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PICARD GROUPS FOR DERIVED MODULE CATEGORIES

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1. *Introduction*

Let k be a commutative ring and A a k-algebra. A bounded complex X of (A, A)-bimodules is *invertible* if there is a bounded complex Y of (A, A)-bimodules such that

$$X \otimes_A^{\mathbf{L}} Y \simeq A \quad \text{in the derived category of } (A, A)\text{-bimodules}$$

and

$$Y \otimes_A^{\mathbf{L}} X \simeq A \quad \text{in the derived category of } (A, A)\text{-bimodules.}$$

We define the group $\mathrm{TrPic}(A)$: its elements are isomorphism classes of invertible complexes in the derived category of (A, A)-bimodules. The product of the classes of X and X' is the class of $X \otimes_A^{\mathbf{L}} X'$. The inverse of the class of X is the class of Y where $X \otimes_A^{\mathbf{L}} Y \simeq A$.

By Rickard's theory [**12**], an equivalence between the derived categories of two k-algebras A and B which are projective as k-modules induces an isomorphism between $\mathrm{TrPic}(A)$ and $\mathrm{TrPic}(B)$. The subgroup of $\mathrm{TrPic}(A)$ given by complexes with homology concentrated in degree 0 is the usual Picard group $\mathrm{Pic}(A)$. As we shall see later, the group $\mathrm{Pic}(A)$ is not an invariant of the derived category.

The paper is organized as follows.

In § 2 we review and prove some results about standard derived equivalences. Flat central base change is dealt with in § 2.4. Then we show that for commutative rings, standard derived equivalences come from Morita equivalences.

In § 3, we study various general properties of TrPic. Some of these are analogs of classical properties of Picard groups such as base change and Fröhlich's localization sequence.

Section 4 is devoted to the study of Brauer tree algebras A with no exceptional vertex. Let n be the number of simple modules of A. We construct a morphism from Artin's braid group on $n+1$ strings to $\mathrm{TrPic}(A)$. When $n = 2$, we show that this morphism is an isomorphism modulo some central subgroup: $\mathrm{TrPic}(A)$ is isomorphic to a central extension of $\mathrm{PSL}_2(\mathbb{Z})$. This applies in particular when A is the group algebra of the symmetric group S_3 over a field of characteristic 3.

The results in this paper were announced by the second author at the ICRA VII conference in August 1994 [**17**] and at the AMS Summer Research Institute 'Cohomology, Representations and Actions on Finite Groups' in July 1996 in Seattle. In [**16**], Yekutieli considered independently the group TrPic, in particular the case of local and commutative algebras, and gave applications to dualizing complexes.

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2. *On Rickard's tilting theory*

2.1. *Notation and terminology*

Let us fix some conventions for the rest of the paper. Let k be a commutative ring and A a k-algebra. By an A-module, we always mean a left A-module. We denote by A° the opposite algebra of A. We denote by A-mod the category of finitely presented A-modules which is an abelian category if A is right coherent.

Let $C = (C^i, d_i)$ be a complex of A-modules where we denote by d_i the differential $C^i \to C^{i+1}$. For n an integer, we denote by $C[n]$ the complex with $C[n]^i = C^{n+i}$ and differential $(-1)^n d$.

We denote by $\mathscr{D}^b(A)$ the full subcategory of the derived category of A-modules consisting of objects with bounded homology. We identify the category of A-modules with the full subcategory of $\mathscr{D}^b(A)$ of complexes concentrated in degree 0. Unless otherwise specified, morphisms are taken in the derived category.

A complex of A-modules is *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projective A-modules. We denote by A-perf the full subcategory of $\mathscr{D}^b(A)$ of perfect complexes.

Given two k-modules M and N, we write $M \otimes N$ for $M \otimes_k N$.

For $C = (C^i, d_i)$ and $D = (D^j, \delta_j)$ two bounded complexes, we denote by $C \otimes D$ the total complex associated with the double tensor complex. This has degree n term $(C \otimes D)^n = \bigoplus_{i+j=n} C^i \otimes D^j$ and the differential in degree n is $\sum_{i+j=n} d_i \otimes 1 + (-1)^i 1 \otimes \delta_j$. Analogously we denote by $\mathscr{H}om(C, D)$ the total complex of the double homomorphism complex. It has degree n term

$$\prod_{i+n=j} \mathrm{Hom}(C^i, D^j)$$

and the differential ∂_n in degree n is

$$\partial_n(f) = \prod_{n+j=i} (d_i \circ f - (-1)^n f \circ \delta_j).$$

Note that given X in $\mathscr{D}^b(A^\circ)$ and Y in $\mathscr{D}^b(A)$, then $X \otimes^{\mathbf{L}}_A Y$ is a complex with bounded homology when X is quasi-isomorphic to a bounded complex of flat A°-modules or Y is quasi-isomorphic to a bounded complex of flat A-modules.

A full triangulated subcategory of a triangulated category is called *épaisse* if it is closed under taking direct summands [**11**, § 1].

The subcategory of a triangulated category *generated* by an object is the smallest épaisse full triangulated subcategory containing that object.

2.2. *Standard derived equivalences*

2.2.1. Let B be a k-algebra.

The following theorem of Rickard gives the essentials of the Morita theory for derived categories [**10**, **12**].

THEOREM 2.1. *The assertions* (i)–(iii) *are equivalent.*

(i) *The bounded categories* $\mathscr{D}^b(A)$ *and* $\mathscr{D}^b(B)$ *are equivalent as triangulated categories.*

(ii) *The categories of perfect complexes* A-perf *and* B-perf *are equivalent as triangulated categories.*

(iii) *There is a perfect complex T of A-modules such that*
  (a) *B is isomorphic to* $\operatorname{End}(T)$*,*
  (b) $\operatorname{Hom}(T, T[i]) = 0$ *for $i \neq 0$,*
  (c) A-perf *is generated by T.*

If A and B are projective over k, then the assertions (i)–(iii) *are equivalent to* (iv).

(iv) *There are a bounded complex X of $(A \otimes B^\circ)$-modules whose restrictions to A and to B° are perfect and a bounded complex Y of $(B \otimes A^\circ)$-modules whose restrictions to B and to A° are perfect such that*

$$X \otimes_B^{\mathbf{L}} Y \simeq A \quad \text{in } \mathscr{D}^b(A \otimes A^\circ) \quad \text{and} \quad Y \otimes_A^{\mathbf{L}} X \simeq B \ \text{ in } \mathscr{D}^b(B \otimes B^\circ).$$

If A and B are right coherent, then the assertions (i)–(iii) *are equivalent to* (v).

(v) *The bounded derived categories of finitely presented modules $\mathscr{D}^b$*(A-mod) *and $\mathscr{D}^b$*(B-mod) *are equivalent as triangulated categories.*

When the assertions (i)–(iii) of the theorem are fulfilled, we say that A and B are *derived equivalent*. A complex T satisfying the conditions in (iii) is called a *tilting complex* for A. Complexes X and Y satisfying the conditions in (iv) are called *two-sided tilting complexes*, inverse to each other. The restriction of X to A is a tilting complex, as well as the restriction of Y to B. Similar statements hold of course restricting X to B° and Y to A°. It is clear from the definition that the functors $Y \otimes_A^{\mathbf{L}} -$ and $X \otimes_B^{\mathbf{L}} -$ are then inverse equivalences between $\mathscr{D}^b(A)$ and $\mathscr{D}^b(B)$, as well as between A-perf and B-perf. Equivalences between $\mathscr{D}^b(A)$ and $\mathscr{D}^b(B)$ of the form $X \otimes_B^{\mathbf{L}} -$ for a complex $X \in \mathscr{D}^b(A \otimes B^\circ)$ are called *standard*.

It is unknown whether every equivalence of derived categories is naturally isomorphic to a standard derived equivalence [**12**, § 3]. It is only known that every equivalence between $\mathscr{D}^b(A)$ and $\mathscr{D}^b(B)$ agrees with a standard equivalence on isomorphism classes of objects when A and B are projective over k [**12**, Corollary 3.5].

2.2.2. For algebras projective over k, two-sided tilting complexes can, up to isomorphism, be chosen to be bounded complexes of modules that are projective as A-modules and projective as B°-modules. This is a consequence of the following lemma (see also [**13**, Lemma 9.2.6]).

LEMMA 2.2. *Assume that A and B are projective over k. Let X be a bounded complex of $(A \otimes B^\circ)$-modules such that the restrictions of X to A and to B° are perfect.*

Then X is isomorphic to a bounded complex of $(A \otimes B^\circ)$-modules all of whose terms are projective as $(A \otimes B^\circ)$-modules except possibly the non-zero term in the smallest degree, which is projective as an A-module and projective as a B°-module.

Proof. Let us start by mentioning that the projectivity assumption of A and B over k ensures that the restriction to A of a projective $(A \otimes B^\circ)$-module is projective, as is the restriction to B°.

Let S be a bounded complex of projective A-modules isomorphic to $\operatorname{Res}_A X$ and let n be an integer such that the terms of S vanish in degrees less than n.

Let Y be a projective resolution of X: this is a right-bounded complex of projective $(A \otimes B^\circ)$-modules isomorphic to X. Let $Z = \tau_{\geqslant n-1} Y$: this is a bounded complex isomorphic to X, with zero terms in degrees less than $n-1$ and with projective terms in degrees greater than $n-1$. We will now show that the degree $n-1$ term of Z is projective as an A-module and as a B°-module.

Since $\mathrm{Res}_A Z$ is isomorphic to the bounded complex of projective A-modules S, there exists a morphism in the category of complexes of A-modules $\alpha\colon S \to \mathrm{Res}_A Z$ which is an isomorphism in $\mathscr{D}^b(A)$. Let D be the cone of α. Then D is an acyclic bounded complex all of whose terms are projective except possibly the non-zero term in the smallest degree. Such a complex is homotopy equivalent to zero. Indeed the largest degree non-zero differential is surjective with image a projective module, and hence splits. So, D is homotopy equivalent to a smaller complex and we continue by induction. This shows that $\mathrm{Res}_A Z$ is a bounded complex of projective A-modules. The same argument shows that $\mathrm{Res}_{B^\circ} Z$ is a bounded complex of projective B°-modules. □

2.2.3. Assume that A and B are projective over k. Let T be a tilting complex for A and $f\colon B \to \mathrm{End}(T)$ be an isomorphism. Then there exists a two-sided tilting complex X for $A \otimes B^\circ$ with the following property: there is an isomorphism between T and the restriction $\mathrm{Res}_A X$ of X to A so that if we denote by $\phi\colon \mathrm{End}(T) \to \mathrm{End}(\mathrm{Res}_A X)$ the induced isomorphism, then ϕf is right multiplication by B. Such a complex X associated with T is unique up to unique isomorphism in $\mathscr{D}^b(A \otimes B^\circ)$ (see [**6**, **12**]).

We denote by ${}_\alpha A_1$ the $(A \otimes A^\circ)$-module equal to A as a right A-module and where the left action of $a \in A$ is given by multiplication by $\alpha(a)$: this is the restriction of the natural structure of an $(A \otimes A^\circ)$-module on A through the morphism $\alpha \otimes 1\colon A \otimes A^\circ \to A \otimes A^\circ$.

Let us consider the set of isomorphism classes of two-sided tilting complexes for $A \otimes B^\circ$ whose restriction to B° is in a given isomorphism class. It is acted on simply transitively by $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$ as shown by the following proposition.

PROPOSITION 2.3. *Let X and X' be two-sided tilting complexes for $A \otimes B^\circ$. The restrictions of X and X' to B° are isomorphic if and only if there exists $\alpha \in \mathrm{Aut}(A)$ such that*

$$X' \simeq {}_\alpha A_1 \otimes_A X.$$

Proof. Assume X and X' have isomorphic restrictions to B°. Let Y be a two-sided tilting complex in $\mathscr{D}^b(B \otimes A^\circ)$ inverse to X and let Y' be a two-sided tilting complex in $\mathscr{D}^b(B \otimes A^\circ)$ inverse to X'.

