Derived categories and algebraic geometry

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Contents

1. Introduction

We describe some basic properties of the derived category of coherent sheaves on a variety (bounded derived category or perfect complexes).

The first chapter considers the problem of extending vector bundles from an open subset. Thomason and Trobaugh provided an answer to this problem by considering extensions for perfect complexes. This has applications to higher K-theory.

In the second chapter, we explain how to characterize subcategories corresponding to objects supported by a given closed subvariety. This permits a reconstruction of the variety (viewed as a ringed space) from a categorical structure. In the case of derived categories, this requires also the tensor structure.

We start with the classical case of the category of coherent sheaves (after Gabriel). We present afterwards a similar approach in the triangulated case, where serious difficulties arise.

Finally, we explain how to deduce that a smooth projective variety with ample or anti-ample canonical bundle is determined by its derived category.

We haven't included proofs of the results on general properties of abelian or triangulated categories (cf [KaScha, Nee3] for proofs). The only difficult part is Lemma 3.9 on compact objects.

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2. Notations

We fix a field k and we call variety a separated scheme of finite type over k. Given a variety X, we denote by X-coh (resp. X-qcoh) the category of coherent (resp. quasi-coherent) sheaves on X.

All functors between triangulated categories are assumed to be triangulated.

We denote by $Z(\mathcal{C})$ the centre of a category $\mathcal{C}$ (=endomorphisms of the identity functor).

Given a ring R, we denote by R-mod the category of finitely generated R-modules.

3. Localisation

3.1. Abelian case

Let us recall some basic properties of categories of coherent sheaves.

Let X be a variety. Given Z closed in X, we denote by $X\text{-coh}_Z$ the full subcategory of X-coh of coherent sheaves with support contained in Z. This is a Serre subcategory of X-coh.

Let us recall that a full subcategory $\mathcal{I}$ of an abelian category $\mathcal{A}$ is a *Serre* subcategory if it is stable under taking subobjects, quotients and extensions. Given such a subcategory, there is a quotient abelian category $\mathcal{A}/\mathcal{I}$ and a functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ with kernel $\mathcal{I}$. It is the solution of the universal problem of taking quotients (for abelian categories). We say that there is an exact sequence of abelian categories $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0$.

Let $j : U = X - Z \to X$ be the open embedding.

Proposition 3.1. *The functor* $j^* : X\text{-coh} \to U\text{-coh}$ *induces an equivalence* $X\text{-coh} / X\text{-coh}_Z \xrightarrow{\sim} U\text{-coh}$, i.e., *there is an exact sequence of abelian categories*

$$0 \to X\text{-coh}_Z \to X\text{-coh} \to U\text{-coh} \to 0.$$

Proof. In the case of quasi-coherent sheaves, we have a functor j_* right adjoint to j^*. The canonical map $j^* j_* \xrightarrow{\sim} \mathbf{1}_{U\text{-qcoh}}$ is an isomorphism and the kernel of j^* is $X\text{-qcoh}_Z$. It follows from Lemma 3.2 below that there is an exact sequence

$$0 \to X\text{-qcoh}_Z \to X\text{-qcoh} \to U\text{-qcoh} \to 0.$$

Let us now deduce the proposition from the characterisation of coherent sheaves as the finitely presented objects in the category of quasi-coherent sheaves. Lemma 3.3 below shows that the canonical functor $X\text{-coh} / X\text{-coh}_Z \to U\text{-coh}$ is fully faithful. We are left with proving that the functor $j^* : X\text{-coh} \to U\text{-coh}$ is essentially surjective. Let G be a coherent sheaf on U and let $F = j_* G$. We have $j^* F \simeq G$. The quasi-coherent sheaf F is an increasing union (filtered colimit) of its coherent subsheaves, $F = \bigcup_{E \text{ coherent} \subset F} E$. It follows that $j^* F = \bigcup_E j^* E$. Since $j^* F$ is coherent and it is an increasing union of a family of subsheaves, one of the members of the family $j^* E$ is equal to $j^* F$ (the filtered colimit stabilizes after finitely many terms). So, $j^* F = j^* E \simeq G$ for a coherent subsheaf E of F. □

Lemma 3.2. *Let* $F : \mathcal{A} \to \mathcal{B}$ *be an exact functor between abelian categories. Assume* F *has a right adjoint* G *and* G *is fully faithful* (i.e., $FG \xrightarrow{\text{can}} \mathbf{1}_{\mathcal{B}}$ *is an isomorphism).*

Then ker F *is a Serre subcategory of* $\mathcal{A}$ *and there is an exact sequence*

$$0 \to \ker F \to \mathcal{A} \to \mathcal{B} \to 0.$$

Lemma 3.3. *Let* $\mathcal{A}$ *be an abelian category,* $\mathcal{A}'$ *a full abelian subcategory of* $\mathcal{A}$ *and* $\mathcal{I}$ *a Serre subcategory of* $\mathcal{A}$. *Assume that for any* $M \in \mathcal{A}'$ *and for any* $N \in \mathcal{I}$ *a subobject or a quotient of* M, *then* $N \in \mathcal{A}'$.

Then the canonical functor $\mathcal{A}'/(\mathcal{I} \cap \mathcal{A}') \to \mathcal{A}/\mathcal{I}$ *is fully faithful.*

3.2. Triangulated case

3.2.1. Derived functors

We start by recalling some properties of derived categories of sheaves and we explain how to construct right derived functors.

