MOTIVES OF DELIGNE-LUSZTIG VARIETIES

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Abstract. Given a reductive algebraic group $\mathbf{G}$ defined over a finite field $\mathbf{F}_q$, Deligne and Lusztig introduced in [DeLu] a family of algebraic varieties acted on by $\mathbf{G}(\mathbf{F}_q)$, in whose ℓ-adic cohomology they realized representations of $\mathbf{G}(\mathbf{F}_q)$. We prove in this note that the ℓ-adic cohomology of these varieties is independent of ℓ. This is a consequence of the validity of Beilinson and Tate conjectures for endomorphism rings of the corresponding equivariant $\mathbf{Q}$-motives. We also explain how this can be used to study rationality properties of unipotent representations.

1. Introduction

In [DiMiRou], we used motivic cohomology to prove the independence in ℓ of certain ℓ-adic cohomology groups of Deligne-Lusztig varieties. We prove here that all cohomology groups of all Deligne-Lusztig varieties are indeed independent of ℓ. This follows from a more precise result on motives of Deligne-Lusztig varieties, based on [Lu5]. Note that in [Lu5], Lusztig was studying the dependence in q, while we are only studying the dependence in ℓ.

Our main general result on motives is that the category of equivariant (unipotent) Deligne-Lusztig motives (with $\mathbf{Q}$-coefficients) is semi-simple and the Frobenius eigenvalue (well defined up to a power of q) attached to a unipotent representation is in the field of character values (Theorem 3.2). This follows from the fact that the indecomposable motives M arising in Deligne-Lusztig varieties have the property that $M^\vee \otimes_{\mathbf{Q}G} M$ is a direct sum of Tate motives: in particular, rational, homological and numerical equivalence coincide, and the cycle map is onto (Beilinson and Tate conjectures, cf [Ka]).

All Deligne-Lusztig motives are (up to numerical equivalence) either of Artin type or supersingular (cf §3.3 for the list of possibilities). We show that, up to a few exceptions, the endomorphism ring of an irreducible rational unipotent representation coincides with the endomorphism ring of the associated indecomposable motive. As a consequence, we deduce the structure of the simple $\mathbf{Q}$-algebras associated with unipotent representations from classical properties of F-isocrystals, as in [Mi].

The use of a group action on a variety and its action on cohomology to prove there are enough algebraic cohomology classes goes back to Tate [Ta]. He deduced that his conjecture holds for Fermat hypersurfaces, which are actually certain (parabolic) Deligne-Lusztig varieties associated with unitary groups [HoMa].

To a large extent, this paper is a mere translation of basic constructions of Lusztig [Lu5, Lu6] to the setting of motives. Note that the relevance of motives in the study of rationality of unipotent representations goes back to Ohmori [Oh3].

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2. Deligne-Lusztig motives

2.1. Motives.

2.1.1. Let p be a prime number and $\bar{\mathbf{F}}_p$ an algebraic closure of $\mathbf{F}_p$. Let $q = p^f$ where f is a positive integer and let $\mathbf{F}_q$ be the subfield with q elements of $\bar{\mathbf{F}}_p$.

We denote by $\mathrm{DM}^{\mathbf{Z}}(\mathbf{F}_q)$ the triangulated category of (geometric) motives over $\mathbf{F}_q$. We put $\mathrm{DM}(\mathbf{F}_q) = \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{DM}(\mathbf{F}_q)$. There is a motivic homology functor

$$H^{mot} = \bigoplus_{i,j} \mathrm{Hom}(\mathbf{Q}(j)[i], -) : \mathrm{DM}(\mathbf{F}_q) \to \mathbf{Q}\text{-bigrmod},$$

where $\mathbf{Q}$-bigrmod denotes the category of finitely generated bigraded $\mathbf{Q}$-vector spaces. Its restriction to the thick subcategory of $\mathrm{DM}(\mathbf{F}_q)$ generated by Tate motives is an equivalence.

The Frobenius endomorphism (relative to $\mathbf{F}_q$) induces an automorphism F of the identity functor of $\mathrm{DM}(\mathbf{F}_q)$, viewed as a triangulated tensor functor. It acts on $\mathbf{Q}(1)$ by q.

We denote by $M(X)$ the motive of a variety and by $M^c(X)$ its motive with compact support (in $\mathrm{DM}(\mathbf{F}_q)$).

2.1.2. Let G be a finite group. We denote by $\mathrm{DM}(\mathbf{F}_q/G)$ the triangulated category of G-equivariant motives over $\mathbf{F}_q$, with coefficients $\mathbf{Q}$. We have a forgetful functor

$$\mathrm{DM}(\mathbf{F}_q/G) \to \mathrm{DM}(\mathbf{F}_q),\ M \mapsto \tilde{M}.$$

We have a bifunctor

$$\mathcal{H}om_G : \mathrm{DM}(\mathbf{F}_q/G)^{\mathrm{opp}} \times \mathrm{DM}(\mathbf{F}_q/G) \to \mathrm{DM}(\mathbf{F}_q),\ \mathcal{H}om_G(M, N) = \mathcal{H}om(\tilde{M}, \tilde{N})^G$$

where $\mathcal{H}om(\tilde{M}, \tilde{N})$ is the internal Hom in $\mathrm{DM}(\mathbf{F}_q)$.

We have a motivic homology functor

$$H^{mot} = \bigoplus_{i,j} \mathrm{Hom}(\mathbf{Q}(j)[i], -) : \mathrm{DM}(\mathbf{F}_q/G) \to \mathbf{Q}G\text{-bigrmod},$$

where $\mathbf{Q}G$-bigrmod denotes the category of finitely generated bigraded $\mathbf{Q}G$-modules.

2.1.3. Let ℓ be a prime, $\ell \neq p$. We have an ℓ-adic realization functor (ℓ-adic homology)

$$H^{et} : \mathrm{DM}(\mathbf{F}_q/G) \to \mathbf{Q}_\ell G[\phi^{\pm 1}]\text{-grmod},$$

where $\mathbf{Q}_\ell G[\phi^{\pm 1}]$-grmod denotes the category of finitely generated graded $\mathbf{Q}_\ell G[\phi^{\pm 1}]$-modules and the action of ϕ is that of the Frobenius.

The ℓ-adic realization gives rise to a natural transformation of functors ("cycle map") $H^{mot} \to \bigoplus_{i \in \mathbf{Z}} \ker_{H^{et}}(\phi - q^i)$. This is conjecturally an isomorphism ("Beilinson-Tate conjecture"). It is known to be so on direct sums of shifted Tate motives, which form a semi-simple thick subcategory.

If M is an indecomposable object of $\mathrm{DM}(\mathbf{F}_q/G)$, then the eigenvalues of F acting on M are conjugate algebraic integers. So, if in addition $\tilde{M}$ is a direct summand of the motive of a smooth projective variety, then $H^{et}(M, \bar{\mathbf{Q}}_\ell)$ is concentrated in a single degree independent of ℓ, for any prime $\ell \neq p$.