The complexes $X' \otimes_B^{\mathbf{L}} Y$ and $X \otimes_B^{\mathbf{L}} Y \simeq A$ have isomorphic restrictions to A°; hence they have both homologies concentrated in degree 0. As a consequence, $X' \otimes_B^{\mathbf{L}} Y$ is isomorphic to its degree 0 homology $M = H^0(X' \otimes_B^{\mathbf{L}} Y)$ and M is free of rank 1 as an A°-module. The complexes $X' \otimes_B^{\mathbf{L}} Y$ and $X \otimes_B^{\mathbf{L}} Y'$ are two-sided tilting complexes for $A \otimes A^\circ$, inverse to each other. Consequently, $M \otimes_A X \otimes_B^{\mathbf{L}} Y' \simeq A$. It follows that $X \otimes_B^{\mathbf{L}} Y'$ has homology concentrated in degree 0. Let $N = H^0(X \otimes_B^{\mathbf{L}} Y')$. Then $M \otimes_A N \simeq A$. Since M is free of rank 1 as an A°-module, we deduce that N is free of rank 1 as an A°-module. Now, $N \otimes_A M \simeq A$. The bimodule M is invertible and free of rank 1 as an A°-module. By [**3**, Theorem 55.12], there exists an automorphism α of A so that M is isomorphic to ${}_\alpha A_1$ as an $A \otimes A^\circ$-module. The result follows. □

2.2.4. Let us give now three results about compositions, products and sums of two-sided tilting complexes.

Standard equivalences can be composed [**12**, Proposition 4.1].

PROPOSITION 2.4. *If C is a k-algebra, X a two-sided tilting complex for $A \otimes B^\circ$ and X' a two-sided tilting complex for $B \otimes C^\circ$, then $X \otimes_B^{\mathbf{L}} X'$ is a two-sided tilting complex for $A \otimes C^\circ$.*

Tensor products of standard derived equivalences give standard derived equivalences [**12**, Lemma 4.3].

PROPOSITION 2.5. *Let C and D be two k-algebras. Assume A and B are flat over k. Let X be a two-sided tilting complex for $A \otimes B^\circ$ and let X' be a two-sided tilting complex for $C \otimes D^\circ$. Then $X \otimes^{\mathbf{L}} X'$ is a two-sided tilting complex for $(A \otimes C) \otimes (B \otimes D)^\circ$. In particular, $X \otimes^{\mathbf{L}} C$ is a two-sided tilting complex for $(A \otimes C) \otimes (B \otimes C)^\circ$.*

The following result is clear.

PROPOSITION 2.6. *Let $\{A_i\}_{i \in I}$ be a finite family of k-algebras, $A = \prod_{i \in I} A_i$ and let e_i be the central idempotent of A such that $e_i A = A_i$.*

Let T be a complex of A-modules. Then T is a tilting complex for A if and only if $e_i T$ is a tilting complex for A_i, for every $i \in I$.

Let B be a k-algebra and X a two-sided tilting complex for $A \otimes B^\circ$. Let f_i be the central idempotent of $B = \mathrm{End}_A(X)$ given by the multiplication by e_i on X and $B_i = f_i B$. Then $X_i = e_i X \simeq X f_i$ is a two-sided tilting complex for $A_i \otimes (B_i)^\circ$, $X = \bigoplus_i X_i$ and $B = \prod_{i \in I} B_i$.

2.3. *Some invariants of a standard derived equivalence*

Let X be a two-sided tilting complex for $A \otimes B^\circ$.

2.3.1. *Hochschild cohomology* [**12**, Proposition 2.5]. Let i be an integer. Let

$$f_i\colon \mathrm{HH}^i(A) = \mathrm{Hom}_{A \otimes A^\circ}(A, A[i]) \to \mathrm{Hom}(X, X[i])$$

be given by

$$(\phi\colon A \to A[i]) \mapsto (\phi \otimes_A 1_X\colon A \otimes_A X = X \to A[i] \otimes_A X = X[i]).$$

Similarly, we have a map $g_i\colon \mathrm{HH}^i(B) \to \mathrm{Hom}(X, X[i])$ given by

$$(\phi\colon B \to B[i]) \mapsto (1_X \otimes_B \phi\colon X \otimes_B B = X \to X \otimes_B B[i] = X[i]).$$

Then f_i and g_i are isomorphisms and we put

$$\mathrm{HH}^i(X) = f_i^{-1} g_i\colon \mathrm{HH}^i(B) \xrightarrow{\sim} \mathrm{HH}^i(A).$$

In particular, we have an isomorphism

$$\mathrm{HH}^0(X)\colon ZB \xrightarrow{\sim} ZA.$$

2.3.2. *Grothendieck groups* [**5**]. Recall that we denote the category of finitely generated projective A-modules by A-proj. Let $K^0(A)$ be the Grothendieck group of

A-proj. The natural embedding $A\text{-proj} \to K^b(A\text{-perf})$ induces an isomorphism of the Grothendieck groups [**15**]. So, the equivalence $X \otimes_B^{\mathbf{L}} -: K^b(B\text{-perf}) \to K^b(A\text{-perf})$ induces an isomorphism $K^0(B) \xrightarrow{\sim} K^0(A)$.

Let us assume now that A and B are right coherent. Let $G^0(A)$ be the Grothendieck group of A-mod. The embedding $A\text{-mod} \to \mathscr{D}^b(A\text{-mod})$ induces an isomorphism of the Grothendieck groups [**15**]. So, the equivalence

$$X \otimes_B^{\mathbf{L}} -: \mathscr{D}^b(B\text{-mod}) \to \mathscr{D}^b(A\text{-mod})$$

induces an isomorphism $G^0(B) \xrightarrow{\sim} G^0(A)$.

2.4. *Flat central base change*

Proposition 2.5 solves trivially the problem of extending a standard derived equivalence through an extension of A coming from an extension of k. Base change with respect to the centers of A and B is more subtle, since the actions of the centers of A and B on a two-sided tilting complex for $A \otimes B^\circ$ are the same only up to homotopy.

Let $Z = ZA$ and let R be a flat commutative Z-algebra. Let X be a two-sided tilting complex for $A \otimes B^\circ$. We identify ZB with Z through the isomorphism $\mathrm{HH}^0(X)$ (cf. § 2.3.1). Let $A' = A \otimes_Z R$ and $B' = B \otimes_Z R$. Let I be the ideal of $R \otimes R$ generated by the elements $x \otimes 1 - 1 \otimes x$ for $x \in R$. We assume that A, B, A' and B' are flat over k.

THEOREM 2.7. *With the assumptions above, there is a pair (X', f) associated with X unique up to unique isomorphism, where*

(a) *X' is a bounded complex of A'-projective and B'°-projective $(A' \otimes B'^\circ)$-modules such that I is contained in the kernel of the canonical map $R \otimes R \to \mathrm{End}_{A' \otimes B'^\circ}(X')$ and*

(b) *$f: \mathrm{Res}_{A' \otimes B^\circ} X' \xrightarrow{\sim} A' \otimes_A X$ is an isomorphism.*

Given such a pair, we have $\mathrm{Res}_{A \otimes B'^\circ} X' \simeq X \otimes_B B'$ and X' is a two-sided tilting complex for $A' \otimes B'^\circ$.

Let C be a k-algebra, Y be a two-sided tilting complex for $B \otimes C^\circ$, and (Y', g) be a pair associated with Y. Let $U = X \otimes_B^{\mathbf{L}} Y$. We identify Z with $Z(C)$ via $\mathrm{HH}^0(U)$. Assume C and $C \otimes_Z R$ are projective over k and let

$$\begin{aligned} h = (f \otimes_B 1_Y)(1_{X'} \otimes_{B'} g)\colon \mathrm{Res}_{A' \otimes C^\circ}(X' \otimes_{B'}^{\mathbf{L}} Y') &\xrightarrow{\sim} X' \otimes_{B'}^{\mathbf{L}} B' \otimes_B^{\mathbf{L}} Y \\ &\xrightarrow{\sim} A' \otimes_A X \otimes_B^{\mathbf{L}} Y = A' \otimes_A U. \end{aligned}$$

Then $(X' \otimes_{B'}^{\mathbf{L}} Y', h)$ is a pair associated with U.

LEMMA 2.8. *Let M and N be two perfect complexes of A-modules. Then the canonical morphism*

$$\mathrm{Hom}_A(M, N) \otimes_Z R \xrightarrow{\sim} \mathrm{Hom}_{A'}(A' \otimes_A M, A' \otimes_A N)$$

is an isomorphism.

Proof. Let $\mathscr{E}$ be the full subcategory of A-perf of complexes N such that the lemma holds for the pair (M, N), for all perfect complexes M.

Note that $\mathscr{E}$ is stable under translation. Let $N_1 \to N \to N_2 \rightsquigarrow$ be a distinguished triangle in A-perf. We put $M' = A' \otimes_A M$, $N' = A' \otimes_A N$, etc. Thanks to the flatness of R over Z, we have a commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{Hom}_A(M, N_2[-1]) \otimes_Z R & \longrightarrow & \mathrm{Hom}_A(M, N_1) \otimes_Z R & \longrightarrow & \mathrm{Hom}_A(M, N) \otimes_Z R \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{A'}(M', N_2'[-1]) & \longrightarrow & \mathrm{Hom}_{A'}(M', N_1') & \longrightarrow & \mathrm{Hom}_{A'}(M', N') \end{array}$$

$$\begin{array}{ccc} \longrightarrow \mathrm{Hom}_A(M, N_2) \otimes_Z R & \longrightarrow & \mathrm{Hom}_A(M, N_1[1]) \otimes_Z R \\ \downarrow & & \downarrow \\ \longrightarrow \mathrm{Hom}_{A'}(M', N_2') & \longrightarrow & \mathrm{Hom}_{A'}(M', N_1'[1]) \end{array}$$

The five lemma shows that, if the lemma holds for $(M, N_1[i])$ and $(M, N_2[i])$ for any i, it will hold for (M, N). Hence, $\mathscr{E}$ is a triangulated subcategory of A-perf. If N is a direct summand of N' and the lemma holds for (M, N'), it holds for (M, N). Hence, $\mathscr{E}$ is an épaisse full triangulated subcategory of A-perf and in order to prove that it is equal to A-perf, it is enough to prove that it contains A.

The same arguments show that the full subcategory of A-perf of complexes M such that the lemma holds for $\mathrm{Hom}_A(M, A[i])$ for any integer i is an épaisse full triangulated subcategory. So, all we have to do now is to prove the lemma for $M = A$ and $N = A[i]$. When $i \neq 0$, both terms in the lemma are zero, whereas for $i = 0$, the result is clear. □

Proof of the theorem. Let T be the restriction of X to A and let $T_1 = A' \otimes_A T$. Consider the canonical morphism

$$\phi_1 \colon B' = \mathrm{End}_A(T) \otimes_Z R \to \mathrm{End}_{A'}(T_1).$$

By Lemma 2.8, ϕ_1 is an isomorphism. Similarly,

$$\mathrm{Hom}_{A'}(T_1, T_1[i]) \simeq \mathrm{Hom}_A(T, T[i]) \otimes_Z R = 0 \quad \text{for } i \neq 0.$$

Since A is in the subcategory of A-perf generated by T, it follows that A' is in the subcategory of A'-perf generated by T_1. Hence T_1 is a tilting complex for A' and $\phi_1 \colon B' \xrightarrow{\sim} \mathrm{End}_{A'}(T_1)$.

Similarly, the restriction T_2 of $X \otimes_B B'$ to B'° is a tilting complex for B'° and we have a canonical isomorphism $\phi_2 \colon A'^\circ \xrightarrow{\sim} \mathrm{End}_{B'^\circ}(T_2)$.

For $i \in \{1, 2\}$, we denote by X_i a two-sided tilting complex for $A' \otimes B'^\circ$ associated with T_i as in § 2.2.3. It comes with an isomorphism $T_i \xrightarrow{\sim} \mathrm{Res}_{A'} X_i$.

We have a canonical isomorphism of B'°-modules:

$$g_1 \colon \mathrm{Hom}_{A'}(A' \otimes_A X, X_1[i]) \xrightarrow{\sim} \mathrm{Hom}_{A'}(\mathrm{Res}_{A'} X_1, X_1[i]).$$

In particular, $\mathrm{Hom}_{A'}(\mathrm{Res}_{A'}(A' \otimes_A X), X_1[i]) = 0$ for $i \neq 0$. On the other hand, right multiplication by B on $A' \otimes_A X$ and X_1 is compatible with the canonical isomorphism $\mathrm{End}_{A'}(A' \otimes_A X) \xrightarrow{\sim} \mathrm{End}_{A'}(X_1)$. So, g_1 is an isomorphism of $(B \otimes B'^\circ)$-modules and $\mathrm{R}\mathscr{H}om_{A'}(A' \otimes_A X, X_1)$ and $\mathrm{R}\mathscr{H}om_{A'}(\mathrm{Res}_{A' \otimes B^\circ} X_1, X_1)$ are isomorphic in $\mathscr{D}^b(B \otimes B'^\circ)$. Since X_1 is a two-sided tilting complex for $A' \otimes B'^\circ$, we know that g comes from an isomorphism

$$f_1 \colon \mathrm{Res}_{A' \otimes B^\circ} X_1 \xrightarrow{\sim} A' \otimes_A X.$$

Similarly, $X \otimes_B B'$ and $\mathrm{Res}_{A \otimes B'^\circ} X_2$ are isomorphic.