Recall that the canonical functor $D(X\text{-coh}) \to D(X\text{-qcoh})$ is fully faithful, *i.e.*, $D(X\text{-coh})$ is equivalent to the full subcategory of $D(X\text{-qcoh})$ of complexes whose cohomology sheaves are coherent. We will identify those two categories.

Let X-inj be the category of quasi-coherent injective sheaves and $\mathrm{Ho}(X\text{-inj})$ the homotopy category of complexes of objects of X-inj. Consider the canonical functor $\mathrm{Ho}(X\text{-qcoh}) \to D(X\text{-qcoh})$. It has a right adjoint ρ ("homotopically injective resolution"). Let $\mathrm{Ho}(X\text{-qcoh})^{hi}$ be its essential image (homotopically injective complexes). The functor ρ is fully faithful, the canonical functor $\mathrm{Ho}(X\text{-qcoh})^{hi} \xrightarrow{\sim} D(X\text{-qcoh})$ is an equivalence with inverse ρ. The intersection of $\mathrm{Ho}(X\text{-qcoh})^{hi}$ with $D^+(X\text{-qcoh})$ is $\mathrm{Ho}^+(X\text{-inj})$, *i.e.*, ρ restricts to an equivalence $D^+(X\text{-qcoh}) \xrightarrow{\sim} \mathrm{Ho}^+(X\text{-inj})$: we recover the classical injective resolutions.

Let us now discuss the derivation of a left exact functor $F : X\text{-qcoh} \to \mathcal{A}$, where $\mathcal{A}$ is an abelian category. We extend F to a functor $\mathrm{Ho}(F) : \mathrm{Ho}(X\text{-qcoh}) \to \mathrm{Ho}(\mathcal{A})$. We restrict this functor to $\mathrm{Ho}(X\text{-qcoh})^{hi}$. We obtain the right derived functor

$$RF : D(X\text{-qcoh}) \underset{\sim}{\xrightarrow{\rho}} \mathrm{Ho}(X\text{-qcoh})^{hi} \xrightarrow{\mathrm{Ho}(F)} \mathrm{Ho}(\mathcal{A}) \xrightarrow{\mathrm{can}} D(\mathcal{A}).$$

The functor RF is triangulated. In particular, the image of a distinguished triangle is a distinguished triangle, while F needs not send an exact sequence to an exact sequence. The left exactness of F shows that $H^0(RF(M)) \xrightarrow{\sim} F(M)$ for $M \in X\text{-qcoh}$.

3.2.2. Open subvarieties and quotients

The functor j_* derives into a functor $Rj_* : D(U\text{-qcoh}) \to D(X\text{-qcoh})$. This is right adjoint to the functor $j^* : D(X\text{-qcoh}) \to D(U\text{-qcoh})$. The kernel of the functor j^* is $D_Z(X\text{-qcoh})$, the full subcategory of $D(X\text{-qcoh})$ of complexes whose cohomology sheaves have their support contained in Z. This is a thick subcategory.

Let us recall that a non-empty full subcategory $\mathcal{I}$ of a triangulated category $\mathcal{T}$ is *thick* if the following two conditions hold

- Given $F \to G \to H \rightsquigarrow$ a distinguished triangle in $\mathcal{T}$, if two objects amongst F, G and H are in $\mathcal{I}$, then the third one is in $\mathcal{I}$ as well.
- Given $F, G \in \mathcal{T}$, if $F \oplus G \in \mathcal{I}$, then $F, G \in \mathcal{I}$.

There is a quotient triangulated category $\mathcal{T}/\mathcal{I}$ and a functor $\mathcal{T} \to \mathcal{T}/\mathcal{I}$ with kernel $\mathcal{I}$, solution of the universal quotient problem (amongst triangulated categories). We say that there is an exact sequence of triangulated categories $0 \to \mathcal{I} \to \mathcal{T} \to \mathcal{T}/\mathcal{I} \to 0$.

Lemma 3.4 gives an exact sequence of triangulated categories

$$0 \to D_Z(X\text{-qcoh}) \to D(X\text{-qcoh}) \to D(U\text{-qcoh}) \to 0$$

Lemma 3.4. *Let $F : \mathcal{T} \to \mathcal{T}'$ be a functor between triangulated categories. Assume F has a right adjoint G and G is fully faithful.*

Then $\ker F$ *is a thick subcategory of $\mathcal{T}$ and there is an exact sequence*

$$0 \to \ker F \to \mathcal{T} \to \mathcal{T}' \to 0.$$

3.2.3. Perfect complexes

We come now to the core of our study. We recall the notion of perfect objects and their basic properties.

An object of $D(X\text{-qcoh})$ is *perfect* if it is locally (quasi)-isomorphic to a bounded complex of free sheaves of finite rank. We denote by X-perf the full subcagegory of $D(X\text{-qcoh})$ of perfect complexes. This is a thick subcategory of $D^b(X\text{-coh})$. If X is quasi-projective, then a complex is perfect if and only if it is quasi-isomorphic to a bounded complex of vector bundles. The variety X is regular if and only if $D^b(X\text{-coh}) = X\text{-perf}$.

Let $\mathcal{T}$ be a triangulated category with infinite direct sums. An object $C \in \mathcal{T}$ is *compact* if given any family $\mathcal{E}$ of objects of $\mathcal{T}$, the canonical map $\bigoplus_{E\in\mathcal{E}} \operatorname{Hom}(C, E) \to \operatorname{Hom}(C, \bigoplus_{E\in\mathcal{E}} E)$ is an isomorphism. We denote by $\mathcal{T}^c$ the full subcategory of $\mathcal{T}$ of compact objects. This is a thick subcategory.

