalent. Hence, we have also that $\mathcal{O}Ge$ and $\mathcal{O}[P \rtimes E]$ are Rickard equivalent. The last result was proved also by Linckelmann [13] as a consequence of the determination of the source algebras. The fact that $\mathcal{O}Ge$ and $\mathcal{O}[P \rtimes E]$ are splendidly Rickard equivalent is a result of the second author.

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