2.1.4. Let $\mathbf{Q}_q$ be the unramified extension of degree f of $\mathbf{Q}_p$. There is a p-adic realization functor (rigid homology) [CiDe, §3.2]

$$H^{rig} : \mathrm{DM}(\mathbf{F}_q/G) \to \big((\mathbf{Q}_q G) \rtimes \langle\sigma\rangle\big)\text{-grmod}$$

where $\big((\mathbf{Q}_q G) \rtimes \langle\sigma\rangle\big)$-grmod denotes the category of finitely generated graded $(\mathbf{Q}_q G)$-modules endowed with a semi-linear invertible endomorphism σ (ie, graded G-equivariant isocrystals over $\mathbf{F}_q$). The endomorphism σ commutes with the G-action, but $\sigma\alpha = F(\alpha)\sigma$ for $\alpha \in \mathbf{Q}_q$.

The p-adic realization gives rise to a natural transformation of functors ("cycle map") $H^{mot} \to \bigoplus_{i\in\mathbf{Z}} \ker_{H^{rig}}(\sigma^f - q^i)$. This is conjecturally an isomorphism ("Beilinson-Tate conjecture"), and known to be so on Tate motives.

2.2. **Deligne-Lusztig varieties.** Let $\mathbf{G}$ be a reductive algebraic group defined over $\bar{\mathbf{F}}_p$ and endowed with an endomorphism F', a power of which is a Frobenius endomorphism. Let $G = \mathbf{G}^{F'}$. Let $\mathcal{B}$ be the variety of Borel subgroups of $\mathbf{G}$. We denote by $\mathcal{W}$ the (finite) set of $\mathbf{G}$-stable locally closed subvarieties of $\mathcal{B} \times \mathcal{B}$, where $\mathbf{G}$ acts diagonally. We denote by δ the smallest positive integer such that F'^{δ} is a Frobenius endomorphism acting trivially on $\mathcal{W}$ and we put $F = F'^{\delta}$. The Frobenius endomorphism F defines a rational structure of $\mathbf{G}$ over a finite field $\mathbf{F}_q$ for some q.

Given $\Omega \in \mathcal{W}^n$, we put

$$\begin{aligned}\mathcal{O}(\Omega) &= \Omega_1 \times_{\mathcal{B}} \Omega_2 \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \Omega_n = \\ &= \{(B_1, B_2, \ldots, B_{n+1}) \in \mathcal{B}^{n+1} \mid (B_i, B_{i+1}) \in \Omega_i \ \forall\ 1 \le i \le n\}.\end{aligned}$$

The Deligne-Lusztig variety $X(\Omega)$ is a closed subvariety of $\mathcal{O}(\Omega)$ defined as

$$X(\Omega) = \{(B_1, B_2, \ldots, B_{n+1}) \in \mathcal{O}(\Omega) \mid B_{n+1} = F'(B_1)\}.$$

It is a variety acted on diagonally by G. It is projective if Ω_i is closed (equivalently, projective), for all i. It is smooth if Ω_i is smooth for all i, for example, if Ω_i is a $\mathbf{G}$-orbit or if $\dim \Omega_i \le 1 + \dim \mathcal{B}$ (cf e.g. [DiMiRou, Proposition 2.3.5]).

We denote by $\mathrm{DL}(G)$ the smallest thick subcategory of $\mathrm{DM}(\mathbf{F}_q/G)$ closed under taking direct summands, under shifts, and under tensoring by Tate motives (with trivial G-action) and containing the motives of all smooth projective Deligne-Lusztig varieties $X(\Omega)$, with Ω_i closed of dimension $1 + \dim \mathcal{B}$. The category $\mathrm{DL}(G)$ contains the compactly supported motives of all Deligne-Lusztig varieties $X(\Omega)$ (cf [DeLu, §9.1]).

The following proposition is the key property for our study.

Proposition 2.1. *Given $M, N \in \mathrm{DL}(G)$, then $\mathcal{H}om_G(M, N)$ is a direct sum of shifted Tate motives.*

Proof. Let S be the subset of Ω of closed orbits of dimension $1 + \dim \mathcal{B}$. Let $m, n \ge 0$, let $s \in S^m$ and $t \in S^n$. The varieties $X(s)$ and $X(t)$ are smooth and projective. The dimension of $X(s)$ is m.

Lusztig [Lu3] showed that there is an F-stable stratification by closed subvarieties $\emptyset = Y_0 \subset \cdots \subset Y_r = X(s) \times_G X(t)$ such that $M^c(Y_i - Y_{i-1})$ is (rationally) a Tate motive (cf [DiMiRou, Proposition 3.4.2]). As a consequence, $\mathcal{H}om_G(M(X(s)), M(X(t))) \simeq M(X(s) \times_G X(t))[2m](m)$ is a direct sum of shifted Tate motives.

Since $\mathrm{DL}(G)$ is generated, as a triangulated categoy closed under taking direct summands and tensoring by Tate motives, by the $M(X(s))$ for $s \in S^n$ for some n, the proposition follows. ☐

Corollary 2.2. *Given $\ell \neq p$ a prime, the ℓ-adic realization functor* $H^{et} : \mathbf{Q}_\ell \otimes \mathrm{DL}(G) \to \mathbf{Q}_\ell G[\phi^{\pm 1}]$-grmod *is fully faithful, with image contained in the full subcategory of objects where ϕ acts semi-simply.*

Similarly, the p-adic realization functor $H^{rig} : \mathbf{Q}_p \otimes \mathrm{DL}(G) \to \big((\mathbf{Q}_q G) \rtimes \langle\sigma\rangle\big)$-grmod *is fully faithful, with image contained in the full subcategory of semi-simple isocrystals.*

In particular, the category $\mathrm{DL}(G)$ *is semi-simple.*

Proof. Let $M, N \in \mathrm{DL}(G)$. We have a commutative diagram whose horizontal maps are canonical isomorphisms

$$\begin{tikzcd}
\mathbf{Q}_\ell \otimes \mathrm{Hom}_{\mathrm{DM}(\mathbf{F}_q)}(\mathbf{Q}, \mathcal{H}om_G(M,N)) \arrow[r, "\sim"] \arrow[d, "H^{et}"'] & \mathbf{Q}_\ell \otimes \mathrm{Hom}_{\mathrm{DM}(\mathbf{F}_q/G)}(M,N) \arrow[d, "H^{et}"] \\
H^{et}(\mathcal{H}om_G(M,N), \mathbf{Q}_\ell)^F \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathbf{Q}_\ell G}(H^{et}(M,\mathbf{Q}_\ell), H^{et}(N,\mathbf{Q}_\ell))^F
\end{tikzcd}$$

Since $\mathcal{H}om_G(M,N)$ is a direct sum of shifted Tate motives, the right vertical arrow is an isomorphism, hence the left vertical arrow is an isomorphism as well.