A key idea in Thomason's approach is the characterisation of perfect complexes as the compact objects of $D(X\text{-qcoh})$ (cf [Rou2] for a study of compactness as a local property for general triangulated categories).

Lemma 3.5. *Let $C \in D(X\text{-qcoh})$. Then C is perfect if and only if it is compact.*

Proof. Let us assume first X is affine, $X = \operatorname{Spec} R$. Since R is compact, we deduce that every perfect object is compact. Let C be a complex of R-modules and let $i \in \mathbf{Z}$ such that $H^iC \neq 0$. Then $\operatorname{Hom}(R, C[i]) \neq 0$. It follows from Lemma 3.9 that the perfect complexes are the same as the compact objects.

We consider now an arbitrary variety X. Let $X = U_1 \cup U_2$ with U_1 an affine open subvariety and U_2 open. We assume that the minimal number of open affine subvarieties in a covering of U_2 is strictly less that the number for X. By induction, we can assume the lemma holds for U_2 and for $U_{12} = U_1 \cap U_2$. Let $j_r : U_r \to X$ and $j_{12} : U_{12} \to X$ be the open immersions. Given

$D \in D(X\text{-qcoh})$, there is a Mayer-Vietoris distinguished triangle:

$$D \to Rj_{1*}j_1^*D \oplus Rj_{2*}j_2^*D \to Rj_{12*}j_{12}^*D \rightsquigarrow$$

Let $C \in D(X\text{-qcoh})$. The triangle above shows that C is compact if j_1^*C, j_2^*C and j_{12}^*C are compact. The converse is clear. On the other hand, C is perfect if and only if j_1^*C, j_2^*C and j_{12}^*C are perfect. The lemma follows by induction. □

An important aspect of the category X-perf is that it provides the "right" K-theory groups, for a variety without enough ample vector bundles. We put $K_0(X) = K_0(X\text{-perf})$.

Recall that given a triangulated category $\mathcal{T}$, we define $K_0(\mathcal{T})$ as the quotient of the free abelian group with basis the isomorphism classes of objects of $\mathcal{T}$ by the relation $[M] = [L] + [N]$ whenever there is a distinguished triangle $L \to M \to N \rightsquigarrow$.

This definition of $K_0(X)$ coincides with the classical one (Grothendieck group of the exact category of vector bundles) when X has an ample family of line bundles (this is the case for a quasi-projective variety).

3.2.4. Extensions of perfect complexes

We put $X\text{-perf}_Z = X\text{-perf} \cap D_Z(X\text{-qcoh})$.

Theorem 3.6 (Thomason-Trobaugh). *The functor j^* induces a fully faithful functor*

$$X\text{-perf}\,/\,X\text{-perf}_Z \to U\text{-perf}.$$

An object of U-perf is the restriction of an object of X-perf if and only if its class in $K_0(U)$ is the restriction of an element of $K_0(X)$.

The occurrence of K_0 in Theorem 3.6 comes from the following lemma of Thomason [Th2, Theorem 2.1] (the proof is tricky).

Lemma 3.7. *Let $\mathcal{T}$ be a triangulated category. There is a bijection from the set of full triangulated subcategories $\mathcal{I}$ of $\mathcal{T}$ that generate $\mathcal{T}$ as a thick subcategory to the set of subgroups of $K_0(\mathcal{T})$ given by sending $\mathcal{I}$ to the image of $K_0(\mathcal{I})$ in $K_0(\mathcal{T})$.*

The calculus of fractions gives a simple criterion for fully faithfulness:

Lemma 3.8. *Let $\mathcal{T}$ be a triangulated category, $\mathcal{I}$ a thick subcategory of $\mathcal{T}$ and $\mathcal{T}'$ a full triangulated subcategory of $\mathcal{T}$.*

Assume every morphism $C \to D$ with $C \in \mathcal{T}'$ and $D \in \mathcal{I}$ factors through an object of $\mathcal{I} \cap \mathcal{T}'$. Then the canonical functor $\mathcal{T}'/(\mathcal{I} \cap \mathcal{T}') \to \mathcal{T}/\mathcal{I}$ is fully faithful.

Given $\mathcal{I}$ a full subcategory of a triangulated category $\mathcal{T}$, we denote by $\bar{\mathcal{I}}$ the smallest thick subcategory of $\mathcal{T}$ closed under taking infinite direct sums and containing $\mathcal{I}$.

Let $\mathcal{I}$ be a thick subcategory of a triangulated category $\mathcal{T}$. The right orthogonal $\mathcal{I}^\perp$ to $\mathcal{I}$ in $\mathcal{T}$ is the full subcategory of $\mathcal{T}$ of objects D with $\operatorname{Hom}(C, D) = 0$ for all $C \in \mathcal{I}$. This is a thick subcategory.

The following lemma is related to Brown-Neeman's representability Theorem and its proof requires some work [Nee2].

Lemma 3.9. *Let $\mathcal{T}$ be a triangulated category with arbitrary direct sums. Let $\mathcal{I}$ be a thick subcategory of $\mathcal{T}^c$. Then every map from an object of $\mathcal{T}^c$ to an object of $\bar{\mathcal{I}}$ factors through an object of $\mathcal{I}$. In particular, we have $\mathcal{T}^c \cap \bar{\mathcal{I}} = \mathcal{I}$.*

We have $\bar{\mathcal{I}} = \mathcal{T}$ if and only if the right orthogonal $\mathcal{I}^\perp$ of $\mathcal{I}$ in $\mathcal{T}$ vanishes.