We have isomorphisms in $\mathcal{D}^b(B \otimes B'^\circ)$:

$$\begin{aligned}\operatorname{Res}_{B\otimes B'^\circ} \mathrm{R}\mathscr{H}om_{A'}(X_1, X_2) &\simeq \mathrm{R}\mathscr{H}om_{A'}(\operatorname{Res}_{A'\otimes B^\circ} X_1, X_2)\\ &\simeq \mathrm{R}\mathscr{H}om_{A'}(A' \otimes_A X, X_2)\\ &\simeq \mathrm{R}\mathscr{H}om_{A}(X, \operatorname{Res}_{A\otimes B'^\circ} X_2)\\ &\simeq \mathrm{R}\mathscr{H}om_{A}(X, X\otimes_B B')\\ &\simeq \mathrm{R}\mathscr{H}om_{A}(X, X)\otimes_B B'\\ &\simeq \operatorname{Res}_{B\otimes B'^\circ} B'.\end{aligned}$$

Note that I is contained in the kernel of the canonical map

$$R \otimes R^\circ \to \operatorname{End}_{A'\otimes B'^\circ}(X_i)$$

for $i \in \{1, 2\}$. It follows that I is contained in the kernel of the canonical map $R \otimes R^\circ \to \operatorname{Hom}_{A'}(X_1, X_2)$. Hence, the action of $B' \otimes B'^\circ$ on $\operatorname{Hom}_{A'}(X_1, X_2)$ factors through the canonical surjection

$$\psi\colon B' \otimes B'^\circ \to (B' \otimes B'^\circ)/I(B' \otimes B'^\circ).$$

Similarly, the action of $B' \otimes B'^\circ$ on B' factors through ψ. Now, the restriction of ψ to $B \otimes B'^\circ$ is surjective. Hence, $\mathrm{R}\mathscr{H}om_{A'}(X_1, X_2) \simeq \operatorname{Hom}_{A'}(X_1, X_2)$ and B' are isomorphic in $\mathcal{D}^b(B' \otimes B'^\circ)$. Since X_2 is a two-sided tilting complex, this shows that X_1 and X_2 are isomorphic and that (X_1, f_1) fulfils the requirements of the theorem.

Now let (X', f) be as in the theorem. We have an isomorphism

$$f_1^{-1}f\colon \operatorname{Res}_{A'\otimes B^\circ} X' \xrightarrow{\sim} \operatorname{Res}_{A'\otimes B^\circ} X_1;$$

hence we have isomorphisms in $\mathcal{D}^b(B \otimes B'^\circ)$:

$$\mathrm{R}\mathscr{H}om_{A'}(\operatorname{Res}_{A'\otimes B^\circ} X', X_1) \simeq \mathrm{R}\mathscr{H}om_{A'}(\operatorname{Res}_{A'\otimes B^\circ} X_1, X_1) \simeq \operatorname{Res}_{B\otimes B'^\circ} B'.$$

Since I is contained in the kernel of the canonical maps $R \otimes R^\circ \to \operatorname{End}_{A'\otimes B'^\circ}(X')$ and $R \otimes R^\circ \to \operatorname{End}_{A'\otimes B'^\circ}(X_1)$, we conclude as above that X' and X_1 are isomorphic.

The centralizer in B' of B is the center of B'; hence the canonical map $\operatorname{End}_{B'\otimes B'^\circ}(B') \to \operatorname{End}_{B'\otimes B^\circ}(\operatorname{Res}_{B'\otimes B^\circ} B')$ is an isomorphism. It follows that the canonical map $\operatorname{End}_{A'\otimes B'^\circ}(X_1) \to \operatorname{End}_{A'\otimes B^\circ}(\operatorname{Res}_{A'\otimes B^\circ}(X_1))$ is an isomorphism. Consequently, there is a unique isomorphism $i\colon X' \to X_1$ such that $f = f_1 i$.

The last part of the theorem is clear. □

Note that we have on our way proven the following result.

PROPOSITION 2.9. *Let A be a k-algebra, $Z = ZA$, R be a flat commutative Z-algebra, and $A' = A \otimes_Z R$. Let T be a tilting complex for A. Then $A' \otimes_A T$ is a tilting complex for A' with endomorphism ring* $\operatorname{End}(T) \otimes_Z R$.

2.5. *Degenerate cases*

We will see in this section that, for local or commutative algebras, Rickard's theory gives nothing more than the usual Morita theory.

LEMMA 2.10. *Let A be an indecomposable k-algebra and T a tilting complex for A. If $H^i(T)$ is projective for every $i \in \mathbb{Z}$, then there are a progenerator P for A and an integer n such that $T \simeq P[n]$.*

Let B be a k-algebra and X a two-sided tilting complex for $A \otimes B^\circ$. Assume $H^i(X)$ is a projective A-module for every $i \in \mathbb{Z}$. Then there is an integer n such that $M = H^n(X)$ induces a Morita equivalence between A and B and $X \simeq M[-n]$.

Proof. We have $T \simeq \bigoplus_{i \in \mathbb{Z}} H^i(T)[-i]$. The module M is finitely generated since T is perfect. Since T generates $K^b(A\text{-perf})$, it follows that $\bigoplus_{i \in \mathbb{Z}} H^i(T)$ is a progenerator for A. If T has non-zero homology in more than one degree, the indecomposability of A gives two distinct integers i and j such that $\mathrm{Hom}(H^i(T), H^j(T)) \neq 0$, and hence such that $\mathrm{Hom}(T, T[j-i]) \neq 0$, which is impossible.

Let us come to the second part of the lemma. The assumption implies that $\mathrm{Res}_A X$ is a tilting complex for A with projective homology. Hence, by the first part of the lemma there are an integer n and an $(A \otimes B^\circ)$-module M such that X is isomorphic to $M[-n]$ and the restriction of M to A is a progenerator. Now, the canonical map $B \to \mathrm{End}_A M$ is an isomorphism. Hence M gives a Morita equivalence between A and B. □

The following result is due to Roggenkamp and the second author for A local (cf. [**17**]).

THEOREM 2.11. *Let A be an indecomposable k-algebra which is local or commutative.*

Let T be a tilting complex for A. Then there are a progenerator P for A and an integer n such that $T \simeq P[n]$.

Let B be a k-algebra and X a two-sided tilting complex for $A \otimes B^\circ$. Then there is an integer n such that $M = H^n(X)$ induces a Morita equivalence between A and B and $X \simeq M[-n]$.

In the local case, thanks to Lemma 2.10, the proposition follows from the following lemma. (A letter, dated 21 October 1993, from J. Rickard to the second author contains the main ideas.)

LEMMA 2.12. *Let A be a local ring. Let X be a bounded complex of projective A-modules such that $\mathrm{Hom}_{\mathcal{D}^b(A)}(X[i], X) = 0$ for $i < 0$. Then X is homotopy equivalent to a projective module translated in some degree.*

Proof. Let us put $X = (X^n, d^n)$. Replacing X by a complex which is homotopy equivalent to X, we may and will assume that the largest n such that $X^n \neq 0$ satisfies $H^n(X) \neq 0$, and that if m is minimal such that $X^m \neq 0$, then d^m is not a split injection.

Let $\psi_1\colon X^m \to A$ be a surjection and $\psi_2\colon A \to X^n$ a split injection with splittings ζ_1 and ζ_2 such that $d^m \zeta_1$ is not split injective and $\zeta_2 d^{n-1}$ is not surjective. We note that the existence of ψ_1 and ψ_2 follows from the fact that the indecomposable projective A-modules are free of rank 1 [**1**, Chapitre II, § 3, Exercise 3]. Let $f = \psi_2 \psi_1$ be the composition

$$f\colon X^m \xrightarrow{\psi_1} A \xrightarrow{\psi_2} X^n.$$

Let $g_m\colon X \to X^m[-m]$ be the morphism of complexes which is the identity in degree m and 0 in the other degrees and $g'_n\colon X^n[-n] \to X$ the morphism of complexes which is the identity in degree n and 0 in the other degrees. Let g be the composition

$$g\colon X[m-n] \xrightarrow{g_m[m-n]} X^m[-n] \xrightarrow{f[-n]} X^n[-n] \xrightarrow{g'_n} X.$$

Assume g is homotopy equivalent to zero. Then there are morphisms $h \in \mathrm{Hom}_A(X^m, X^{n-1})$ and $h' \in \mathrm{Hom}_A(X^{m+1}, X^n)$ such that

$$f = d^{n-1}h + h'd^m.$$

Therefore

$$1_A = \zeta_2 d^{n-1} h \zeta_1 + \zeta_2 h' d^m \zeta_1.$$

Since $\zeta_2 d^{n-1}$ is not surjective, $\zeta_2 d^{n-1} h \zeta_1$ is not invertible, and hence lies in the radical of A. Similarly, $d^m \zeta_1$ is not split injective; hence $\zeta_2 h' d^m \zeta_1$ is in the radical of A. So, 1_A is in the radical of A. This is impossible.

It follows that g is not homotopy equivalent to zero and consequently $\mathrm{Hom}_{\mathscr{D}^b(A)}(X[m-n], X) \neq 0$. This shows that $m = n$. □

Proof of the theorem. We assume now that A is commutative. Let $\mathfrak{m}$ be a maximal ideal of A. We denote by $A_{\mathfrak{m}}$ the localization of A at $\mathfrak{m}$. By Proposition 2.9, $T_{\mathfrak{m}} = A_{\mathfrak{m}} \otimes_A T$ is a tilting complex for $A_{\mathfrak{m}}$. Since $A_{\mathfrak{m}}$ is local, it follows that $H^i(T_{\mathfrak{m}})$ is finitely generated and projective for all integers i. Since $H^i(T) \otimes_A A_{\mathfrak{m}}$ is finitely generated and projective for all maximal ideals $\mathfrak{m}$ of A, we conclude that $H^i(T)$ is a finitely generated projective A-module [**1**, Chapitre I, § 3, Proposition 12] for every i and the result follows from Lemma 2.10. □

Together with Proposition 2.6, Theorem 2.11 has the following consequence.

COROLLARY 2.13. *Let A be a local k-algebra or a commutative k-algebra, and B be a k-algebra derived equivalent to A. Then A and B are Morita equivalent.*

REMARK 1. There are nevertheless non-trivial equivalences of derived categories in commutative algebra and algebraic geometry involving non-affine varieties or some extra structures. An example is the Koszul duality between the exterior algebra $\Lambda(V)$ of a vector space V and the symmetric algebra $S(V^*)$ of the dual vector space V^*, where there is an equivalence between the derived categories of bounded complexes of finitely generated graded modules. There are also derived equivalences of Mukai type involving, in particular, Calabi–Yau varieties.

2.6. *Stable equivalences*

We call a k-algebra A a *Gorenstein* k-algebra if $A^* = \mathrm{Hom}_k(A, k)$ is a projective A-module. In this section, we assume that A is a finitely generated projective k-module and a Gorenstein k-algebra. All modules will be assumed to be finitely generated. Let B be a Gorenstein k-algebra finitely generated projective as a k-module.

The following proposition and corollary show that for such algebras, Rickard theory can be made more precise: equivalences of the derived categories can be lifted to equivalences of homotopy categories, which induce stable equivalences [**12**, Corollary 5.5; **7**].

PROPOSITION 2.14. *Let X be a two-sided tilting complex for $A \otimes B^\circ$ and Y an inverse to X.*

Then there exist a bounded complex C of $(A \otimes B^\circ)$-modules and a bounded complex D of $(B \otimes A^\circ)$-modules with the following properties:

(1) *X and C are isomorphic;*

(2) *Y and D are isomorphic;*

(3) *there is an integer n such that*

  (a) *$C^{-i} = 0$ and $D^i = 0$ for $i > n$,*

  (b) *C^{-i} and D^i are projective for $i < n$,*

  (c) *C^{-n} is projective as an A-module and projective as a B°-module,*

  (d) *D^n is projective as a B-module and projective as an A°-module;*

(4) *$C \otimes_B D$ is homotopy equivalent to A as a complex of (A, A)-bimodules;*

(5) *$D \otimes_A C$ is homotopy equivalent to B as a complex of (B, B)-bimodules.*

Proof. By Lemma 2.2, there are a bounded complex C of $(A \otimes B^\circ)$-modules and an integer n such that C is isomorphic to X, $C^i = 0$ for $i < -n$, C^i is projective for $i > -n$, and C^{-n} is projective as an A-module and projective as a B°-module.