Since F acts semi-simply on the motive $H^{et}(\mathcal{H}om_G(M,M), \mathbf{Q}_\ell)$, it follows that it acts semi-simply on $\mathrm{End}_{\mathbf{Q}_\ell G}(H^{et}(M,\mathbf{Q}_\ell))$, hence it acts semi-simply on $H^{et}(M,\mathbf{Q}_\ell)$.

The statement in the p-adic case is proven in the same way. □

Our independence of ℓ result is the following.

Theorem 2.3. *Let $\Omega \in \mathcal{W}^n$. The character of $G \times \langle F\rangle$ acting on $H^i_c(X(\Omega), \mathbf{Q}_\ell)$ is rational valued and independent of the prime $\ell \neq p$.*

Proof. Consider a simple object M of $\bar{\mathbf{Q}} \otimes \mathrm{DL}(G)$ (cf Corollary 2.2). Its endomorphism ring is $\bar{\mathbf{Q}}$ and the action of $Z(\bar{\mathbf{Q}}G)$ factors through a central character associated with a simple $\bar{\mathbf{Q}}G$-module V. Fix an embedding $\bar{\mathbf{Q}} \subset \bar{\mathbf{Q}}_\ell$. Since there is an isomorphism $\bar{\mathbf{Q}}_\ell \otimes_{\bar{\mathbf{Q}}} \mathrm{End}(M) \xrightarrow{\sim} \mathrm{End}_{\bar{\mathbf{Q}}_\ell G}(H^{et}(M, \bar{\mathbf{Q}}_\ell))$ compatible with the actions of $Z(\bar{\mathbf{Q}}_\ell G)$ and F, it follows that $H^{et}(M, \bar{\mathbf{Q}}_\ell)$ is concentrated in a single degree and isomorphic to the $\bar{\mathbf{Q}}_\ell G$-module $\bar{\mathbf{Q}}_\ell \otimes_{\bar{\mathbf{Q}}} V$. Furthermore, F acts by on $H^{et}(M, \bar{\mathbf{Q}}_\ell)$ by the same scalar as on M.

Let now N be an object of $\mathrm{DL}(G)$. There is a decomposition $\bar{\mathbf{Q}} \otimes N = \bigoplus_{i=1}^n M_i$ with M_i simple in $\bar{\mathbf{Q}} \otimes \mathrm{DL}(G)$, and the multiset of isomorphism classes of M_i's is invariant under $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. Let V_i be a simple $\bar{\mathbf{Q}}G$-module such that $Z(\bar{\mathbf{Q}}G)$ acts on M_i through the central character of V_i and let λ_i be the eigenvalue of F acting on M_i. We extend the action of G on V_i to an action of $G \times \langle F\rangle$ by letting F act as λ_i. The multiset of isomorphism classes of V_i's is invariant under $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ and $\bar{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}_\ell} H^{et}(N, \mathbf{Q}_\ell) \simeq \bigoplus_i \bar{\mathbf{Q}}_\ell \otimes_{\bar{\mathbf{Q}}} V_i$, for any embedding $\bar{\mathbf{Q}} \subset \bar{\mathbf{Q}}_\ell$. So, the character of $G \times \langle F\rangle$ on $H^{et}(N, \mathbf{Q}_\ell)$ is $\sum_i \chi_{V_i} \otimes \lambda_i$, and that character is rational valued.

It follows that the character of $H^{et}_i(M^c(X(\Omega)), \bar{\mathbf{Q}}_\ell)$ is rational and independent of ℓ. The theorem follows by duality. □

Remark 2.4. Theorem 2.3 in the smooth projective case is a general property of ℓ-adic cohomology. In general, only the Lefschetz character of a non-smooth or non-projective variety is known to be independent of ℓ [DeLu, Proposition 3.3].

3. Rationality of Deligne-Lusztig motives and unipotent representations

The character fields and realization fields of unipotent representations of finite groups of Lie type are known. Cf [Ge1, §5] for a general discussion of character fields (when F' is a Frobenius). For the realization fields, cf [Lu6] for general results concerning rational valued characters and [Ge1] for the reduction to cuspidal characters. The Schur indices for ${}^2A_n$ are determined in [Oh1], for types A_n, B_n, C_n, D_n and ${}^2D_n$ in [Lu6, Corollary 1.12 and §1.13]. For the exceptional groups, cf [Go] for ${}^2B_2$ and ${}^2G_2$, and [Ge1, Ge2, Ge3] for all other cases, as well as [Oh3] for cases in E_7 with small p.

We show here how the consideration of motives gives a more uniform approach to those results.

3.1. **Decomposition of categories of equivariant motives.**

3.1.1. Let again G be an arbitrary finite group.

Let K be a number field. Denote by $\mathrm{Irr}_K(G)$ the set of isomorphism classes of simple KG-modules. Given $V \in \mathrm{Irr}_K(G)$, there is a unique idempotent $e_V \in Z(KG)$ such that e_V acts by the identity on V and by 0 on any $V' \in \mathrm{Irr}_K(G)$ not isomorphic to V.

We have a morphism from $Z(KG)$ to the algebra of endomorphisms of the identity functor of $K \otimes \mathrm{DM}(\mathbf{F}_q/G)$, viewed as a triangulated functor. This induces a decomposition

$$K \otimes \mathrm{DM}(\mathbf{F}_q/G) = \bigoplus_{V \in \mathrm{Irr}_K(G)} (K \otimes \mathrm{DM}(\mathbf{F}_q/G))_V,$$

where $(K \otimes \mathrm{DM}(\mathbf{F}_q/G))_V$ is the image of e_V. There is an equivalence of categories

$$\mathcal{H}om_G(V,-) : (K \otimes \mathrm{DM}(\mathbf{F}_q/G))_V \xrightarrow{\sim} \mathrm{End}_{KG}(V)^{\mathrm{opp}} \otimes_K (K \otimes \mathrm{DM}(\mathbf{F}_q)),$$

where V is viewed as a multiple of the trivial motive K. An inverse is given by $V \otimes_{\mathrm{End}_{KG}(V)^{\mathrm{opp}}} -$.

3.1.2. Let $M \in (K \otimes \mathrm{DM}(\mathbf{F}_q/G))_V$ be an indecomposable object and let $N = \mathcal{H}om_G(V, M)$. We have a canonical map $\mathrm{End}_{KG}(V)^{\mathrm{opp}} \to \mathrm{End}_{\mathrm{DM}(\mathbf{F}_q)}(N)$. Note that $M \simeq V \otimes_{\mathrm{End}_{KG}(V)^{\mathrm{opp}}} N$, hence $\bar{M}$ is isomorphic to $N^{\oplus d}$ in $K \otimes \mathrm{DM}(\mathbf{F}_q)$, where $d = \dim_{\mathrm{End}_{KG}(V)}(V)$.