Lemma 3.10. *Let Y be a closed subvariety of X. We have $D_Y(X\text{-qcoh}) = \overline{X\text{-perf}_Y}$.*

Given Z' closed in X, we put $K_0(X \text{ on } Z') = K_0(X\text{-perf}_{Z'})$. We will show a version "with supports" of Theorem 3.6:

Theorem 3.11. *Let Z' be a closed subvariety of X. The functor j^* induces a fully faithful functor $X\text{-perf}_{Z'}/X\text{-perf}_{Z\cap Z'} \to U\text{-perf}_{U\cap Z'}$. An object of $U\text{-perf}_{U\cap Z'}$ is the image of an object of $X\text{-perf}_{Z'}$ if and only if its class in $K_0(U \text{ on } U \cap Z')$ is the image of an element of $K_0(X \text{ on } Z')$.*

Proof of Theorem 3.11 and Lemma 3.10. Let us show first that the lemma for X implies the theorem for X. The combination of Lemmas 3.8, 3.9 and 3.10 shows that j^* induces a fully faithful functor $X\text{-perf}_{Z'}/X\text{-perf}_{Z\cap Z'} \to U\text{-perf}_{U\cap Z'}$. Let $\mathcal{I}$ be the image of that functor: this is a full triangulated subcategory. Since $\overline{X\text{-perf}_{Z'}} = D_{Z'}(X\text{-qcoh})$ (lemma 3.10), we have $\bar{\mathcal{I}} = D_{U\cap Z'}(U\text{-qcoh})$. It follows from Lemma 3.9 that $U\text{-perf}_{Z'\cap U}$ is the thick subcategory generated by $\mathcal{I}$. The theorem follows now from Lemma 3.7.

Lemma 3.9 shows that Lemma 3.10 will follow from the fact that $(X\text{-perf}_Y)^\perp = 0$.

Let us assume first that X is affine. Let $\{y_1, \dots, y_r\}$ be a family of generators of the defining ideal of Y and let

$$G_r = \bigotimes_{i=1}^{r} (0 \to \mathcal{O}_X \xrightarrow{y_i} \mathcal{O}_X \to 0)$$

be the associated Koszul complex (the non-zero terms are in degrees $-r, \dots, 0$). We will show by induction on r that an object $C \in D_Y(X\text{-qcoh})$ vanishes if $G_r \otimes C = 0$. The lemma will follow, since $\mathrm{Hom}(G_r^\vee, C[i]) \simeq H^i(G_r \otimes C)$, where $G_r^\vee = R\,\mathrm{Hom}(G_r, \mathcal{O}_X)$ is the dual of G_r. The case $r = 0$ is clear.

Consider $r > 0$ and $C \in D_Y(X\text{-qcoh})$ a non-zero object. By induction, there exists i such that $H^i(G_{r-1} \otimes C) \neq 0$. The distinguished triangle

$$G_{r-1} \otimes C \xrightarrow{y_r} G_{r-1} \otimes C \to G_r \otimes C \rightsquigarrow$$

gives an exact sequence $H^{i-1}(G_r \otimes C) \to H^i(G_{r-1} \otimes C) \xrightarrow{y_r} H^i(G_{r-1} \otimes C)$. Since $H^i(G_{r-1} \otimes C)$ is supported by the closed subvariety $(y_r = 0)$, we deduce that multiplication by y_r has a non-zero kernel, hence $H^{i-1}(G_r \otimes C) \neq 0$. This completes the proof of Lemma 3.10 in the affine case.

We will now prove the lemma by induction on the minimal number of open affine subsets in a covering of X.

Let $X = U_1 \cup U_2$ with U_1 open affine and U_2 open for which the lemma holds. We put $Z_i = X - U_i$. Let $C \in D_Y(X\text{-qcoh})$ with $\mathrm{Hom}(D, C) = 0$ for all $D \in X\text{-perf}_Y$.

Let $D \in U_1\text{-perf}_{Y \cap Z_2}$. The functor $Rj_{1*} : D(U_1\text{-qcoh}) \to D(X\text{-qcoh})$ restricts to equivalences $D_{Y\cap Z_2}(U_1\text{-qcoh}) \xrightarrow{\sim} D_{Y \cap Z_2}(X\text{-qcoh})$ and $U_1\text{-perf}_{Y\cap Z_2} \xrightarrow{\sim} X\text{-perf}_{Y \cap Z_2}$. It follows that $\mathrm{Hom}(Rj_{1*}D, C) = 0$. Let C' be the cocone of the adjunction morphism $C \xrightarrow{\text{can}} Rj_{2*}j_2^*C$. We have

$$\mathrm{Hom}(Rj_{1*}D, Rj_{2*}j_2^*C[n]) \simeq \mathrm{Hom}(j_2^*Rj_{1*}D, j_2^*C[n]) = 0$$

for all n, hence $\mathrm{Hom}(Rj_{1*}D, C') \simeq \mathrm{Hom}(Rj_{1*}D, C) = 0$. Since C' is supported by $Y \cap Z_2$, there exists $C'' \in D_{Y\cap Z_2}(U_1\text{-qcoh})$ such that $C' = Rj_{1*}C''$. We have $\mathrm{Hom}(D, C'') \simeq \mathrm{Hom}(Rj_{1*}D, C') = 0$. The affine case of the lemma shows that $C'' = 0$, hence $C \simeq Rj_{2*}j_2^*C$.