Let $D = \mathscr{H}om_A(C, A)$: this is a bounded complex of $(B \otimes A^\circ)$-modules isomorphic to Y and $D^i = 0$ for $i > n$, D^i is projective for $i < n$, and D^n is projective as a B-module and projective as an A°-module.

All terms of the bounded complex $C \otimes_B D$ are projective, except the degree 0 term, which is projective over k. Since $C \otimes_B D$ has homology only in degree 0, it is homotopy equivalent to a bounded complex Z with no terms in positive degrees, whose terms in negative degrees are projective and whose degree 0 term is k-projective. Since Z has homology only in degree 0 and this homology module $H^0(Z) \simeq A$ is projective over k, the restriction to k of Z is homotopy equivalent to $H^0(Z)$. Since $A \otimes A^\circ$ is Gorenstein, an injection of a projective module inside a module splits if it splits when restricted to k. We deduce that Z is homotopy equivalent to $H^0(Z)$ as a complex of $(A \otimes A^\circ)$-modules.

Similarly, $D \otimes_A C$ is homotopy equivalent to B. □

Let M be an $(A \otimes B^\circ)$-module, projective as an A-module and as a B°-module. Let N be a $(B \otimes A^\circ)$-module, projective as a B-module and as an A°-module. We say that M induces a *stable equivalence* between A and B with inverse N if

$$M \otimes_B N \oplus \text{projective module} \simeq A \oplus \text{projective module} \quad \text{as } (A, A)\text{-bimodules}$$

and

$$N \otimes_A M \oplus \text{projective module} \simeq B \oplus \text{projective module} \quad \text{as } (B, B)\text{-bimodules}.$$

Let $\Omega_{A \otimes A^\circ} A$ be the kernel of the multiplication map $A \otimes A^\circ \to A$. This module $\Omega_{A \otimes A^\circ} A$ induces a stable equivalence of A. We denote by $\Omega^{-1}_{A \otimes A^\circ} A$ an indecomposable $(A \otimes A^\circ)$-module which is an inverse of $\Omega_{A \otimes A^\circ} A$.

For V an A-module, $\varepsilon = \pm 1$ and n a non-negative integer, we put

$$\Omega_A^{\varepsilon n} V = (\Omega^{\varepsilon}_{A \otimes A^\circ} A)^{\otimes_A n} \otimes_A V.$$

COROLLARY 2.15. *Let X be a two-sided tilting complex for $A \otimes B^\circ$ and Y an inverse to X. Let C and D be as in Proposition 2.14, $M = \Omega^{-n}_{A \otimes B^\circ} C^{-n}$ and*

$N = \Omega^n_{B\otimes A^\circ} D^n$. Then M and N induce inverse stable equivalences between A and B. Furthermore, up to projective direct summands, the isomorphism classes of M and N are independent of the choice of C and D.

Proof. The complex $C \otimes_B D$ is a bounded complex all of whose terms are projective, except the degree 0 term which is isomorphic to

$$C^{-n} \otimes_B D^n \oplus \text{projective module}.$$

By Proposition 2.14(4), we have

$$C^{-n} \otimes_B D^n \simeq A \oplus \text{projective module}.$$

Since

$$(\Omega^{-n}_{A\otimes B^\circ} C^{-n}) \otimes_B (\Omega^n_{B\otimes A^\circ} D^n) \oplus \text{projective module} \simeq C^{-n} \otimes_B D^n \oplus \text{projective module},$$

it follows that $M \otimes_B N \oplus$ projective module $\simeq A \oplus$ projective module. Similarly, $N \otimes_A M \oplus$ projective module $\simeq B \oplus$ projective module. So, M and N induce inverse stable equivalences between A and B.

Assume C_1 and C_2 are two complexes with the properties of C in Proposition 2.14: they are quasi-isomorphic, $C_1^{-i} = 0$ for $i > m$, $C_2^{-i} = 0$ for $i > n$, C_1^{-i} is projective for $i < m$ and C_2^{-i} is projective for $i < n$. Assume $n \geqslant m$. Then $C_2^{-n}[n]$ is isomorphic to the cone of a morphism $E \to C_2$ where E is a bounded complex of finitely generated projective modules. So, $C_2^{-n}[n]$ is isomorphic to the cone of a morphism $E \to C_1$. Since $A \otimes B^\circ$ is Gorenstein, this morphism, *a priori* in the derived category, comes from a genuine morphism of complexes f. The cone of f is homotopy equivalent to a bounded complex with $C_1^{-m} \oplus$ projective module in degree $-m$ and zero or projective terms elsewhere. It follows that

$$C_2^{-n} \oplus \text{projective module} \simeq \Omega^{-m+n}_{A\otimes B^\circ} C_1^{-m} \oplus \text{projective module}$$

and the unicity statement is proved. □

3. *Picard groups*

3.1. *Definitions*

Definition 3.1. We denote by $\mathrm{TrPic}(A)$ the group of isomorphism classes of two-sided tilting complexes for $A \otimes A^\circ$ where the product of the classes of X and Y is given by the class of $X \otimes_A Y$.

That this is indeed a group follows from Proposition 2.4.

Note that a standard derived equivalence between two algebras A and B induces an isomorphism between $\mathrm{TrPic}(A)$ and $\mathrm{TrPic}(B)$.

By § 2.3, we have canonical morphisms

$$\mathrm{TrPic}(A) \longrightarrow \mathrm{Aut}\, ZA,$$
$$\mathrm{TrPic}(A) \longrightarrow \mathrm{Aut}\, G_0(A) \quad \text{(if } A \text{ is right coherent)},$$
$$\mathrm{TrPic}(A) \longrightarrow \mathrm{Aut}\, K_0(A).$$

The usual Picard group $\mathrm{Pic}(A)$ is the group of isomorphism classes of invertible

(A, A)-bimodules. Hence, we have a canonical injection

$$\mathrm{Pic}(A) \longrightarrow \mathrm{TrPic}(A).$$

Note that $\mathrm{Pic}(A)$ is not a normal subgroup of $\mathrm{TrPic}(A)$ nor an invariant of $\mathscr{D}^b(A)$. For example, two Brauer tree algebras with the same numerical invariants are standardly derived equivalent [**10**] but they have non-isomorphic Picard groups if the trees have non-isomorphic automorphism groups.

We denote by $\mathrm{TrPic}^0(A)$ the subgroup of $\mathrm{TrPic}(A)$ given by those elements X whose induced automorphism of ZA fixes the idempotents, that is, such that $eX \simeq Xe$ for every idempotent e of ZA. Recall that the k-algebra A is *indecomposable* if $Z(A)$ has no non-trivial idempotent.

Thanks to Proposition 2.6, we have the following lemma.

LEMMA 3.2. *Let $\{A_i\}_{i \in I}$ be a finite family of indecomposable k-algebras, $A = \prod_{i \in I} A_i$ and let e_i be the central idempotent of A such that $e_i A = A_i$. The map $X \mapsto \{e_i X e_i\}_i$ induces an isomorphism*

$$\mathrm{TrPic}^0(A) \xrightarrow{\sim} \prod_i \mathrm{TrPic}(A_i).$$

We denote by $\mathrm{Sh}(A)$ the subgroup of $\mathrm{TrPic}(A)$ generated by $A[1]$. It is clear that $\mathrm{Sh}(A)$ is central in $\mathrm{TrPic}(A)$. The group $\mathrm{Sh}(A)$ is an infinite cyclic group and the direct product $\mathrm{Pic}(A) \times \mathrm{Sh}(A)$ is a subgroup of $\mathrm{TrPic}(A)$.

Theorem 2.11 has the following consequence.

PROPOSITION 3.3. *If A is a matrix algebra over an indecomposable commutative k-algebra or over a local k-algebra, then* $\mathrm{TrPic}(A) = \mathrm{Pic}(A) \times \mathrm{Sh}(A)$.

3.2. *Base change*

In this section, we assume A is flat over k.

Let R be a commutative k-algebra. Then Proposition 2.5 gives a canonical morphism

$$\mathrm{TrPic}(A) \longrightarrow \mathrm{TrPic}(A \otimes R).$$

The next two lemmas help to reduce the study of $\mathrm{TrPic}(A)$ to the case of algebras over fields.

LEMMA 3.4. *Assume k is a local ring with maximal ideal $\mathfrak{m}$. Then the kernel of the canonical map*

$$\mathrm{TrPic}(A) \xrightarrow{\phi} \mathrm{TrPic}(A \otimes k/\mathfrak{m})$$

is contained in $\mathrm{Out}(A)$.

Proof. Let T be a bounded complex of finitely generated projective A-modules such that $T \otimes k/\mathfrak{m}$ is homotopy equivalent to its 0-homology. Then T is homotopy equivalent to its 0-homology: this is a consequence of the following fact. Let f be a morphism between two finitely generated projective A-modules. By Nakayama's lemma f is a surjection if and only if $f \otimes 1_{k/\mathfrak{m}} = 1_{A/\mathfrak{m}A} \otimes_A f$ is a surjection. Similarly, f is a split injection if and only if $f \otimes 1_{k/\mathfrak{m}} = 1_{A/\mathfrak{m}A} \otimes_A f$ is a split injection.

If X is in the kernel of the map ϕ of the lemma, then the restriction of X to A is isomorphic to a projective A-module N and X is in $\mathrm{Pic}(A)$. Since $N \otimes k/\mathfrak{m}$ is a free $A \otimes k/\mathfrak{m}$-module of rank 1, it follows that N is a free A-module of rank 1. Hence, X is actually in $\mathrm{Out}(A)$, by Proposition 2.3. □

For $\mathfrak{m}$ a maximal ideal of k, we put $A_{\mathfrak{m}} = A \otimes k_{\mathfrak{m}}$.

Let H be a set of maximal ideals of k such that, given a maximal ideal $\mathfrak{m}$ of k outside H, the restriction to $A_{\mathfrak{m}}$ of a two-sided tilting complex for $A_{\mathfrak{m}} \otimes A^{\circ}_{\mathfrak{m}}$ has finitely generated and projective homology.

LEMMA 3.5. *Assume A is indecomposable. Then the kernel of the canonical map*

$$\mathrm{TrPic}(A)/\mathrm{Sh}(A) \longrightarrow \prod_{\mathfrak{m} \in H} \mathrm{TrPic}(A_{\mathfrak{m}})/\mathrm{Sh}(A_{\mathfrak{m}}),$$

is contained in $\mathrm{Pic}(A)$.

Proof. Let X be a two-sided tilting complex such that $X \otimes k_{\mathfrak{m}} \simeq A_{\mathfrak{m}}[n_{\mathfrak{m}}]$ as an $(A_{\mathfrak{m}} \otimes A^{\circ}_{\mathfrak{m}})$-module for every $\mathfrak{m} \in H$, where $n_{\mathfrak{m}}$ is an integer. Then $H^i(X) \otimes k_{\mathfrak{m}}$ is a finitely generated projective $A_{\mathfrak{m}}$-module for $\mathfrak{m} \in H$. It is also finitely generated projective for $\mathfrak{m} \notin H$ by assumption. So, X has finitely generated projective homology. Now, Lemma 2.10 says that X is in $\mathrm{Pic}(A) \times \mathrm{Sh}(A)$. □

Let $Z = ZA$ be the centre of A, let R be a flat commutative Z-algebra and assume A and $A \otimes_Z R$ are flat over k. Then Theorem 2.7 gives a canonical morphism

$$\mathrm{TrPic}(A) \longrightarrow \mathrm{TrPic}(A \otimes_Z R).$$

3.3. *A localization sequence*

Assume k is a Dedekind domain with field of fractions K. Recall that an algebra B over a field K is separable if $B \otimes_K L$ is semisimple for any field extension L of K. The k-algebra A is a hereditary order if it is a finitely generated projective k-module, if every left ideal of A is a projective A-module, and if $A \otimes K$ is separable.

When k is a discrete valuation ring, the following result follows from the classification of tilting complexes by S. König and the second author [**8**].