Note that M is a direct summand of $V \otimes_K N$, hence M is a direct summand of $V \otimes_K \tilde{N}$ for some indecomposable object $\tilde{N}$ of $K \otimes \mathrm{DM}(\mathbf{F}_q)$ that is a direct summand of N. It follows that N is a direct summand of $\mathrm{End}_{KG}(V) \otimes_K \tilde{N}$, hence $N \simeq \tilde{N}^{\oplus r}$ for some $r \leq \dim_K \mathrm{End}_{KG}(V)$.

Let $Z_V = Z(\mathrm{End}_{KG}(V))$ and $A_V = C_{\mathrm{End}_{K\otimes \mathrm{DM}(\mathbf{F}_q)}(N)}(Z_V)$. Note that the central simple Z_V-algebra $\mathrm{End}_{KG}(V)^{\mathrm{opp}}$ is contained in A_V, hence

$$\begin{aligned} A_V &= \mathrm{End}_{KG}(V)^{\mathrm{opp}} \otimes_{Z_V} C_{A_V}(\mathrm{End}_{KG}(V)^{\mathrm{opp}}) \\ &\simeq \mathrm{End}_{KG}(V)^{\mathrm{opp}} \otimes_{Z_V} \mathrm{End}_{K\otimes \mathrm{DM}(\mathbf{F}_q/G)}(M). \end{aligned}$$

In particular, the canonical map $\mathrm{End}_{KG}(V)^{\mathrm{opp}} \to \mathrm{End}_{K\otimes \mathrm{DM}(\mathbf{F}_q)}(N)$ is an isomorphism if and only if $Z_V \subset Z(\mathrm{End}_{K\otimes \mathrm{DM}(\mathbf{F}_q)}(N))$ and $Z_V \xrightarrow{\sim} \mathrm{End}_{K\otimes \mathrm{DM}(\mathbf{F}_q/G)}(M)$.

We have a commutative diagram, all of whose maps are injective. We will identify below all the vector spaces involved to subspaces of $\mathrm{End}(N)$.

$$\begin{tikzcd}
Z\,\mathrm{End}(V) \arrow[r] \arrow[d] & \mathrm{End}(V)^{\mathrm{opp}} \arrow[ddr] & \\
Z\,\mathrm{End}(M) \arrow[r] & \mathrm{End}(M) \arrow[dr] & \\
K[F] \arrow[r] & Z\,\mathrm{End}(N) \arrow[r] \arrow[ul] & \mathrm{End}(N)
\end{tikzcd}$$

In the lemma below, the assumption of the existence of an ℓ-adic realization fully faithful on M and on N, with semi-simple action of F should always hold (Tate + Beilinson conjecture). Similarly, it should always be true that $K[F] = Z\,\mathrm{End}(N)$ (Tate + Beilinson conjecture again).

Lemma 3.1. *Assume there is a prime $\ell \neq p$ and an embedding $K \subset \bar{\mathbf{Q}}_\ell$ such that F acts semisimply on $H^{et}(M, \bar{\mathbf{Q}}_\ell)$ and ℓ-adic realization gives an isomorphism $\bar{\mathbf{Q}}_\ell \otimes_K \mathrm{End}_{K\otimes\mathrm{DM}(\mathbf{F}_q/G)}(M) \xrightarrow{\sim} \mathrm{End}_{\bar{\mathbf{Q}}_\ell G}(H^{et}(M, \bar{\mathbf{Q}}_\ell))^F$.*

Then

- $Z\,\mathrm{End}(M)$ *is generated by* $Z\,\mathrm{End}(V)$ *and* $K[F]$
- $\mathrm{End}(M)$ *is a division algebra.*

The following assertions are equivalent:

(a1) $Z\,\mathrm{End}(V) = \mathrm{End}(M)$
(a2) *The $\bar{\mathbf{Q}}_\ell G$-module $H^{et}(M, \bar{\mathbf{Q}}_\ell)$ is multiplicity-free.*

The following assertions are equivalent:

(b1) $Z\,\mathrm{End}(V) \subset Z\,\mathrm{End}(N)$ *and* $K[F] = Z\,\mathrm{End}(N)$
(b2) *The eigenspaces of F on $H^{et}(M, \bar{\mathbf{Q}}_\ell)$ are isotypic $\bar{\mathbf{Q}}_\ell G$-modules*
(b3) $Z\,\mathrm{End}(V) \subset K[F]$
(b4) $Z\,\mathrm{End}(M) = K[F]$.
(b5) $Z\,\mathrm{End}(V) \subset Z\,\mathrm{End}(N)$ *and ℓ-adic realization gives an isomorphism*

$$\bar{\mathbf{Q}}_\ell \otimes_K \mathrm{End}_{K\otimes\mathrm{DM}(\mathbf{F}_q)}(N) \xrightarrow{\sim} \mathrm{End}_{\bar{\mathbf{Q}}_\ell}(H^{et}(N, \bar{\mathbf{Q}}_\ell))^F.$$

The following assertions are equivalent:

(c1) *The canonical map* $\mathrm{End}_{KG}(V) \to \mathrm{End}_{K\otimes\mathrm{DM}(\mathbf{F}_q)}(N)$ *is an isomorphism and* $K[F] = Z\,\mathrm{End}(N)$
(c2) *The eigenspaces of F on $H^{et}(M, \bar{\mathbf{Q}}_\ell)$ are non-isomorphic simple $\bar{\mathbf{Q}}_\ell G$-modules.*
(c3) $\mathrm{End}(M) = K[F]$.

When the equivalent conditions (c1) and (c2) hold, we have

$$Z\,\mathrm{End}_{KG}(V) = Z\,\mathrm{End}_{K\otimes\mathrm{DM}(\mathbf{F}_q/G)}(M) = \mathrm{End}_{K\otimes\mathrm{DM}(\mathbf{F}_q/G)}(M) = K[F] = Z\,\mathrm{End}_{K\otimes\mathrm{DM}(\mathbf{F}_q)}(N)$$

and the simple algebra $\mathrm{End}_{KG}(V)$ splits after completion at any prime ideal above a prime $\ell \neq p$.

Proof. The assumption shows that $\mathrm{End}(M)$ is a semi-simple algebra, hence a division algebra since M is indecomposable. So, $Z\,\mathrm{End}(M)$ is a field and $K[F]$ a subfield. The assumption shows also that $Z\,\mathrm{End}(M)$ is generated by its subalgebras $Z\,\mathrm{End}(V)$ and $K[F]$.