Let $E' \in U_2\text{-perf}_{Y\cap U_2}$, $E = E' \oplus E'[1]$ and $G = E_{|U_1\cap U_2}$. The affine case of the theorem shows that there exists $F \in U_1\text{-perf}_{Y\cap U_1}$ and an isomorphism $F_{|U_1 \cap U_2} \xrightarrow{\sim} G$. Let now D be the cocone of the morphism sum of the adjunction morphisms $Rj_{2*}E \oplus Rj_{1*}F \to Rj_{12*}G$. Then $j_1^*D \simeq F$ and $j_2^*D \simeq E$, hence $D \in X\text{-perf}_Y$. We have $\mathrm{Hom}(E, j_2^*C) \simeq \mathrm{Hom}(D, Rj_{2*}j_2^*C) = 0$. By induction, we deduce that $j_2^*C = 0$, hence $C = 0$. This proves Lemma 3.10 for X. □

Exercice 3.1. Let $C \in X\text{-qcoh}$ such that for any set I of objects of X-qcoh, the canonical map $\bigoplus_{D\in I} \mathrm{Hom}(C, D) \to \mathrm{Hom}(C, \bigoplus_{D\in I} D)$ is an isomorphism. Show that C is coherent.

A striking special case of Theorem 3.6 is given by the following corollary.

Corollary 3.12. *Let $\mathcal{L}$ be a vector bundle on U. Then there exists a perfect complex on X whose restriction to U is quasi-isomorphic to $\mathcal{L} \oplus \mathcal{L}[1]$.*

Remark 3.13. Let us show, following Serre [Se, 5.a, p.371], that Theorem 3.6 doesn't hold for vector bundles.

Let $X = \mathbf{A}^3$ and $U = X - \{0\}$. Let $\mathcal{F}$ be the vector bundle on U which is the pullback of the tangent bundle on $\mathbf{P}^2$. The restriction map $K_0(X) \to K_0(U)$ is an isomorphism. Since $\mathcal{F}$ is not the direct sum of two line bundles, it is not the restriction of a vector bundle on X.

Let $\mathcal{G}$ be a coherent sheaf on X extending $\mathcal{F}$. The second syzygy $\Omega^2\mathcal{G}$ of $\mathcal{G}$ is locally free (hence free) and this provides a perfect complex extending $\mathcal{F}$: there is a complex of free sheaves

$$0 \to \Omega^2\mathcal{G} \to P^{-1} \to P^0 \to 0$$

with homology concentrated in degree 0 and isomorphic to $\mathcal{G}$.

Remark 3.14. A proof similar to that of Proposition 3.1 shows that there is an exact sequence $0 \to D^b_Z(X\text{-coh}) \to D^b(X\text{-coh}) \to D^b(U\text{-coh}) \to 0$. When X is smooth, then X-perf $= D^b(X\text{-coh})$, hence Theorem 3.6 is a consequence of that exact sequence. In this case, the canonical map $K_0(X) \to K_0(U)$ is surjective.

3.2.5. Applications to K-theory

From Theorem 3.6, Thomason deduces a long exact sequence for higher K-theory, via Waldhausen's theory [Th1]:

Theorem 3.15. *There is a long exact sequence*

$$\cdots \to K_i(X \text{ on } Z) \to K_i(X) \to K_i(U) \to K_{i-1}(X \text{ on } Z) \to \cdots$$

Thomason deduces also an excision result.

Theorem 3.16. *If $X = U \cup V$ with V open and $Z \subset V$, then there are isomorphisms $K_i(X \text{ on } Z) \xrightarrow{\sim} K_i(V \text{ on } Z)$.*

Finally, he obtains a Mayer-Vietoris Theorem.

Theorem 3.17. *Let U and V be open subsets of X. Then there is a long exact sequence*

$$\cdots \to K_i(U \cup V) \to K_i(U) \oplus K_i(V) \to K_i(U \cap V) \to K_{i-1}(U \cup V) \to \cdots$$

These sequences can be extended to negative i, via a version of Bass' fundamental Theorem:

Theorem 3.18. *There is an exact sequence*

$$0 \to K_i(X) \to K_i(X[T]) \oplus K_i(X[T^{-1}]) \to K_i(X[T, T^{-1}]) \to K_{i-1}(X) \to 0$$

Classical methods based on exact categories had led to similar results under restrictive assumptions. Thomason obtains also a local-global principle for K_i's.

4. Reconstruction

4.1. Abelian case

We will start with the classical case of coherent sheaves, following Gabriel [Ga].

4.1.1. Classification of Serre subcategories

We say that a Serre subcategory $\mathcal{I}$ of an abelian category $\mathcal{A}$ is of *finite type* if it is generated by an object (*i.e.*, the smallest Serre subcategory of $\mathcal{A}$ containing the object is $\mathcal{I}$). We say that a Serre subcategory $\mathcal{I}$ is *irreducible* if it is not equal to 0 and if it is not generated by two proper Serre subcategories of $\mathcal{I}$.

Theorem 4.1 (Gabriel). *The map $Z \mapsto X\text{-coh}_Z$ from the set of closed subsets of X to the set of Serre subcategories of finite type of X-coh is a bijection.*

The closed irreducible subsets correspond to the irreducible Serre subcategories.