LEMMA 3.6. *Let A be a hereditary order. Then the restriction to A of a two-sided tilting complex for $A \otimes A^{\circ}$ has projective homology. If A is in addition indecomposable, then* $\mathrm{TrPic}(A) = \mathrm{Pic}(A) \times \mathrm{Sh}(A)$.

Proof. Let X be a two-sided tilting complex for $A \otimes A^{\circ}$. As A is hereditary, every indecomposable direct summand of the restriction T of X to A has non-zero homology in at most one degree, and this homology group is a projective A-module or a torsion module. Since $\mathrm{End}_A(T) \simeq A$ is torsion free, it follows that there is no indecomposable direct summand of T whose non-zero homology group is a torsion module. Hence, $H^i(T)$ is projective for every i and the second part of the lemma follows from Lemma 2.10. □

Assume A is an indecomposable k-algebra, finitely generated and projective as a k-module. We assume also that $A \otimes K$ is separable. Let H be the set of maximal ideals $\mathfrak{m}$ of k for which $A_{\mathfrak{m}}$ is not a maximal order. It is known that H is finite [**2**, § 29A].

Let TrPicent(A) be the kernel of the canonical morphism $\mathrm{TrPic}(A) \to \mathrm{Aut}\, ZA$.

The following theorem generalizes Fröhlich's localization sequence for Picard groups [**3**, Theorem 55.25].

THEOREM 3.7. *There is an exact sequence*

$$1 \longrightarrow \mathrm{TrPicent}(ZA) \xrightarrow{f} \mathrm{TrPicent}(A) \xrightarrow{g} \prod_{\mathfrak{m}\in H} \mathrm{TrPicent}(A_{\mathfrak{m}})/\mathrm{Sh}(A_{\mathfrak{m}}).$$

Here, g is the product of the canonical maps

$$\mathrm{TrPicent}(A) \longrightarrow \mathrm{TrPicent}(A_{\mathfrak{m}}) \longrightarrow \mathrm{TrPicent}(A_{\mathfrak{m}})/\mathrm{Sh}(A_{\mathfrak{m}})$$

and $f\colon \mathrm{TrPicent}(ZA) \to \mathrm{TrPicent}(A)$ is given by $X \mapsto \mathrm{Res}_{ZA^\circ}^{ZA\otimes ZA^\circ} X \otimes_{ZA} A$, where A is viewed as a $ZA \otimes (A \otimes A^\circ)$-module by the action $(z \otimes (a_1 \otimes a_2)) \cdot a = z a_1 a a_2$ for $a_1, a \in A$, $a_2 \in A^\circ$ and $z \in ZA$.

Proof. Fröhlich's theorem [**3**, Theorem 55.25] says that the restriction of f to a map $\mathrm{Picent}(ZA) \to \mathrm{Picent}(A)$ is well defined in the sense that $X \otimes_{ZA} A$ is an invertible (A, A)-bimodule when X is an invertible (ZA, ZA)-bimodule. Fröhlich's theorem states moreover that the sequence

$$1 \longrightarrow \mathrm{Picent}(ZA) \xrightarrow{f} \mathrm{Picent}(A) \xrightarrow{g} \prod_{\mathfrak{m}\in H} \mathrm{Picent}(A_{\mathfrak{m}})$$

is exact.

Recall from Proposition 3.3 that

$$\mathrm{TrPicent}(ZA) = \mathrm{Picent}(ZA) \times \mathrm{Sh}(ZA).$$

It follows that f is well defined and injective and that $gf = 0$.

When $\mathfrak{m} \notin H$, the order $A_{\mathfrak{m}}$ is maximal, and hence hereditary [**2**, § 26B]. It follows from Lemmas 3.6 and 3.5 that $\ker g$ is contained in $\mathrm{Picent}(A) \times \mathrm{Sh}(A)$, and hence that $\ker g = \mathrm{im}\, f$. □

REMARK 2. (1) In general, g will not be surjective: for example, if A is indecomposable and commutative but there is a maximal ideal $\mathfrak{m}$ of k such that $k_{\mathfrak{m}} \otimes A$ is not indecomposable, then g is not surjective. Examples for such rings are group rings $\mathbb{Z}G$ for an abelian group G over the integers $\mathbb{Z}$.

(2) We do not know any example of an element in

$$\mathrm{TrPicent}(\mathbb{Z}G) - \mathrm{Picent}(\mathbb{Z}G) \times \mathrm{Sh}(\mathbb{Z}G)$$

for a finite group G.

3.4. *Stable Picard groups*

We assume here that A is a finitely generated Gorenstein k-algebra, projective as a k-module.

We say that an A-module M is projective-free if it has no projective direct summand. Given a finitely generated A-module M, there is a projective-free A-module N, unique up to isomorphism, such that $M \simeq N \oplus$ projective module. We call N the projective-free part of M.

DEFINITION 3.8. We denote by $\mathrm{StPic}(A)$ the group of isomorphism classes of projective-free $(A \otimes A^\circ)$-modules inducing a self-stable equivalence of A. The product of the classes of M and N is the class of the projective-free part of $M \otimes_A N$.

A stable equivalence induced by a bimodule between two algebras A and B gives rise to an isomorphism between $\mathrm{StPic}(A)$ and $\mathrm{StPic}(B)$.

We have a natural inclusion $\mathrm{Pic}(A) \to \mathrm{StPic}(A)$. Corollary 2.15 gives a canonical map $\mathrm{TrPic}(A) \to \mathrm{StPic}(A)$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Pic}(A) & \longrightarrow & \mathrm{TrPic}(A) \\ & \searrow \quad \swarrow & \\ & \mathrm{StPic}(A) & \end{array}$$

The bimodule $\Omega_{A \otimes A^\circ} A$ defines a central element of $\mathrm{StPic}(A)$: this is the image of $A[-1] \in \mathrm{TrPic}(A)$.

Let $\mathrm{Out}_0(A)$ be the subgroup of $\mathrm{Out}(A)$ of those automorphisms fixing the isomorphism classes of non-projective indecomposable modules. The group $\mathrm{Out}_0(A)$ is invariant under stable equivalence [**9**].

4. *Brauer tree algebras*

4.1. *Definition*

Let Γ be a finite connected tree with a cyclic ordering of the edges adjacent to a given vertex and with a particular vertex v, the *exceptional* vertex, and a positive integer m, the *multiplicity* of the exceptional vertex. Let k be a field.

To this data (Γ, v, m) one associates a finite-dimensional symmetric k-algebra, called a *Brauer tree* algebra, characterized up to Morita equivalence by the following properties.

The isomorphism classes of simple modules are parametrized by the edges of Γ. Denote by P_j a projective cover of a simple module S_j corresponding to an edge j. Then $\mathrm{rad}(P_j)/\mathrm{soc}(P_j)$ is the direct sum of two uniserial modules U_a and U_b where a and b are the vertices of j. For $c \in \{a, b\}$, let $j = j_0, j_1, \ldots, j_r$ be the cyclic ordering of the $r+1$ edges around c. Then the composition factors of U_c, starting from the top, are

$$S_{j_1}, S_{j_2}, \ldots, S_{j_r}, S_{j_0}, S_{j_1}, \ldots, S_{j_r}$$

where the number of composition factors is $m(r+1) - 1$ if c is the exceptional vertex, and r otherwise. Note that when $m = 1$, the choice of an exceptional vertex is irrelevant.

Associated to a Brauer tree algebra are two numerical invariants: the number of edges of the tree Γ and the multiplicity of the exceptional vertex.

By [**12**, Theorem 4.2], two Brauer tree algebras with the same numerical invariants are derived equivalent. So, a Brauer tree algebra associated with (Γ, v, m) is derived equivalent to a Brauer tree algebra associated with a line with the same number of edges as Γ and with an exceptional vertex at an end having multiplicity m. Hence, the study of TrPic for a Brauer tree algebra reduces to this last case.

4.2. *Some elements of* TrPic

We now restrict ourselves to the case of Brauer tree algebras where the multiplicity m is 1.

Let A be a basic Brauer tree algebra associated to a line with n edges numbered $1, \dots, n$ such that i is adjacent to $i+1$, and with no exceptional vertex, so that the multiplicity m is 1. We assume $n > 1$.

The Loewy series of the projective indecomposable modules are as follows:

$$P_1 = \begin{matrix} S_1 \\ S_2 \\ S_1 \end{matrix}, \quad P_n = \begin{matrix} S_n \\ S_{n-1} \\ S_n \end{matrix} \quad \text{and} \quad P_i = \begin{matrix} & S_i & \\ S_{i-1} & & S_{i+1} \\ & S_i & \end{matrix} \quad \text{for } i \neq 1, n.$$

The dimensions of the vector spaces of homomorphisms between projective modules is given by

$$\dim_k \operatorname{Hom}_A(P_i, P_j) = \begin{cases} 0 & \text{if } |i-j| > 1, \\ 1 & \text{if } |i-j| = 1, \\ 2 & \text{if } i = j. \end{cases}$$

By [**14**, Lemma 2] a projective cover of the $(A \otimes A^\circ)$-module A is given by

$$\bigoplus_{1 \leqslant i \leqslant n} P_i \otimes P_i^* \xrightarrow{f} A,$$

where P_i^* is the A°-module $\operatorname{Hom}_k(P_i, k)$.

Let

$$X_i = (0 \longrightarrow P_i \otimes P_i^* \xrightarrow{f} A \longrightarrow 0),$$

where A is in degree 0. The isomorphism class of this complex does not depend on the choice of f.

THEOREM 4.1. *The complex X_i is a two-sided tilting complex, and hence defines an element t_i of* $\operatorname{TrPic}(A)$.

Proof. To prove that X_i is a two-sided tilting complex, we follow the method of [**14**, Theorem 6]. We have

$X_i \otimes_A X_i^* =$

$$\left(0 \to P_i \otimes P_i^* \xrightarrow{1 \otimes_A f^* + f} P_i \otimes P_i^* \otimes_A P_i \otimes P_i^* \oplus A \xrightarrow{f \otimes_A 1 - f^*} P_i \otimes P_i^* \to 0\right).$$

The map

$$f \otimes_A 1 \colon \bigoplus_j P_j \otimes P_j^* \otimes_A P_i \otimes P_i^* \to P_i \otimes P_i^*$$

is surjective. Now, the projective A-modules $\bigoplus_{j \neq i} P_j \otimes P_j^* \otimes_A P_i \otimes P_i^*$ and $P_i \otimes P_i^* \otimes_A P_i \otimes P_i^*$ have no common non-zero direct summand. By [**14**, Lemma 1], this implies that the restriction of $f \otimes_A 1$ to $P_i \otimes P_i^* \otimes_A P_i \otimes P_i^*$ remains surjective. Since $P_i \otimes P_i^*$ is projective, the map

$$f \otimes_A 1 - f^* \colon P_i \otimes P_i^* \otimes_A P_i \otimes P_i^* \oplus A \to P_i \otimes P_i^*$$

is a split surjection. By duality, the map

$$1 \otimes_A f^* + f\colon\ P_i \otimes P_i^* \to P_i \otimes P_i^* \otimes_A P_i \otimes P_i^* \oplus A$$

is a split injection. Hence, $X_i \otimes_A X_i^*$ is homotopy equivalent to a module V which satisfies

$$P_i \otimes P_i^* \oplus P_i \otimes P_i^* \oplus V \simeq P_i \otimes P_i^* \otimes_A P_i \otimes P_i^* \oplus A.$$

As $P_i \otimes_A P_i^* \simeq \mathrm{Hom}_A(P_i, P_i)$ has dimension 2, we obtain $V \simeq A$ and finally $X_i \otimes_A X_i^*$ is homotopy equivalent to A. □

The action of the functor

$$F_i := X_i \otimes_A -$$

on simple modules is easily described. One has

$$F_i(S_i) \simeq \Omega(S_i)[1] \quad \text{and} \quad F_i(S_j) \simeq S_j \ \text{ for } j \neq i,$$

where $\Omega(S_i)$ is the kernel of a surjective map $P_i \to S_i$.

Let us now describe the action of F_i on projective modules.

Given i and j with $|i-j| = 1$, we denote by $P_i \to P_j$ a non-zero map from P_i to P_j; such a map is unique up to a scalar. From now on, complexes with zero terms in positive degrees will be written as $C^r \to \ldots \to C^0$, where C^0 is in degree 0.

LEMMA 4.2. *We have*

$$F_i(P_j) \simeq \begin{cases} P_j & \text{if } |i-j| > 1, \\ P_i \to 0 & \text{if } i = j, \\ P_i \to P_j & \text{if } |i-j| = 1. \end{cases}$$

Proof. When $|i-j| > 1$, we have

$$P_i^* \otimes_A P_j \simeq \mathrm{Hom}_A(P_i, P_j) = 0;$$

hence $F_i(P_j) \simeq P_j$.