Since F acts semisimply on $H^{et}(M, \bar{\mathbf{Q}}_\ell)$, it acts semisimply on $H^{et}(N, \bar{\mathbf{Q}}_\ell) \simeq \mathrm{Hom}_{\bar{\mathbf{Q}}_\ell G}(V, H^{et}(M, \bar{\mathbf{Q}}_\ell))$. Note that $Z\,\mathrm{End}(N) \subset Z\,\mathrm{End}(M)$ acts faithfully on $H^{et}(N, \bar{\mathbf{Q}}_\ell)$ and Z_V acts faithfully on $H^{et}(M, \bar{\mathbf{Q}}_\ell)$.

We have $Z_V = \mathrm{End}(M)$ if and only if $\bar{\mathbf{Q}}_\ell \otimes_K Z_V = \mathrm{End}_{\bar{\mathbf{Q}}_\ell G}(H^{et}(M, \bar{\mathbf{Q}}_\ell))^F$, hence if and only if $H^{et}(M, \bar{\mathbf{Q}}_\ell)$ is a direct sum of non-isomorphic simple $\bar{\mathbf{Q}}_\ell G$-modules.

Note that (b2) is equivalent to $\bar{\mathbf{Q}}_\ell \otimes_K Z\,\mathrm{End}(V) \subset \bar{\mathbf{Q}}_\ell[F]$, hence equivalent to (b3). Since $Z\,\mathrm{End}(M)$ is generated by $Z\,\mathrm{End}(V)$ and $K[F]$, it follows that (b3) is equivalent to (b4). It is clear that (b4) is equivalent to (b1).

Assume (b3). Then

$$\mathrm{End}_{\bar{\mathbf{Q}}_\ell}(H^{et}(N, \bar{\mathbf{Q}}_\ell))^F = \mathrm{End}_{KG}(V) \otimes_{Z_V} \mathrm{End}_{\bar{\mathbf{Q}}_\ell G}(H^{et}(M, \bar{\mathbf{Q}}_\ell))^F$$

and (b5) follows.

Assume (b5). We have $\mathrm{End}_{K\otimes \mathrm{DM}(\mathbf{F}_q)}(N) = \mathrm{End}_{KG}(V) \otimes \mathrm{End}(M)$, hence $Z\,\mathrm{End}(V) \subset Z\,\mathrm{End}_{\bar{\mathbf{Q}}_\ell}(H^{et}(N, \bar{\mathbf{Q}}_\ell))^F$. Assertion (b3) follows.

Note that (c1) is equivalent to (a1) and (b1), hence to (a2) and (b2), which is equivalent to (c2). The assertion (c3) is equivalent to (a1) and (b4), hence equivalent to (c1). $\square$

The case $\mathrm{End}(N) = \mathrm{End}(V)$ is the optimal scenario: the division ring $\mathrm{End}(V)$ measures the obstruction for V to be absolutely simple. When $\mathrm{End}(N) = \mathrm{End}(V)$, this obstruction is "absorbed" by the motive N. The condition $\mathrm{End}(N) = \mathrm{End}(V)$ holds for $K = \mathbf{Q}$ for most Deligne-Lusztig motives, but fails in a few cases. In those cases, either one of (a2) or (b2) can fail (but not both). Note that (b3) holds trivially if $Z\,\mathrm{End}(V) = K$.

3.2. **Unipotent representations.** We assume from now on that $G = \mathbf{G}^{F'}$ as in §2.2.

3.2.1. *Unipotent motives.* Let K be a number field and $V \in \mathrm{Irr}_K(G)$. We say that V is *unipotent* if $(K \otimes \mathrm{DM}(\mathbf{F}_q/G))_V \neq 0$. This is equivalent to the requirement that $\bar{\mathbf{Q}}_\ell \otimes_K V$ is a direct summand of $H^i_c(X(\Omega), \bar{\mathbf{Q}}_\ell)$ for some Ω and some embedding $K \subset \bar{\mathbf{Q}}_\ell$ (cf Corollary 2.2). We denote by $\mathrm{Unip}_K(G)$ the set of unipotent representations in $\mathrm{Irr}_K(G)$.

Let $V \in \mathrm{Unip}_K(G)$. Consider an indecomposable object $M \in (K \otimes \mathrm{DM}(\mathbf{F}_q/G))_V$. It follows from Proposition 2.1 that every indecomposable object of $(K \otimes \mathrm{DM}(\mathbf{F}_q/G))_V$ is isomorphic to a shift of a Tate twist of M. We denote by M_V the unique indecomposable object of $(K \otimes \mathrm{DM}(\mathbf{F}_q/G))_V$ such that the eigenvalues of F on $K \otimes M_V$ have absolute value 1 (weight 0) or $q^{1/2}$ (weight 1) and such that $H^{et}(M_V, \bar{\mathbf{Q}}_\ell)$ is concentrated in degree 0 (for some ℓ, or any ℓ, cf Theorem 2.3). We put $N_V = \mathcal{H}om_G(V, M_V) \in K \otimes \mathrm{DM}(\mathbf{F}_q)$. We denote by $\tilde{N}_V$ an indecomposable direct summand of N_V: there is an integer r_V such that $N_V \simeq \tilde{N}_V^{\oplus r_V}$ and $\tilde{N}_V$ is well defined up to isomorphism.

Theorem 3.2. *We have*

$$K \otimes \mathrm{DM}(\mathbf{F}_q/G) = \bigoplus_{V \in \mathrm{Unip}_K(G)} (K \otimes \mathrm{DM}(\mathbf{F}_q/G))_V$$

and $(K\otimes \mathrm{DM}(\mathbf{F}_q/G))_V$ *is equivalent to the category of bigraded modules over the division algebra* $\mathrm{End}(M_V)$.

Let $V \in \mathrm{Unip}_K(G)$.

- *We have* $K[F] \subset Z\,\mathrm{End}(V)$ *and* $\mathrm{End}(M_V)$ *is a division algebra with center* $Z\,\mathrm{End}(V)$.
- *If* $\bar{\mathbf{Q}} \otimes V$ *is isotypic and* K *is the extension of* $\mathbf{Q}$ *generated by the character values of* V*, then the unique eigenvalue of* F *acting on* M_V *is in* K.
- *Given any prime* $\ell \neq p$ *and any embedding* $K \subset \bar{\mathbf{Q}}_\ell$*, no simple* $\bar{\mathbf{Q}}_\ell G$*-module can appear as a submodule of two different eigenspaces of* F *on* $H^{et}(M_V, \bar{\mathbf{Q}}_\ell)$.

Proof. The decomposition is given in §3.1.1. The equivalence of categories follows from the structure of objects of $(K \otimes \mathrm{DM}(\mathbf{F}_q/G))_V$ discussed above and the semi-simplicity of $K \otimes \mathrm{DM}(\mathbf{F}_q/G)$ (Corollary 2.2).