This follows immediately from the next lemma:

Lemma 4.2. *A coherent sheaf with support Z generates $X\text{-coh}_Z$ as a Serre subcategory.*

Proof. Let F be a coherent sheaf with support Z and let $\mathcal{I}$ be the Serre subcategory of X-coh generated by F. Let $i : Y \to X$ be the closed embedding of a subvariety. Every coherent sheaf on X supported by Y is an extension of sheaves of the form i_*G. Let $\mathcal{J}$ be the Serre subcategory of Y-coh generated by i^*F. Since $i_*i^*F \in \mathcal{I}$, we have $i_*(\mathcal{J}) \subset \mathcal{I}$. If $\mathcal{J} = Y\text{-coh}_{Y\cap Z}$, then $X\text{-coh}_{Y\cap Z} \subset \mathcal{I}$. If in addition $Z \subset Y$, then $\mathcal{I} = X\text{-coh}_Z$.

It follows that it is enough to prove the lemma for X reduced and $Z = X$. We proceed by induction on the dimension n of X, then on the number of irreducible components of dimension n of X, then on the number of irreducible components of dimension $n-1$ of X, etc.

Let Y be a proper closed subset of X. The discussion above shows that by induction we can assume $X\text{-coh}_Y \subset \mathcal{I}$.

Let M be a coherent sheaf on X. Let U be an irreducible open affine subset of X and $j : U \to X$ the open immersion. Shrinking U if necessary, the sheaves j^*M and j^*F are free, of respective ranks r and $s > 0$. Let $f : j^*F^r \xrightarrow{\sim} j^*M^s$ be an isomorphism. By Proposition 3.1, there is a coherent sheaf M' on X, there are $\psi : F^r \to M'$ and $\phi : M^s \to M'$ such that $j^*(\psi) = j^*(\phi)f$ and $j^*(\phi)$ is an isomorphism. The kernels and cokernels of ϕ and ψ have their supports contained in $X - U$, hence they are in $\mathcal{I}$. It follows that $M \in \mathcal{I}$. □

4.1.2. Centres

Lemma 4.3. *Let R be a ring. The canonical map $Z(R) \to Z(R\text{-mod})$ is an isomorphism.*

Proof. Evaluation at R gives a left inverse. Let $\alpha \in Z(R\text{-mod})$ with $\alpha(R) = 0$. Since any R-module is a quotient of a free R-module, it follows that $\alpha = 0$. □

Corollary 4.4 (Gabriel). *The abelian category X-coh determines the variety X.*

Proof. We define a ringed space $\mathcal{E}$. Its points are the irreducible Serre subcategories of finite type of X-coh. The open subsets are the $D(\mathcal{I})$, defined as the set of those subcategories $\mathcal{J}$ that are not contained in a given Serre subcategory $\mathcal{I}$.

Theorem 4.1 shows that the map sending a point $x \in X$ to $X\text{-coh}_{\overline{\{x\}}}$ defines a homeomorphism $X \to \mathcal{E}$.

Consider the presheaf of rings on $\mathcal{E}$ given by $\mathcal{O}'_{\mathcal{E}}(D(\mathcal{I})) = Z(X\text{-coh}/\mathcal{I})$. If $D(\mathcal{I}') \subset D(\mathcal{I})$, then the quotient functor $X\text{-coh}/\mathcal{I} \to X\text{-coh}/\mathcal{I}'$ induces a map $Z(X\text{-coh}/\mathcal{I}) \to Z(X\text{-coh}/\mathcal{I}')$. We denote by $\mathcal{O}_{\mathcal{E}}$ the associated sheaf. The canonical map $\Gamma(U) \to Z(U\text{-coh})$ induces a morphism of ringed spaces $X \to \mathcal{E}$. To check that this is an isomorphism, it is enough to consider its restriction to an open affine subset: Lemma 4.3 provides the conclusion in the affine case. □

Remark 4.5. Actually, Lemma 4.3 holds for non-affine varieties: the canonical map $\Gamma(\mathcal{O}_X) \to Z(X\text{-coh})$ is an isomorphism of rings. That follows from the construction of the category X-coh by gluing the abelian categories U_i-coh along the quotient categories $(U_i \cap U_j)$-coh, given a finite open covering of X by open affine subsets U_i.

As a consequence, the presheaf in the proof of Corollary 4.4 is a sheaf.

4.2. Triangulated case

4.2.1. Classification of thick subcategories

Inspired by work on the stable homotopy category (description of the chromatic tower), Hopkins and Neeman [Ho, Nee1] have given a classification of thick subcategories of the category of perfect complexes over an affine variety. This result was generalised later by Thomason [Th2].

Let $\mathcal{I}$ be a thick subcategory of X-perf. We say that $\mathcal{I}$ is of *finite type* if it is generated by an object (*i.e.*, if $\mathcal{I}$ is the smallest thick subcategory of X-perf containing the object). We say that $\mathcal{I}$ is an *ideal* if it is thick and if given any $C \in \mathcal{I}$ and $D \in X$-perf then $C \otimes^{\mathbf{L}} D \in \mathcal{I}$. We say that an ideal $\mathcal{I}$ is *irreducible* if it is non-zero and it is not generated by two proper ideals.