The morphism

$$f \otimes_A 1\colon \bigoplus_{j \neq i} P_j \otimes P_j^* \otimes_A P_i \to P_i$$

is not surjective. Since P_i is projective indecomposable, the morphism

$$f \otimes_A 1\colon P_i \otimes P_i^* \otimes_A P_i \to P_i$$

is surjective by [**14**, Lemma 1] and therefore $X_i \otimes_A P_i$ has homology concentrated in degree -1. As $P_i^* \otimes_A P_i$ is two-dimensional, we obtain $F_i(P_i) \simeq P_i[1]$.

The last case is clear. □

4.3. *Determination of the automorphism group*

Now, we need to understand the automorphisms of A.

LEMMA 4.3. *Let B be a Brauer tree algebra with n edges and multiplicity $m = 1$. Then $\mathrm{Out}_0(B) \simeq k^\times/\mu_n(k)$, where $\mu_n(k)$ is the group of nth roots of unity of k.*

Proof. As explained in § 4.1, the algebra B is derived equivalent to a basic Brauer tree algebra whose tree is a star. By § 3.4, we are reduced to proving the lemma for such a B. Let B be a basic star Brauer tree algebra with n edges and multiplicity 1.

Let us define an algebra A_l for $l \geqslant 0$ with generators e_i, with $i \in \mathbb{Z}/n\mathbb{Z}$, and t, with relations

$$t^{l+1} = 0, \quad e_i^2 = e_i, \quad e_i e_j = 0 \text{ if } i \neq j, \quad 1 = \sum_i e_i \quad \text{and} \quad te_i = e_{i+1}t.$$

It is well known that $B \simeq A_n$ (cf. [**4**]).

The algebra A_l is serial; hence indecomposable modules are determined (up to isomorphism) by their Loewy series. So $\mathrm{Out}_0(A_l)$ is the subgroup of $\mathrm{Out}(A_l)$ given by the automorphisms fixing the isomorphism classes of simple modules.

For $x \in k^\times$, let $\alpha_l(x)$ be the automorphism of A_l given by $\alpha_l(x)(e_i) = e_i$ and $\alpha_l(x)(t) = xt$. Then $\alpha_l(x)$ gives an element of $\mathrm{Out}_0(A_l)$. Assume $x \in \mu_n(k)$ and let $y = \sum_i x^i e_i \in A_l$. Then $yty^{-1} = xt$ and $ye_iy^{-1} = e_i$ for all i; hence $\alpha_l(x)$ is an inner automorphism.

We will prove by induction on l that for $1 \leqslant l \leqslant n$, the morphism $k^\times \to \mathrm{Out}_0(A_l)$, $x \mapsto \alpha_l(x)$ is surjective and has kernel $\mu_n(k)$.

Note that t is a generator of the Jacobson radical $J(A_l)$ of A_l and that the map sending the generators t, e_i of A_l onto the generators t, e_i of A_{l-1} induces an isomorphism $A_l/J(A_l)^l \xrightarrow{\sim} A_{l-1}$.

An automorphism φ of A_1 inducing the trivial automorphism on $A_0 \simeq A_1/J(A_1)$ has to fix the elements e_i. Indeed, we have $\varphi(e_i) = e_i + t\sum_j \varphi_{ij}e_j$ where $\varphi_{ij} \in k$ for such an automorphism. Since $\varphi(e_i)\varphi(e_j) = 0$ for $i \neq j$, we get $\varphi_{ij} = 0$ if $i \neq j$. As $1 = \varphi(1) = \sum_i \varphi(e_i)$, we also get $\varphi_{ii} = 0$. This implies that $\varphi(e_i) = e_i$.

Let $y = \sum_i a_i e_i + \sum_i b_i e_i t$ be an arbitrary invertible element of A_1 (here, $a_i \in k^\times$ and $b_i \in k$). Then an elementary calculation shows that

$$y^{-1} = \sum_i \frac{1}{a_i} e_i - \sum_i \frac{b_i}{a_{i-1}a_i} e_i t.$$

Hence, $yty^{-1} = \left(\sum_i c_i e_i\right)t$, where $c_i = a_i/a_{i-1}$. Note that $\prod_i c_i = 1$. It follows that $\alpha_1(x)$ is not inner, for $x^n \neq 1$.

Assume the result holds for A_{l-1} with $l \geqslant 2$. Let φ be an automorphism of A_l in $\mathrm{Out}_0(A_l)$. Then φ induces an automorphism of A_{l-1} in $\mathrm{Out}_0(A_{l-1})$. By the induction hypothesis, we may assume that this induced automorphism is trivial, multiplying if necessary by some $\alpha_l(x)$ and by an inner automorphism. Then $\varphi(e_i) = e_i + t^l \sum_j \varphi_{ij} e_j$ for some $\varphi_{i,j} \in k$. As $\varphi(e_i)\varphi(e_j) = 0$ for $i \neq j$, we get $\varphi_{ij} = 0$ if $i \neq j$. Since $1 = \varphi(1) = \sum_i \varphi(e_i)$, we also get $\varphi_{ii} = 0$. This implies that $\varphi(e_i) = e_i$. We now have $\varphi(t) = t + t^l \sum_i \varphi_i e_i$ for some $\varphi_i \in k$. So,

$$\varphi(t)\varphi(e_i) = e_{i+1}t + \varphi_i t^l e_i \quad \text{and} \quad \varphi(e_{i+1})\varphi(t) = e_{i+1}t + t^l \varphi_{i+l} e_{i+1-l}.$$

As $1 < l \leqslant n$, we have $e_{i+1-l} \neq e_i$; hence $\varphi_i = 0$ for $1 \leqslant i \leqslant n$. Therefore, $\varphi(t) = t$ and φ is trivial. Hence, the result is true for A_l.

It follows that $\mathrm{Out}_0(A_n) = \langle \alpha_n(x) \rangle_{x \in k^\times} \simeq k^\times/\mu_n(k)$. □

Up to isomorphism, $\Omega^n_{A \otimes A^\circ}(A)$ has a unique non-zero and non-projective direct summand. We denote it by M. It induces a self-stable equivalence of Morita type. Since it is indecomposable, the module $M \otimes_A V$ is indecomposable for any simple A-module V [**9**, Theorem 2.1]. We have then $M \otimes_A V_i \simeq \Omega^n V_i \simeq V_{n+1-i}$. So, $M \otimes_A -$ sends simple modules to simple modules. It now follows from [**9**, Theorem

2.1] that M induces a self-Morita equivalence. In other words, M is an invertible bimodule and we denote by ω the element (of order 2) of $\mathrm{Out}(A)$ it induces. Note that the image of ω in $\mathrm{StPic}(A)$ is central; hence ω is central in $\mathrm{Out}(A)$.

The image of ω in $\mathrm{Aut}\, G_0(A)$ corresponds to the non-trivial automorphism of the tree of A.

We denote by Γ the subgroup of $\mathrm{TrPic}(A)$ generated by $t_1, \ldots, t_n$ and by G the subgroup of $\mathrm{TrPic}(A)$ generated by $t_1, \ldots, t_n, \omega$ and $[1]$.

PROPOSITION 4.4. *We have*

$$\mathrm{Out}(A) = \langle \omega \rangle \times \mathrm{Out}_0(A).$$

The group $\mathrm{Out}_0(A)$ *centralizes* Γ *and* $\omega t_i \omega = t_{n+1-i}$. *Furthermore,* $\Gamma \cap \mathrm{Out}(A) = 1$ *and* $G \cap \mathrm{Out}_0(A) = 1$.

Proof. Since indecomposable A-modules are determined by their radical series, an automorphism of A which fixes the isomorphism classes of simple modules will fix the isomorphism classes of all modules. Hence, $\mathrm{Out}_0(A)$ is the kernel of the canonical map $\mathrm{Out}(A) \to \mathrm{Aut}\, G_0(A)$. This map actually factors through the group of automorphisms of the tree of A. The group of automorphisms of the tree has order 2 and is generated by the image of ω. It follows that $\mathrm{Out}(A) = \langle \omega \rangle \times \mathrm{Out}_0(A)$. Note that we have followed [**9**, Theorem 4.7].

Let us consider the complex

$$\begin{aligned} &{}_1A_\omega \otimes_A \left(\bigoplus_{1 \leqslant i \leqslant n} P_i \otimes P_i^* \xrightarrow{f} A \right) \otimes_A ({}_\omega A_1) \\ &\qquad \simeq \left(\bigoplus_{1 \leqslant i \leqslant n} ({}_1A_\omega \otimes_A P_i) \otimes ({}_1A_\omega \otimes_A P_i)^* \xrightarrow{g} A \right). \end{aligned}$$

This complex defines again a projective cover of A, and hence is isomorphic, as a complex, to the complex

$$\bigoplus_{1 \leqslant i \leqslant n} P_i \otimes P_i^* \xrightarrow{f} A.$$

As ${}_1A_\omega \otimes_A P_i \simeq P_{n+1-i}$, the complex

$${}_1A_\omega \otimes_A X_i \otimes_A ({}_\omega A_1) \simeq (P_{n+1-i} \otimes P_{n+1-i}^* \xrightarrow{g} A)$$

is isomorphic to X_{n+1-i}; hence, $\omega t_i \omega = t_{n+1-i}$.

Similarly, one proves that $\mathrm{Out}_0(A)$ centralizes each of the elements t_i of Γ.

Let us prove that $\mathrm{Out}(A) \cap \Gamma = 1$. Since the canonical map $\mathrm{Out}(A) \to \mathrm{StPic}(A)$ is injective, we can check this property in $\mathrm{StPic}(A)$. But, the image of t_i in $\mathrm{StPic}(A)$ is trivial, and hence the property holds. Finally, the image of G in $\mathrm{StPic}(A)$ intersects trivially the image of $\mathrm{Out}_0(A)$, and so we conclude that $G \cap \mathrm{Out}_0(A) = 1$. □

REMARK 3. By Propositions 2.3 and 4.4, when doing calculations inside G, it is enough to look at the action on projective indecomposable modules: for $\sigma, \sigma' \in G$, we have $\sigma = \sigma'$ if and only if $\sigma(P) \simeq \sigma'(P)$ for any indecomposable projective module P. Our main tool will then be Lemma 4.2.

4.4. *Braid relations*

Denote by B_{n+1} the Artin braid group on $n+1$ strings, generated by $\sigma_1, \ldots, \sigma_n$ with the relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. We put $w_0 = \sigma_1(\sigma_2\sigma_1) \ldots (\sigma_{n-1} \ldots \sigma_1)(\sigma_n \ldots \sigma_1)$.

THEOREM 4.5. *There is a surjective group morphism*

$$B_{n+1} \to \Gamma, \quad \sigma_i \mapsto t_i.$$

The image of w_0 *is* $\omega[n]$.

Proof. The first braid relation $t_i t_j = t_j t_i$ if $|i-j| > 1$ is immediate, since $P_i^* \otimes_A P_j \simeq \mathrm{Hom}_A(P_i, P_j) = 0$ for $|i-j| > 1$.

By Remark 3, we have only to check that $F_i F_{i+1} F_i(P) \simeq F_{i+1} F_i F_{i+1}(P)$ for every projective indecomposable module P.

Since F_{i+1} preserves cones, the complex $F_{i+1}(P_i \to P_{i-1})$ is isomorphic to the cone of a non-zero morphism $(P_{i+1} \to P_i) \to P_{i-1}$. Therefore this complex is isomorphic to a three-term complex $P_{i+1} \to P_i \to P_{i-1}$. Note that the notation follows the convention above and that such a complex is well defined up to isomorphism.

The complex $F_i(P_{i+1} \to P_i)$ is isomorphic to the cone of a non-zero morphism $(P_i \to P_{i+1}) \to (P_i \to 0)$ and hence is isomorphic to $P_{i+1} \to 0$.

Similarly, $F_{i+1}(P_i \to P_{i+1}) \simeq (P_i \to 0)$.

Now we have done the necessary computations to determine $F_{i+1} F_i(P_j)$ for all j. Two more are necessary to determine $F_i F_{i+1} F_i(P_j)$ for all j.

The complex $F_i(P_{i+1} \to P_i \to P_{i-1})$ is isomorphic to the cone of a non-zero morphism $(P_{i+1} \to 0) \to (P_i \to P_{i-1})$, and hence to $P_{i+1} \to P_i \to P_{i-1}$.