Let K' be a finite extension of K and let $\lambda_1, \lambda_2 \in K'$ be two distinct eigenvalues of F acting on M_V. There are $V_1, V_2 \in \mathrm{Unip}_{K'}(G)$ such that M_{V_i} is a direct summand of the λ_i-eigenspace of M_V. Since M_{V_1} is not isomorphic to a Tate twist of a shift of M_{V_2}, it follows that $V_1 \not\simeq V_2$. As a consequence, $K'[F] \subset Z(\mathrm{End}(K' \otimes_K V))$, hence $Z\,\mathrm{End}(K' \otimes_K M_V) = Z(\mathrm{End}(K' \otimes_K V))$ (cf Lemma 3.1). We deduce that $Z\,\mathrm{End}(M_V) = Z\,\mathrm{End}(V)$ and $K[F] \subset Z\,\mathrm{End}(V)$. Note that $Z\,\mathrm{End}(V)$ is the extension of $\mathbf{Q}$ generated by character values of elements of G acting on V. If $\bar{\mathbf{Q}} \otimes V$ is isotypic, then $Z\,\mathrm{End}(V) = K$, hence F acts as a scalar in K.

The last statement follows from the discussion above and from Corollary 2.2. □

Remark 3.3. The property that the Frobenius eigenvalue is contained in the field of character values can be deduced from the case by case determination of all those fields in [Ge1, §5]. It builds on results on eigenvalues of Frobenius going back to [DiMi, Lu1]. Our proof seems to be the first direct explanation of that fact.

Remark 3.4. Note that the group $\mathbf{G}$ has always a split structure over $\mathbf{F}_{p^\delta}$, with Frobenius endomorphism F_0 such that F is a power of F_0. In particular, the varieties $X(\Omega)$ have an $\mathbf{F}_{p^\delta}$-structure. Since the simple unipotent representations of G are F_0-stable, it follows that given V, there is a motive over $\mathbf{F}_{p^\delta}$ giving, after extension to $\mathbf{F}_q$, a multiple of N_V.

3.2.2. *Harish-Chandra induction.* Let $\mathbf{P}$ be an F'-stable parabolic subgroup of $\mathbf{G}$. Let $\mathbf{U}$ be its unipotent radical and let $\mathbf{L}$ be a Levi complement. Let $L = \mathbf{L}^{F'}$ and $U = \mathbf{U}^{F'}$.

We have a Harish-Chandra induction functor

$$R_L^G : K \otimes \mathrm{DM}(\mathbf{F}_q/L) \to K \otimes \mathrm{DM}(\mathbf{F}_q/G),\ M \mapsto M(G/U) \times_L M$$

and a left and right adjoint Harish-Chandra restriction functor

$${}^*R_L^G : K \otimes \mathrm{DM}(\mathbf{F}_q/G) \to K \otimes \mathrm{DM}(\mathbf{F}_q/L),\ M \mapsto \mathcal{H}om_G(M(G/U), M).$$

These functors are compatible with the corresponding functors on $\bar{\mathbf{Q}}_\ell$-representations, via ℓ-adic realizations.

Let $V \in \mathrm{Unip}_K(G)$ and $V' \in \mathrm{Unip}_K(L)$. If $\mathrm{Hom}_{KG}(V, R_L^G(V')) \neq 0$, then $\bar{N}_V \simeq \bar{N}_{V'}$. In particular, the isomorphism type of $\bar{N}_V$ does not change inside Harish-Chandra series.

Lemma 3.5. *Let $V \in \mathrm{Unip}_K(G)$ and let (L, V') be a cuspidal pair such that $\mathrm{Hom}_{KG}(V, R_L^G(V')) \neq 0$. Then*

- *M_V is a direct summand of $R_L^G(M_{V'})$,*
- *$\bar{N}_V \simeq \bar{N}_{V'}$*
- *and there is a morphism of fields $Z\,\mathrm{End}(V') \to Z\,\mathrm{End}(V)$.*

Proof. We have

$$\mathcal{H}om_G(V, R_L^G(M_{V'})) \simeq N_{V'} \otimes_{\mathrm{End}_{KL}(V')} \mathrm{Hom}_{KL}({}^*R_L^G(V), V').$$

We deduce that M_V is a direct summand of $R_L^G(M_{V'})$, hence $\bar{N}_V = \bar{N}_{V'}$.

Let $K' = Z\,\mathrm{End}(V')$ and consider a decomposition $K' \otimes V' = \bigoplus_i V'_i$ into simple (non-isomorphic) $K'L$-modules. By [Lu2, §3.25], we have $\mathrm{Hom}_{K'G}(R_L^G(V'_i), R_L^G(V'_j)) = 0$ if $i \neq j$.

It follows that $K' \otimes \operatorname{End}_{KG}(R_L^G(V')) = \bigoplus_i \operatorname{End}_{K'G}(R_L^G(V_i'))$. Consequently, the canonical map $K' \otimes Z\operatorname{End}(V') \to K' \otimes \operatorname{End}_{KG}(R_L^G(V'))$ has its image contained in the center of $K' \otimes \operatorname{End}_{KG}(R_L^G(V'))$. We deduce that the canonical map $Z\operatorname{End}(V') \to \operatorname{End}_{KG}(R_L^G(V'))$ has its image contained in the center of $\operatorname{End}_{KG}(R_L^G(V'))$.

Since $\operatorname{End}_{KG}(R_L^G(V'))$ has a quotient isomorphic to a matrix algebra over $\operatorname{End}_{KG}(V)$, we obtain a morphism of algebras $Z\operatorname{End}(V') \to Z\operatorname{End}(V)$. □

Let us recall an important result from the representation theory of Hecke algebras [Ge1, Proposition 5.6].

Theorem 3.6. *Assume* $\mathbf{G}$ *is simple. Let* $V \in \operatorname{Unip}_{\mathbf{Q}}(G)$ *and let* (L, V') *be a cuspidal pair such that* $\operatorname{Hom}_{\mathbf{Q}G}(V, R_L^G(V')) \neq 0$. *We have* $\operatorname{End}(V') \simeq \operatorname{End}(V)$ *unless* $\mathbf{L}$ *is a maximal torus and*

- $\mathbf{G} = E_7$ *and* $V = V_{1024}$ *corresponds to the sum of the two irreducible representations of dimension* 512 *of the Weyl group.*
- $\mathbf{G} = E_8$ *and* $V \in \{V_{8192_1}, V_{8192_2}\}$ *corresponds to the sum of two irreducible representations of dimension* 4096 *of the Weyl group.*

3.2.3. *Unipotent motives and realizations.* We recall here some basic facts of Lusztig's theory (Lemmas 3.8 and 3.7) and derive consequences for unipotent motives.