Theorem 4.6 (Hopkins, Neeman, Thomason). *The map $Z \mapsto X\text{-perf}_Z$ from the set of closed subsets of X to the set of ideals of finite type of X-perf is a bijection.*

Irreducible closed subsets correspond to irreducible subcategories.

The starting point is the following Lemma.

Lemma 4.7. *Let Z be a closed subset of X. Then there exists a perfect complex on X with support Z.*

Proof. Assume first Z is irreducible and $X = \operatorname{Spec} R$ is affine. Consider equations $f_1 = 0, \ldots, f_n = 0$ defining Z. The support of $\bigotimes_i (0 \to R \xrightarrow{f_i} R \to 0)$ is Z: this solves the lemma in that case.

We assume now that Z is irreducible but X is not necessarily affine. Let U be an open affine subset of X containing the generic point of Z. Then there exists $C \in U$-perf with support $U \cap Z$. The version "with supports" of the localisation theorem (Theorem 3.11) shows that there exists $D \in X\text{-perf}_Z$ such that $D_{|U} \simeq C \oplus C[1]$. We have $\operatorname{Supp}(D) \cap U = Z \cap U$, hence $\operatorname{Supp}(D) = Z$.

We consider finally the general case. Let $Z = Z_1 \cup \cdots \cup Z_r$ be the decomposition in irreducible components and consider $C_i \in X$-perf with support Z_i. Then the support of $\bigoplus C_i$ is Z. □

Theorem 4.6 follows now from the next lemma.

Lemma 4.8. *A perfect complex on X with support Z generates $X\text{-perf}_Z$ as an ideal.*

Proof. Let $C \in X$-perf with support Z and let $\mathcal{I}$ be the ideal of X-perf generated by C. Lemmas 3.9 and 3.10 show that $\mathcal{I} = X\text{-perf}_Z$ if and only if $D^b_Z(X\text{-coh}) \subset \bar{\mathcal{I}}$.

Given a closed immersion $i : Y \to X$, we have $i_* Li^* C \simeq C \otimes^{\mathbf{L}} \mathcal{O}_Y \in \bar{\mathcal{I}}$ since $\mathcal{O}_Y \in D(X\text{-qcoh}) = \overline{X\text{-perf}}$ (lemma 3.9). In addition, $Li^*C \in Y\text{-perf}_{Y\cap Z}$. Every object of $D^b_{Y\cap Z}(X\text{-coh})$ is a finite extension (=iterated cone) of objects i_*G with $G \in D^b_{Y\cap Z}(Y\text{-coh})$. Let $\mathcal{J}$ be the ideal of $Y\text{-perf}_{Y\cap Z}$ generated by Li^*C. Since $i_*(Li^*C \otimes^{\mathbf{L}} M) \simeq C \otimes^{\mathbf{L}} i_*M$ for all $M \in D(Y\text{-qcoh})$, we have $i_*(\bar{\mathcal{J}}) \subset \bar{\mathcal{I}}$. If $\bar{\mathcal{J}} = D_{Y\cap Z}(Y\text{-qcoh})$, then $D^b_{Y\cap Z}(X\text{-coh}) \subset i_*(\bar{\mathcal{J}})$. If in addition $Z \subset Y$, then $\mathcal{I} = X\text{-perf}_Z$.

It follows that it is enough to prove the lemma for X reduced and $Z = X$. We proceed by induction on the dimension n of X, then on the number of irreducible components of dimension n of X, then on the number of irreducible components of dimension $n-1$ of X, etc.

Let Y be a proper closed subset of X. The discussion above shows that $X\text{-perf}_Y \subset \mathcal{I}$.

Let $M \in X\text{-perf}$. There is a non-empty open affine subset $j : U \to X$ such that j^*M and j^*C are finite sums of complexes $\mathcal{O}_U[r]$. So, there are finite-dimensional graded vector spaces (viewed as complexes with vanishing differential) $V \neq 0$ and W and there is an isomorphism $f : j^*(C \otimes_k W) \xrightarrow{\sim} j^*(M \otimes_k V)$. Theorem 3.6 shows that there is an object $M' \in X\text{-perf}$, maps $\psi : C \otimes_k W \to M'$ and $\phi : M \otimes_k V \to M'$ such that $j^*(\psi) = j^*(\phi) f$ and $j^*(\phi)$ is an isomorphism. The cones of ϕ and ψ have a support contained in the closed subset $X - U$ of X, so they are in $\mathcal{I}$ by induction. It follows that $M \in \mathcal{I}$. □

Remark 4.9. The classical proof of Theorem 4.6 uses the following result ("tensor nilpotence Theorem").

Let $C \in X\text{-perf}$, let $D \in D(X\text{-qcoh})$ and let $f : C \to D$. We assume that for every point x of X, we have $f \otimes k(x) = 0$ in $D(k(x)\text{-Mod})$.

Then there is an integer n such that $\otimes^n f : \otimes^n C \to \otimes^n D$ vanishes in $D(X\text{-qcoh})$.

4.2.2. Centres

We proceed as in §4.1.2 to obtain a reconstruction Theorem [Ba1, Rou1].

Let X be a variety. We define $Z(X\text{-perf})_{\text{lnil}}$ as the subring of $Z(X\text{-perf})$ given by elements α such that $\alpha(C)$ is nilpotent for all $C \in X\text{-perf}$. We put $Z(X\text{-perf})_{\text{lred}} = Z(X\text{-perf})/Z(X\text{-perf})_{\text{lnil}}$.