We have $F_i(P_{i+1} \to P_{i+2}) \simeq P_i \to P_{i+1} \to P_{i+2}$.

Summarizing, we have:

$$\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_{i-2} \\ P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \\ P_{i+3} \\ \vdots \\ P_n \end{pmatrix} \xrightarrow{F_i} \begin{pmatrix} & P_1 \\ & P_2 \\ & \vdots \\ & P_{i-2} \\ P_i \to & P_{i-1} \\ P_i \to & 0 \\ P_i \to & P_{i+1} \\ & P_{i+2} \\ & P_{i+3} \\ & \vdots \\ & P_n \end{pmatrix} \xrightarrow{F_{i+1}} \begin{pmatrix} & & & P_1 \\ & & & P_2 \\ & & & \vdots \\ & & & P_{i-2} \\ P_{i+1} \to & P_i & \to & P_{i-1} \\ P_{i+1} \to & P_i & \to & 0 \\ & P_i & \to & 0 \\ & P_{i+1} & \to & P_{i+2} \\ & & & P_{i+3} \\ & & & \vdots \\ & & & P_n \end{pmatrix}$$

$$\xrightarrow{F_i} \begin{pmatrix} & & & & P_1 \\ & & & & P_2 \\ & & & & \vdots \\ & & & & P_{i-2} \\ P_{i+1} & \to & P_i & \to & P_{i-1} \\ P_{i+1} & \to & 0 & \to & 0 \\ P_i & \to & 0 & \to & 0 \\ P_i & \to & P_{i+1} & \to & P_{i+2} \\ & & & & P_{i+3} \\ & & & & \vdots \\ & & & & P_n \end{pmatrix}$$

and

$$\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_{i-2} \\ P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \\ P_{i+3} \\ \vdots \\ P_n \end{pmatrix} \xrightarrow{F_{i+1}} \begin{pmatrix} & P_1 \\ & P_2 \\ & \vdots \\ & P_{i-2} \\ & P_{i-1} \\ P_{i+1} \to & P_i \\ P_{i+1} \to & 0 \\ P_{i+1} \to & P_{i+2} \\ & P_{i+3} \\ & \vdots \\ & P_n \end{pmatrix} \xrightarrow{F_i} \begin{pmatrix} & & P_1 \\ & & P_2 \\ & & \vdots \\ & & P_{i-2} \\ & P_i \to & P_{i-1} \\ & P_{i+1} \to & 0 \\ P_i \to & P_{i+1} \to & 0 \\ P_i \to & P_{i+1} \to & P_{i+2} \\ & & P_{i+3} \\ & & \vdots \\ & & P_n \end{pmatrix}$$

$$\xrightarrow{F_{i+1}} \begin{pmatrix} & & P_1 \\ & & P_2 \\ & & \vdots \\ & & P_{i-2} \\ P_{i+1} \to & P_i \to & P_{i-1} \\ P_{i+1} \to & 0 \to & 0 \\ P_i \to & 0 \to & 0 \\ P_i \to & P_{i+1} \to & P_{i+2} \\ & & P_{i+3} \\ & & \vdots \\ & & P_n \end{pmatrix}.$$

Hence we have indeed $F_i F_{i+1} F_i(P_j) \simeq F_{i+1} F_i F_{i+1}(P_j)$ for all j.

By induction on i, we have

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_i \\ P_{i+1} \\ P_{i+2} \\ \vdots \\ P_n \end{pmatrix} \xrightarrow{F_i F_{i-1} \dots F_1} \begin{pmatrix} P_i \to P_{i-1} \to \dots \to P_2 \to & P_1 & \to 0 \\ & P_1 & \to 0 \\ & P_2 & \to 0 \\ & & \vdots \\ & P_{i-1} & \to 0 \\ & P_i & \to P_{i+1} \\ & & P_{i+2} \\ & & \vdots \\ & & P_n \end{pmatrix}$$

and, in particular,

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \end{pmatrix} \xrightarrow{F_n F_{n-1} \dots F_1} \begin{pmatrix} P_n \to P_{n-1} \to \dots \to P_2 \to & P_1 & \to 0 \\ & P_1 & \to 0 \\ & P_2 & \to 0 \\ & & \vdots \\ & P_{n-1} & \to 0 \end{pmatrix}.$$

Now, by induction on i, we have

$$F_i F_{i-1} \dots F_1(P_r \to P_{r-1} \to \dots \to P_2 \to P_1) \simeq (P_r \to P_{r-1} \to \dots \to P_{i+1})[i]$$

for $i < r$. In particular,

$$F_{r-1} \dots F_1(P_r \to P_{r-1} \to \dots \to P_2 \to P_1) \simeq P_r[r-1].$$

We deduce that

$$F_1(F_2 F_1) \dots (F_n \dots F_1)(P_i) \simeq P_{n-i+1}[n]. \qquad \square$$

In the next section, we will prove that for $n = 2$, the morphism $B_3 \to \Gamma$ is bijective and in § 4.6 that $\mathrm{TrPic}(A)$ is generated by Γ and $\mathrm{Pic}(A) \times \mathrm{Sh}(A)$.

4.5. *Faithfulness of the braid group action*

We assume now that A is a Brauer tree algebra associated to a line with two edges and no exceptional vertex. An example of such an algebra is the group algebra of the symmetric group $\mathfrak{S}_3$ over a field of characteristic 3.

Put $\phi = t_1 \omega$. A tilting complex corresponding to ϕ acts as

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} P_1 \to P_2 \\ P_1 \to 0 \end{pmatrix}.$$

As is shown in Theorem 4.5, we have $\omega = t_1 t_2 t_1[-2]$. Note that $\phi^3 = t_1 t_2 t_1 \omega = [2]$.

THEOREM 4.6. *The map*

$$S = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \phi \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto \omega$$

induces an isomorphism $\chi\colon \mathrm{PSL}_2(\mathbb{Z}) \xrightarrow{\sim} G/\mathrm{Sh}(A)$. *Hence the subgroup* G *of* $\mathrm{TrPic}(A)$ *is isomorphic to a central extension of* $\mathrm{PSL}_2(\mathbb{Z})$.

Since $\mathrm{PSL}_2(\mathbb{Z})$ is generated by S and T with the relations $S^3 = T^2 = 1$, we have indeed a morphism $\mathrm{PSL}_2(\mathbb{Z}) \to G/\mathrm{Sh}(A)$. This morphism is surjective, since G is generated by ϕ, ω and [1].

Note that the morphism $B_3/Z(B_3) \to \mathrm{PSL}_2(\mathbb{Z})$ given by

$$\sigma_1 \mapsto ST \quad \text{and} \quad \sigma_2 \mapsto TS$$

is an isomorphism. As a consequence, we have the following corollary.

COROLLARY 4.7. *The morphism* $B_3 \to \mathrm{TrPic}(A)$ *given by* $\sigma_i \mapsto t_i$ *is injective.*

Let C be a bounded complex of projective modules. Then we have a decomposition $C = C_r \oplus C_a$ in the category of complexes, where C_a is homotopy equivalent to 0 and C_r has no non-zero direct summand which is homotopy equivalent to 0. We call C_r the *reduced part* of C. This is well defined up to isomorphism in the category of complexes.

For X a complex of k-modules, we denote by $\dim X$ the dimension of X, viewed as a k-module by forgetting the differential and the grading.

Let $C \in \mathrm{TrPic}(A)$ and C_i be the reduced part of $C \otimes_A P_i$. For $\{i, j\} = \{1, 2\}$, we denote by $\mathrm{Cone}(C_i \to C_j)$ the reduced part of the cone of a non-zero morphism from C_i to C_j. Since $\mathrm{Hom}_A(C_i, C_j) \simeq \mathrm{Hom}_A(P_i, P_j)$ is one-dimensional, the morphism is well defined up to a scalar in the homotopy category; hence $\mathrm{Cone}(C_i \to C_j)$ is well defined up to isomorphism in the category of complexes.

We deduce Theorem 4.6 from the following more precise result.

PROPOSITION 4.8. *Let $C \in G$ equal $\chi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ up to shift.*

Then forgetting the differential and the grading, we find that C_1 is isomorphic to $P_1^{|a|} \oplus P_2^{|c|}$ and C_2 is isomorphic to $P_1^{|b|} \oplus P_2^{|d|}$.

Assume ab and cd are not both zero. Let $C_{12} = \mathrm{Cone}(C_1 \to C_2)$ and $C_{21} = \mathrm{Cone}(C_2 \to C_1)$.

If $ab \leqslant 0$ and $cd \leqslant 0$, then

$$\dim C_{12} = |\dim C_1 - \dim C_2| \quad \textit{and} \quad \dim C_{21} = \dim C_1 + \dim C_2.$$

If $ab \geqslant 0$ and $cd \geqslant 0$, then

$$\dim C_{12} = \dim C_1 + \dim C_2 \quad \textit{and} \quad \dim C_{21} = |\dim C_1 - \dim C_2|.$$

REMARK 4. Note that, for example when $ab \leqslant 0$, $cd \leqslant 0$, $|b| \leqslant |a|$ and $|d| \leqslant |c|$, the statement of the proposition is that every morphism $C_1 \to C_2$ which is not homotopy equivalent to zero is surjective. It is an obvious fact that any morphism in the homotopy category can be represented by a monomorphism or an epimorphism in the category of complexes by adding a large enough complex which is homotopy equivalent to 0. The statement in the proposition is concerned with monomorphisms or epimorphisms between *reduced* complexes.

Proof. Note first that an element $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathrm{PSL}_2(\mathbb{Z})$ is determined by $|a|$, $|b|$, $|c|$, $|d|$ and by the signs of ab and cd. Note that if both ab and cd are non-zero, then these signs are equal.

The proposition is clear when $ab = cd = 0$, since then C is isomorphic, up to shift, to A or to M.

So, we assume $(ab, cd) \neq (0, 0)$. We will prove the proposition by induction on $|a| + |b| + |c| + |d|$.

Conjugating if necessary x by T, we may assume that $ab \leqslant 0$ and $cd \leqslant 0$. Let us assume that $|b| \leqslant |a|$ and $|d| \leqslant |c|$. The other case can be dealt with by using the same proof as below, conjugating all matrices by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We have

$$x = \begin{pmatrix} -b & a+b \\ -d & c+d \end{pmatrix} S.$$

When $b|a+b| = d|c+d| = 0$, we have two cases: if $x = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ then $C \simeq X_2$, up to a shift; if $x = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ then $C \simeq X_1 \otimes_A M$, up to a shift.

In the first case, we have

$$C_1 \simeq P_2 \to P_1, \quad C_{12} \simeq P_1[1],$$
$$C_2 \simeq P_2 \to 0, \quad C_{21} \simeq P_2 \to P_2 \to P_1,$$

and we have finished.

In the second case, we have

$$C_1 \simeq P_1 \to P_2, \quad C_{12} \simeq P_2[1],$$
$$C_2 \simeq P_1 \to 0, \quad C_{21} \simeq P_1 \to P_1 \to P_2.$$

and we also have the required result.

Assume now that $b|a+b|$ and $d|c+d|$ are not both zero. Denote by C' a two-sided tilting complex such that the class of C in $\mathrm{TrPic}(A)$ is the product of the class of C' by ϕ. Let C'_1 be the reduced part of $C' \otimes_A P_1$ and let C'_2 be the reduced part of $C' \otimes_A P_2$. The image of C' in $G/\mathrm{Sh}(A)$ is equal to $\chi\left(\begin{smallmatrix} -b & a+b \\ -d & c+d \end{smallmatrix}\right)$.

Note that our assumptions on the sign of ab and cd, on the absolute value of a compared to that of b and on the absolute value of c compared to that of d imply that $|a+b|+|c+d| < |a|+|c|$. Hence by induction we have

$$\dim \mathrm{Cone}(C'_1 \to C'_2) = \dim C'_1 + \dim C'_2$$

and

$$\dim \mathrm{Cone}(C'_2 \to C'_1) = |\dim C'_1 - \dim C'_2|.$$

We have $C_1 \simeq \mathrm{Cone}(C'_1 \to C'_2)$ and $C_2 \simeq C'_1[1]$. Consider the canonical map $\mathrm{Hom}(C'_1[1], C'_1[1]) \to \mathrm{Hom}(C_1, C_2)$. Then, the morphism $C_1 \to C_2$ which is the image of the identity morphism on $C'_1[1]$ under this canonical map is not homotopy equivalent to zero. So, $\dim \mathrm{Cone}(C_1 \to C_2) = \dim C'_2 = \dim C_1 - \dim C_2$.