The next lemma, going back to [Lu4, §14.2], states a result that is true for all (simple) groups and all unipotent representations, except for one case. Given (G, F) of type ${}^2F_4$, we denote by $V_{21} \in \operatorname{Unip}_{\mathbf{Q}}(G)$ the unique unipotent representation whose character is a multiple of the unique cuspidal unipotent character of degree $\frac{1}{3}q^4\Phi_1^2(q)\Phi_2(q)^2\Phi_{12}(q)\Phi_{24}(q)$. Here, Φ_d denotes the d-th cyclotomic polynomial.

Lemma 3.7. *Assume* $\mathbf{G}$ *is simple. Let* $V \in \operatorname{Unip}_{\mathbf{Q}}(G)$. *Then* $H^{et}(M_V, \bar{\mathbf{Q}}_\ell)$ *is a multiplicity-free* $\mathbf{Q}_\ell G$*-module and* $\operatorname{End}(M_V) = Z\operatorname{End}(V)$, *unless* $(\mathbf{G}, F)$ *has type* ${}^2F_4$ *and* $V = V_{21}$.

Proof. Note that the statement about $\operatorname{End}(M_V)$ follows from the multiplicity-free property, thanks to Lemma 3.1.

Assume F is a Frobenius endomorphism. By [Lu6, Lemma 1.2 and §1.13], there is Ω closed and i such that every irreducible component of $\bar{\mathbf{Q}}_\ell \otimes V$ occurs with multiplicity 1 in the j-th ℓ-adic intersection cohomology group of $X(\Omega)$. Consequently, by purity, every such component occurs with multiplicity 1 in the weight j part L of the intersection cohomology (*i.e.*, the sum of the eigenspaces of F with eigenvalues of absolute value $q^{j/2}$). By [Lu4, §3], the character of L is a linear combination with integer coefficients of weight parts of cohomology groups of smooth projective Deligne-Lusztig varieties (change from the Kazhdan-Lusztig basis in the Hecke algebra to the generating family given by products of Kazhdan-Lusztig basis elements associated to simple reflections). Since the motive of a smooth projective variety is the direct sum of submotives associated to different weights, it follows that the character of L is an integral linear combination of ℓ-adic homology groups of unipotent motives. Projecting onto $\operatorname{DL}(G)_V$, we obtain the multiplicity-free result for a direct sum of shifts and Tate twists of M_V. It follows that the result holds for M_V itself.

Let us now consider the case where $(\mathbf{G}, F)$ has type ${}^2B_2$, ${}^2G_2$ or ${}^2F_4$. By [Lu4, Appendix, pp. 373–376], V occurs with multiplicity 1 in the character of the complex of cohomology of some Deligne-Lusztig variety, and we can conclude as above. □

Lemma 3.8. *Let $V \in \mathrm{Irr}_{\mathbf{Q}}(G)$ be a cuspidal unipotent simple representation. Then, the eigenspaces of F on $H^{et}(M_V, \bar{\mathbf{Q}}_\ell)$ are isotypic $\bar{\mathbf{Q}}_\ell G$-modules, for all $\ell \neq p$.*

We have $Z\,\mathrm{End}(M_V) = Z\,\mathrm{End}(V) = \mathbf{Q}[F]$.

Proof. If $Z\,\mathrm{End}(V) = \mathbf{Q}$, then $H^{et}(M_V, \bar{\mathbf{Q}}_\ell)$ is a direct summand of a multiple of the isotypic module $\bar{\mathbf{Q}}_\ell \otimes V$, hence $H^{et}(M_V, \bar{\mathbf{Q}}_\ell)$ is isotypic.

Note that the result holds for those V that occur in the cohomology of a Coxeter Deligne-Lusztig variety, cf [Lu1, Table 7.3].

We proceed case by case for the remaining V's, using Lusztig's classification (cf for example [Ca, §13.7]) and Geck's description of character fields [Ge1, §5]. We assume $Z\,\mathrm{End}(V) \neq \mathbf{Q}$ and V does not occur in the cohomology of a Coxeter Deligne-Lusztig variety. Looking at tables, this happens only in type ${}^2F_4$. There is one such V, and there are two non-isomorphic simple modules occuring in $\bar{\mathbf{Q}} \otimes V$, with distinct λ's equal to $\pm i$ (cf [Ge1, §]).

The last part of the lemma follows from Lemma 3.1 and Theorem 3.2. ☐

The following proposition would be a consequence of Belinson+Tate conjectures.

Proposition 3.9. *Let $V \in \mathrm{Unip}_{\mathbf{Q}}(G)$. The algebra $\mathrm{End}(\tilde{N}_V)$ is a central simple $\mathbf{Q}[F]$-algebra that splits at places above $\ell \neq p$ and at complex places.*

The motive $\tilde{N}_V$ has weight 0 or 1. If it has weight 0, then $\mathrm{End}(\tilde{N}_V) = \mathbf{Q}[F]$. If it has weight 1, then $\mathrm{End}_{\mathbf{Q}G}(V)$ has Hasse invariants $\frac{1}{2}$ at real places and $\frac{1}{2}[\mathbf{Q}[F]_\nu : \mathbf{Q}_p]$ at places ν above p.

Given any $\ell \neq p$, the ℓ-adic realization gives an isomorphism

$$\mathbf{Q}_\ell \otimes \mathrm{End}(\tilde{N}_V) \xrightarrow{\sim} \mathrm{End}_{\mathbf{Q}_\ell}(H^{et}(\tilde{N}_V, \mathbf{Q}_\ell))^F.$$

Proof. Thanks to Lemma 3.5, it is enough to consider the case where V is cuspidal. In that case, the result follows from Lemmas 3.8 and 3.1. ☐

Theorem 3.10. *Let $V \in \mathrm{Unip}_{\mathbf{Q}}(G)$. The algebras $\mathrm{End}_{\mathbf{Q}G}(V)$, $\mathrm{End}(M_V)$ and $\mathrm{End}(N_V)$ split at all places above $\ell \neq p$ and at all complex places.*

Assume $\mathbf{G}$ is simple and $((\mathbf{G}, F), V)$ is not $({}^2F_4, V_{21})$, (E_7, V_{1024}) nor (E_8, V_{8192_i}) for some $i \in \{1, 2\}$.

Then $\mathrm{End}_{\mathbf{Q}G}(V) = \mathrm{End}(N_V)$ and $Z\,\mathrm{End}_{\mathbf{Q}G}(V) = \mathbf{Q}[F]$.

If N_V has weight 0, then $\mathrm{End}_{\mathbf{Q}G}(V) = \mathbf{Q}[F]$.

If N_V has weight 1, then $\mathrm{End}_{\mathbf{Q}G}(V)$ has Hasse invariants $\frac{1}{2}$ at real places and $\frac{1}{2}[\mathbf{Q}[F]_\nu : \mathbf{Q}_p]$ at places ν above p.

