THE DERIVED CATEGORY AS A QUOTIENT OF THE HOMOTOPY CATEGORY OF PERMUTATION MODULES

Letter to Alexander Beilinson, October 21, 2006

Let $\mathcal{A}$ be an abelian category, $\mathcal{A}$-proj its full subcategory of projective objects.

Lemma 0.1. *Let $\mathcal{C}$ be a full additive subcategory of $\mathcal{A}$ containing $\mathcal{A}$-*proj*. Let $\mathcal{I}$ be the full subcategory of $K^b(\mathcal{C})$ of acyclic complexes.*

Assume that for every $M \in \mathcal{C}$ and every $d \geq 0$, there is $r \geq 0$, $M' \in \mathcal{C}$ and an exact sequence

$$0 \to Q^{-r-d} \to \cdots \to Q^0 \to M \oplus M' \to 0$$

with $Q^i \in \mathcal{C}$ for all i and Q^i projective for $i \geq -d$.

Then, the canonical functor $K^b(\mathcal{C})/\mathcal{I} \to D^b(\mathcal{A})$ is fully faithful.

Proof. Consider $M \in \mathcal{C}$, $C \in K^b(\mathcal{A})$ acyclic and $f : M \to C$. Let $d \geq 0$ such that $C^i = 0$ for $i \leq -d$. We will show that f factors through an acyclic complex in $K^b(\mathcal{C})$. We choose a resolution as provided by assumption. Without loss of generality, we may assume that $M' = 0$ (replace M by $M \oplus M'$ and f by its composition with the injection of M into $M \oplus M'$).

Consider $D = 0 \to Q^{-r-d} \to \cdots \to Q^0 \to 0$. Since the stupid truncation $\sigma^{\geq -d}D$ is a bounded complex of projectives, we have $\mathrm{Hom}_{K^b(\mathcal{A})}(\sigma^{\geq -d}D, C) \simeq \mathrm{Hom}_{D^b(\mathcal{A})}(\sigma^{\geq -d}D, C) = 0$ because C is acyclic. So, the composite map $g : D \xrightarrow{\mathrm{can}} M \xrightarrow{f} C$ factors through $\sigma^{<-d}D$. But $\mathrm{Hom}_{K^b(\mathcal{A})}(\sigma^{<-d}D, C) = 0$, hence $g = 0$.

$$\begin{tikzcd}
\sigma^{\geq -d}D \arrow[r] \arrow[dr, "0"'] & D \arrow[r] \arrow[d] & \sigma^{<-d}D \arrow[r, squiggly] \arrow[dl, dashed, "0"] & {} \\
& M \arrow[d, "f"'] & & \\
& C & &
\end{tikzcd}$$

It follows that f factors through the cone L of the canonical map $D \to M$. That cone is an acyclic object of $K^b(\mathcal{C})$.

It follows now that the canonical map $\mathrm{Hom}_{K^b(\mathcal{C})/\mathcal{I}}(M, X) \to \mathrm{Hom}_{D^b(\mathcal{C})}(M, X)$ is an isomorphism for any $X \in K^b(\mathcal{C})$. Since this holds for every $M \in \mathcal{C}$, it holds for all $M \in K^b(\mathcal{C})$. $\square$

Let k be a commutative noetherian regular ring. All k-modules considered below are supposed to be finitely generated.

Lemma 0.2. *Let G be a finite group. There exists an integer r with the following property: given M a kG-module and d a non-negative integer, there is a kG-module M' and an exact sequence of kG-modules*

$$0 \to Q^{-r-d} \to \cdots \to Q^0 \to M \oplus M' \to 0$$

where Q^i is a direct summand of a permutation module for every i and Q^i is projective for $i \geq -d$. Furthermore, if M is a direct summand of a permutation module, then M' can be chosen as a direct summand of a permutation module.

Proof. Let us show that it is enough to prove the Lemma for G a symmetric group. There is an inclusion $G \subset \mathfrak{S}_n$ for some n. Fix r such that the Lemma holds for $\mathfrak{S}_n$. Let d be a non-negative integer, M be a kG-module and $N = \mathrm{Ind}_G^{\mathfrak{S}_n} M$. Let $0 \to Q^{-r-d} \to \cdots \to Q^0 \to N \to 0$ be an exact sequence as provided by the Lemma. Note that $\mathrm{Res}_G^{\mathfrak{S}_n} N = M \oplus M'$ for some kG-module M'. So, by restriction, we obtain an exact sequence as needed (note that restriction maps $k\mathfrak{S}_n$-perm to kG-perm).

Let us now prove the Lemma for $G = \mathfrak{S}_n$, with the stronger statement that M' can be chosen to be 0. Let $\cdots \to P^{-i} \xrightarrow{d^{-i}} P^{-i-1} \to \cdots \to P^0 \to M \to 0$ be a projective resolution of M. Replacing M by $\ker d^{-d}$, we see that it is enough to prove the (stronger) Lemma for $d = 0$.

Let V be the permutation representation of $\mathfrak{S}_n$ on n-points with coefficients in k and let $T = V^{\otimes n}$, an object of kG-perm (the action is the tensor product action). Let $A = \mathrm{End}_{kG}(T)$. This is a Schur algebra and it has finite global dimension. We denote by r the global dimension of A. Furthermore, T is a projective right A-module. Let M be a kG-module and $L = \mathrm{Hom}_{kG}(T, L)$, an A-module. There is a projective resolution

$$0 \to R^{-r} \to \cdots \to R^0 \to L \to 0$$

Applying the exact functor $T \otimes_A -$, we obtain the required exact sequence. □

The following result follows immediately from Lemmas 0.1 and 0.2.

Proposition 0.3. *Let G be a finite group. Let kG-perm be the full subcategory of kG-mod whose objects are direct summands of permutation modules. Let $\mathcal{I}$ be the full subcategory of acyclic complexes in $K^b(kG\text{-perm})$. Then, the canonical functor $K^b(kG\text{-perm})/\mathcal{I} \to D^b(kG)$ is fully faithful. Furthermore, every object of $D^b(kG)$ is a direct summand of an object in the image.*

The functor $K^b(kG\text{-perm})/\mathcal{I} \to D^b(kG)$ is an equivalence if and only if the classes of objects in kG-perm generate $K_0(kG\text{-mod})$ (by Thomason).

This does not hold in general: consider for example a prime p, $k = \mathbf{Z}_p$ the ring of p-adic numbers and G cyclic of order p. Then, $K_0(kG) \xrightarrow{\sim} K_0(\mathbf{Q}_p G)$ is a free abelian group of rank p, while there are only two isomorphism classes of indecomposable kG-modules that are direct summands of permutation modules, namely k and kG. The functor won't be an equivalence either for $k = \mathbf{Z}$.

On the other hand, the functor is an equivalence in the following cases:

- $|G| \in k^\times$, since in that case every kG-module that is projective over k is a direct summand of a permutation module
- k is a field of characteristic p: indeed, in that case, denoting by P a Sylow p-subgroup of G, then, every kG-module is a direct summand of a module induced from P and $K_0(kP\text{-mod}) = \mathbf{Z}[k]$.

Remark 0.4. Lemma 0.2 and Proposition 0.3 (and their proofs) hold if the k-modules are not assumed to be finitely generated over k.