We need to prove now that $\dim \mathrm{Cone}(C_2 \to C_1) = \dim C_1 + \dim C_2$.

Let $z \in Z(A)$ such that multiplication by z induces a non-zero but non-invertible endomorphism of P_1; such an element is obtained as follows. Since $\mathrm{End}_{A \otimes A^\circ}(A) \simeq Z(A)$ and the head and the socle of A as an $(A \otimes A^\circ)$-module are isomorphic to $S_1 \otimes S_1^* \oplus S_2 \otimes S_2^*$, we take for z an $(A \otimes A^\circ)$-endomorphism of A with image isomorphic to $S_1 \otimes S_1^*$.

Let z' be the image of z by the automorphism of $Z(A)$ induced by C'. Then under the isomorphism $\mathrm{End}(P_1) \simeq \mathrm{End}(C'_1)$ induced by C', the image of the endomorphism given by multiplication by z is the endomorphism given by multiplication by z'.

Multiplication by z' on a projective module has image in the socle of this module. Hence, the morphism $f\colon C'_1[1] \to C'_1[1]$ given by multiplication by z' extends to a morphism $g\colon C'_1[1] \to \mathrm{Cone}(C'_1 \to C'_2)$. Now, the identity map $C'_1[1] \to C'_1[1]$ extends to a map $h\colon \mathrm{Cone}(C'_1 \to C'_2) \to C'_1[1]$ and we have $f = hg$. As f is not zero, g is not zero either. The reduced part of the cone of g has dimension $\dim C'_1 + \dim \mathrm{Cone}(C'_1 \to C'_2)$. Now, a non-zero morphism $C_2 \to C_1$ is equal to g up to a scalar. Hence, its cone has dimension $\dim C_2 + \dim C_1$. So, the second part of the proposition holds for x.

Finally, we know by induction that C'_1 is isomorphic to $P_1^{|b|} \oplus P_2^{|d|}$ and C'_2 is isomorphic to $P_1^{|a+b|} \oplus P_2^{|c+d|}$, when the differential and the grading are omitted. As $\dim C_1 = \dim C'_1 + \dim C'_2$ and $C_1 = \mathrm{Cone}(C'_1 \to C'_2)$, we deduce that C_1 is isomorphic to $P_1^{|a|} \oplus P_2^{|c|}$ when the differential and the grading are omitted. Since

$C_2 \simeq C_1'[1]$, when omitting the differential and the grading, C_2 becomes isomorphic to $P_1^{|b|} \oplus P_2^{|d|}$. So, the first part of the proposition holds for x. □

4.6. *Transitivity of the braid group action*

In this section we prove that, for A a Brauer tree algebra with two edges and no exceptional vertex, the group $\mathrm{TrPic}(A)$ is generated, by t_1, t_2 and $\mathrm{Pic}(A) \times \mathrm{Sh}(A)$, and we deduce the structure of $\mathrm{TrPic}(A)$.

Let us start with some general properties of the image of simple modules by a derived equivalence.

Let A be a finite-dimensional algebra over a field k.

For X a bounded complex with non-zero homology, we denote by $\mathrm{lb}(X)$ the smallest integer i with $H^i(X) \neq 0$. Similarly, we denote by $\mathrm{rb}(X)$ the largest integer i with $H^i(X) \neq 0$. We define the amplitude of X as $\Lambda(X) = \{\mathrm{lb}(X), \mathrm{lb}(X)+1, \ldots, \mathrm{rb}(X)\}$. Finally, the length $\ell(X)$ of X is the cardinality of its amplitude.

LEMMA 4.9. *Let $U \to V \to W \rightsquigarrow$ be a distinguished triangle in $\mathscr{D}^b(A)$. Then $\Lambda(V) \subseteq \Lambda(U) \cup \Lambda(W)$.*
If $\mathrm{lb}(U) \neq \mathrm{lb}(W)+1$ and $\mathrm{rb}(U) \neq \mathrm{rb}(W)+1$, then we have

$$\Lambda(V) = \Lambda(U) \cup \Lambda(W).$$

Proof. This follows immediately from the long exact sequence

$$\ldots \longrightarrow H^i(U) \longrightarrow H^i(V) \longrightarrow H^i(W) \longrightarrow H^{i+1}(U) \longrightarrow \ldots.$$ □

Let C be a two-sided tilting complex in $\mathscr{D}^b(A \otimes A^\circ)$. Replacing C by an isomorphic complex, we may and will assume that $C^i = 0$ for $i \notin \Lambda(C)$. To avoid trivialities, we assume furthermore that $\Lambda(C)$ has more than one element, or in other words that C is not a shifted module.

Denote by F: $\mathscr{D}^b(A) \to \mathscr{D}^b(A)$ the functor $C \otimes_A -$.

The following lemma is clear.

LEMMA 4.10. *If M is an A-module, then $\Lambda(F(M)) \subseteq \Lambda(C)$.*

LEMMA 4.11. *We have $\Lambda(C) = \bigcup_V \Lambda(F(V))$ where V runs over the simple A-modules.*

Proof. Since, as an A-module, $C \simeq F(A)$, and as A has a composition series of simple modules, Lemma 4.9 gives the inclusion $\Lambda(C) \subseteq \bigcup_V \Lambda(F(V))$. The reverse inclusion follows from Lemma 4.10. □

The next lemma is crucial; when $\ell(C) = 2$, this had been pointed out to us by J. Rickard.

LEMMA 4.12. *If V is simple, then $\Lambda(F(V)) \neq \Lambda(C)$.*

Proof. Let V be a simple module with $\Lambda(F(V)) = \Lambda(C)$. Let T be the restriction of C^* to A. Then the complex of k-modules $\mathscr{H}om_A(T, V) \simeq C \otimes_A V$ has amplitude $\Lambda(C) = \Lambda(T^*)$. Let $m = \mathrm{lb}(T)$ and $n = \mathrm{rb}(T)$. There is a non-zero morphism $T \to V[-n]$, and hence a non-zero morphism $f \colon P[-n] \to T$, which is injective, where P is a projective cover of V. There is also a non-zero morphism $T \to V[-m]$. This means that T^m has a direct summand isomorphic to P whose intersection with $H^m(T)$ is non-zero. So, there is a non-zero morphism $g \colon T \to P[-m]$ which is surjective. Now, the morphism $f[n-m] \circ g \colon T \to T[n-m]$ is non-zero:

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & T^m & \longrightarrow & T^{m+1} \longrightarrow \dots \\
 & & & & \big\downarrow & & \\
 & & & & P & & \\
 & & & & \big\downarrow & & \\
\dots \longrightarrow & & T^{n-1} & \longrightarrow & T^n & \longrightarrow & 0
\end{array}$$

But, T being a tilting complex, that is not possible unless $m = n$, which has been excluded, so we get a contradiction. $\square$

From now on, A is a Brauer tree algebra with two edges and no exceptional vertex.

The algebra A has two simple modules S_1 and S_2 and we may assume the indexing is chosen so that

$$\mathrm{lb}(F(S_1)) > \mathrm{lb}(F(S_2)) \quad \text{and} \quad \mathrm{rb}(F(S_1)) > \mathrm{rb}(F(S_2)) \tag{1}$$

since by Lemmas 4.11 and 4.12, there is no inclusion between the sets $\Lambda(F(S_1))$ and $\Lambda(F(S_2))$. Note that, we then have $\mathrm{lb}(C) = \mathrm{lb}(F(S_2))$ and $\mathrm{rb}(C) = \mathrm{rb}(F(S_1))$.

Denote by $X = P_1 \to P_2$ a complex with P_2 in degree 0 and where the differential $P_1 \to P_2$ is non-zero. We have $H^0(X) \simeq S_2$ and $H^{-1}(X) \simeq S_1$, and hence we have a distinguished triangle

$$S_1[1] \longrightarrow X \longrightarrow S_2 \rightsquigarrow$$

and, applying F, a distinguished triangle

$$F(S_1)[1] \longrightarrow F(X) \longrightarrow F(S_2) \rightsquigarrow.$$

By Lemma 4.9, this implies that $\Lambda(F(X)) \subseteq \{\mathrm{lb}(F(S_2)), \dots, \mathrm{rb}(F(S_1)) - 1\}$, using (1). In particular, $\ell(F(X)) < \ell(C)$.

Let L be the kernel of a surjective map $P_2 \to S_2$. We have an exact sequence

$$0 \longrightarrow S_2 \longrightarrow L \longrightarrow S_1 \longrightarrow 0,$$

and hence a distinguished triangle

$$F(S_2) \longrightarrow F(L) \longrightarrow F(S_1) \rightsquigarrow.$$

By Lemma 4.9, we obtain $\Lambda(F(L)) = \Lambda(C)$. The distinguished triangle

$$F(L) \longrightarrow F(P_2) \longrightarrow F(S_2) \rightsquigarrow$$

shows that $\mathrm{lb}(F(P_2)) = \mathrm{lb}(C)$. The distinguished triangle

$$F(S_1) \longrightarrow F(P_1) \longrightarrow F(L) \rightsquigarrow$$

shows that $\mathrm{rb}(F(P_1)) = \mathrm{rb}(C)$.

If $\mathrm{rb}(F(P_2)) < \mathrm{rb}(C)$, then $\Lambda(F(X) \oplus F(P_2))$ is strictly contained in $\Lambda(C)$. We use the notation of § 4.2 for the complex X_2. Let $C' = C \otimes_A X_2^*[1]$. Then $C' \otimes_A P_1 \simeq F(X)$ and $C' \otimes_A P_2 \simeq F(P_2)$. Consequently, $\Lambda(C')$ is strictly contained in $\Lambda(C)$.

If $\mathrm{rb}(F(P_2)) = \mathrm{rb}(C)$, then

$$\Lambda(F(X)[-1] \oplus F(P_1)) \subseteq \Lambda(C) \quad \text{and} \quad \ell(F(X)) + \ell(F(P_1)) < \ell(F(P_2)) + \ell(F(P_1)).$$

Let $C' = C \otimes_A X_1[-1]$. Then

$$C' \otimes_A P_1 \simeq F(P_1) \quad \text{and} \quad C' \otimes_A P_2 \simeq F(X)[-1].$$

So, $\Lambda(C') \subseteq \Lambda(C)$ and

$$\ell(C' \otimes_A P_1) + \ell(C' \otimes_A P_2) < \ell(C \otimes_A P_1) + \ell(C \otimes_A P_2).$$

It follows by induction first on $\ell(C)$, then on $\ell(C \otimes_A P_1) + \ell(C \otimes_A P_2)$, that, modulo the subgroup generated by t_1 and t_2, every element of $\mathrm{TrPic}(A)$ is in $\mathrm{Pic}(A) \times \mathrm{Sh}(A)$.

Denote by $\widetilde{B}_3$ the extension of $B_3 = \langle \sigma_1, \sigma_2 \rangle$ generated by z, σ_1 and σ_2 with the relations $z^4 = (\sigma_1 \sigma_2)^3$ and $z\sigma_1 z^{-1}\sigma_1^{-1} = z\sigma_2 z^{-1}\sigma_2^{-1} = 1$. We have an injective morphism $\widetilde{B}_3 \to \mathrm{TrPic}(A)$ given by $\sigma_i \mapsto t_i$ and $z \mapsto [1]$.

We have completed our description of TrPic.

THEOREM 4.13. *Let A be a Brauer tree algebra over a field k, with two edges and without exceptional vertex. Then*

$$\mathrm{TrPic}(A) \simeq \widetilde{B}_3 \times (k^\times/\{\pm 1\}).$$

REMARK 5. The results of § 4 have a counterpart for Green orders as defined by Roggenkamp. Details and proofs are given in [**18**].

Let $\mathcal{O}$ be a complete discrete valuation ring with residue field k and A a Brauer tree algebra over k with n edges and no exceptional vertex. For Green orders Λ over $\mathcal{O}$, such that $\Lambda \otimes_{\mathcal{O}} k \simeq A$, one can construct a morphism $B_{n+1} \to \mathrm{TrPic}(\Lambda)$ lifting the morphism $B_{n+1} \to \mathrm{TrPic}(A)$ constructed in § 4.4. Moreover, one proves that when $n = 2$, there is an isomorphism $\widetilde{B}_3 \simeq \mathrm{TrPic}(\Lambda)$. The canonical map $\mathrm{TrPic}(\Lambda) \to \mathrm{TrPic}(A)$ will not be surjective in general.

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