Proof. Proposition 3.9 shows that $\mathrm{End}(N_V)$ has trivial Hasse invariants at complex places and a places above $\ell \neq p$.

Lemma 3.7 shows that $\mathrm{End}(M_V) = Z\,\mathrm{End}(V)$

When V is cuspidal, Lemma 3.8 shows that $Z\,\mathrm{End}(V) = \mathbf{Q}[F]$. Theorem 3.6 shows that this property remains true for any V.

Lemma 3.1 shows that $\mathrm{End}(V) = \mathrm{End}(N_V)$. The theorem follows now from Proposition 3.9. ☐

Remark 3.11. In [BrMa], it is conjectured that the algebra of $\mathbf{Q}_\ell G$-endomorphisms of the ℓ-adic cohomology of certain Deligne-Lusztig varieties is the Hecke algebra of a complex reflection reflection group. We conjecture that, in the same setting, the endomorphism ring of the G-equivariant motive of that variety is that same Hecke algebra, now taken over $\mathbf{Q}$. We actually expect a similar integral statement to hold.

3.3. **Cuspidal motives.** We assume $\mathbf{G}$ is simple. We give here, for each unipotent rational cuspidal representation V of G, the structure of the various associated endomorphism rings, and the minimal polynomial of the Frobenius endomorphism.

The cuspidal unipotent representations are given in [Lu4]. The information on Frobenius eigenvalues is provided in [Lu1, Lu2] and [Lu4, §11 and 14.2] (cf also [GeMa]).

Recall that $Z\operatorname{End}(V) = Z\operatorname{End}(M_V) = K[F]$ for any cuspidal V. Recall also that $\operatorname{End}_{KG}(V)$ splits at finite places that are not above p.

3.3.1. *Weight* 0 *case (Artin case).* We assume N_V has weight 0.

- Assume $(G,F) \neq ({}^2F_4, V_{21})$. We have $\operatorname{End}_{\mathbf{Q}G}(V) = \mathbf{Q}[F]$ and the table below describes the minimal polynomial of F.

B_{n^2+n}	Φ_2
C_{n^2+n}	Φ_2
$D_{(n+1)^2}$	Φ_2
${}^2D_{(n+1)^2}$	Φ_2
${}^2A_{m(m+1)/2}$,	Φ_2
E_6	Φ_3
E_8	$\Phi_1, \Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6$
F_4	$\Phi_1, \Phi_1, \Phi_2, \Phi_3, \Phi_4$
G_2	Φ_1, Φ_2, Φ_3
${}^3D_4$	Φ_1, Φ_2
${}^2E_6$	Φ_1, Φ_3
${}^2F_4$	$\Phi_2, \Phi_2, \Phi_2, \Phi_4, \Phi_4, \Phi_6$

Here, $n \geq 1$, $m \geq 2$ and $[m(m+1)/4]$ is assumed to be even.

- Assume $(G,V) = ({}^2F_4, V_{21})$. We have $\Phi_2(F) = 0$ and $\operatorname{End}(\tilde{N}_V) = \mathbf{Q}$. It is known that $m_{\mathbf{Q}}(\chi_{21}) = 2$ [Ge1, Theorem 1.6]. Also $m_{\mathbf{Q}_\ell}(\chi_{21}) = 1$ for all $\ell \neq 2$ [Oh2, Proposition 5]. So $\operatorname{End}_{\mathbf{Q}G}(V)$ is the quaternion algebra over $\mathbf{Q}$ that splits at all finite places except at 2. We have $\operatorname{End}(M_V) \simeq \operatorname{End}_{\mathbf{Q}G}(V)$.

We sketch now a conjectural geometrical explanation for the occurence of this quaternion algebra, via the action of a braid monoid. Following [BrMa, 5.11 Folgerung], we expect there is a faithful action of the Hecke algebra

$$\mathcal{H} = \mathbf{Q}\langle T_1, T_2\rangle/(T_1T_2T_1 = T_2T_1T_2, \{T_i^2 + q\sqrt{2}T_i + q^2\}_{i=1,2})$$

on M_V, coming from a geometrical action of the classical Artin braid group B_3 (up to inverting purely inseparable morphisms) on a Deligne-Lusztig variety. Putting $t_i = q^{-1}\sqrt{2}T_i$, we obtain a description of $\mathcal{H}$ as

$$\mathcal{H} = \mathbf{Q}\langle t_1, t_2\rangle/(t_1t_2t_1 = t_2t_1t_2, \{t_i^2 + 2t_i + 2\}_{i=1,2}).$$

There is an algebra decomposition $\mathcal{H} = \mathcal{H}' \times \mathcal{H}''$, where $\mathcal{H}'' \simeq \mathbf{Q}(\sqrt{-2})$ and $\mathcal{H}''$ is a quaternion algebra over $\mathbf{Q}$ splitting at all finite places except 2. We expect the action above induces an isomorphism $\mathcal{H}'' \xrightarrow{\sim} \operatorname{End}(M_V)$. Note that we have deformed the rational group algebra of $\mathfrak{S}_3$ in such a way that the simple 2-dimensional representation is not defined over $\mathbf{Q}$ anymore.

3.3.2. *Weight* 1 *case (supersingular case).* In that case, the invariant of $\mathrm{End}_{\mathbf{Q}G}(V) \simeq \mathrm{End}(\tilde{N}_V)$ is 0 at complex infinite places and is 1/2 at real places.

The possible cases, together with the minimal polynomial of F are given below.

$$\begin{array}{cl} E_7 & X^2+q \\ {}^2A_{m(m+1)/2} & X+\sqrt{q} \\ {}^2B_2 & X^2+\sqrt{2q}X+q \\ {}^2G_2 & X^2+q \\ {}^2G_2 & X^2+\sqrt{3q}X+q \\ {}^2G_2 & X^2+q \end{array}$$

Here $m \geq 2$, $[m(m+1)/4]$ is assumed to be odd and q is an even prime power in the case ${}^2A_{m(m+1)/2}$.

Hasse Invariants of $\mathrm{End}_{\mathbf{Q}G}(V)$:

- $X+\sqrt{q}$, q even prime power. $\mathrm{inv}_\infty = \mathrm{inv}_p = \frac{1}{2}$
- X^2+q
  - $\mathrm{inv}_\nu = \frac{1}{2}$ for ν above p if q is an even power of p and $p \equiv 1 \pmod 4$
  - $\mathrm{End}_{\mathbf{Q}G}(V) = \mathbf{Q}[F]$ otherwise.
- $X^2+2^nX+2^{2n-1}$, $q = 2^{2n-1}$. $\mathrm{End}_{\mathbf{Q}G}(V) = \mathbf{Q}[F]$.
- $X^2+3^nX+3^{2n-1}$, $q = 3^{2n-1}$. $\mathrm{End}_{\mathbf{Q}G}(V) = \mathbf{Q}[F]$.

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