Lemma 4.10. *The canonical morphism $\Gamma(\mathcal{O}_X) \to Z(X\text{-perf})$ induces an isomorphism $\Gamma(\mathcal{O}_X)_{\text{red}} \xrightarrow{\sim} Z(X\text{-perf})_{\text{lred}}$.*

Proof. Evaluation at $\mathcal{O}_X$ gives a left inverse to the canonical map $\Gamma(\mathcal{O}_X) \to Z(X\text{-perf})$.

Assume first $X = \operatorname{Spec} R$ is affine. Let $\alpha \in Z(R\text{-perf})$ such that $\alpha(R) = 0$. A perfect complex is quasi-isomorphic to a bounded complex C of finitely generated projective R-modules. Let $n = \max\{i | C^i \neq 0\} - \min\{i | C^i \neq 0\}$. By induction on n, one sees that $\alpha(C)^{n+1} = 0$ and the lemma follows in the affine case.

Consider now an arbitrary variety X and $\alpha \in Z(X\text{-perf})$ such that $\alpha(\mathcal{O}_X) = 0$. Let U be an open affine subset of X and let $\alpha_U \in Z(U\text{-perf})$ be the element induced by α. We have $\alpha_U(\mathcal{O}_U) = 0$, hence for any $C \in U\text{-perf}$, the endomorphism $\alpha_U(C)$ is nilpotent. Let $X = U_1 \cup \cdots \cup U_r$ be a covering by open affine subsets and let $V = U_2 \cup \cdots \cup U_r$. We prove the lemma by induction on r. Let $C \in X\text{-perf}$ and $n > 0$ such that $\alpha_{U_1}(C_{|U_1})^n = 0$. Then $\alpha(C)^n$ factors through an object $C' \in X\text{-perf}_Z$, where $Z = X - U_1$ and $\alpha(C)^{(d+1)n}$ factors through $\alpha(C')^{dn}$ for all $d \geq 0$. Since $Z \subset V$, the restriction functor $X\text{-perf}_Z \to V\text{-perf}_Z$ is fully faithful. By induction, $\alpha_V(C'_{|V})$ is nilpotent, hence $\alpha(C')$ is nilpotent and $\alpha(C)$ is nilpotent as well. □

Theorem 4.11 (Balmer, R.). *If X is a reduced variety, then the category X-perf, viewed as a tensor triangulated category, determines X.*

It would be more satisfactory not to use the tensor structure. Unfortunately, there are too many thick subcategories in general. Balmer has developed a general approach to the study of the geometry of tensor triangulated categories [Ba2, BaFl].

Example 4.12. Let $X = \mathbf{P}^1$. Then the thick subcategory of X-perf generated by $\mathcal{O}$ is equivalent to $D^b(k\text{-mod})$. It is not of the form $X\text{-perf}_Z$.

4.2.3. Remarks on centres

Let us discuss in more details centres of categories of perfect complexes. Note that the difficulties are due to the weakness of the axioms of triangulated categories and can be solved using dg-categories [To]. The next two Propositions show that the centre of the category of perfect complexes has no non-zero nilpotent elements in some cases.

Proposition 4.13. *Let R be a ring without zero divisors. Then the canonical map $Z(R) \to Z(R\text{-perf})$ is an isomorphism.*

Proof. Evaluation at R defines a morphism $\alpha : Z(R\text{-perf}) \to Z(R)$. The canonical map $Z(R) \to Z(R\text{-perf})$ is a right inverse. We will show that α is injective.

Let $z \in Z(R\text{-perf})$ with $z(R) = 0$. Let C be a non-zero bounded complex of finitely generated projective R-modules. Consider r minimal such that $C^r \neq 0$ and s maximal such that $C^s \neq 0$. We will show by induction on $s - r$ that $z(C) = 0$. This is clear when $s = r$. Assume $s > r$.

Since C^r is a finitely generated projective module, there is a finitely generated projective module P such that $C^r \oplus P$ is a free module of finite rank. Let $D = C \oplus (0 \to P \xrightarrow{\text{id}} P \to 0)$, where the non-zero terms of the complex $0 \to P \xrightarrow{\text{id}} P \to 0$ are in degrees r and $r+1$. Then D^r is free of finite rank and C is homotopy equivalent to D. So, it is enough to prove that $z(C) = 0$ when C^r is free.

We proceed now by induction on the rank of C^r. Let $C^r = L_1 \oplus \cdots \oplus L_n$ be a decomposition into free modules of rank 1. Consider an integer i with $1 \le i \le n$. Let D be the subcomplex of C given by $D^l = C^l$ pour $l \neq r$ and $D^r = L_1 \oplus \cdots \oplus L_{i-1} \oplus L_{i+1} \oplus \cdots \oplus L_n$. By induction, we have $z(D) = 0$. It follows that the composition $f : D \xrightarrow{\text{can}} C \xrightarrow{z(C)} C$ is homotopic to 0. Consider $\{h^l : D^l \to C^{l-1}\}_l$ with $f = d_C h + h d_D$. The morphism h extends uniquely into a graded endomorphism k of degree -1 of C that vanishes on L_i. Let $\psi = z(C) - (d_C k + k d_C)$, a map homotopic to $z(C)$. The composition $D \xrightarrow{\text{can}} C \xrightarrow{\psi} C$ vanishes, hence $\psi^l = 0$ for $l \neq r$ and $D^r \subset \ker \psi^r$. Note that $\operatorname{im} \psi^r \subset \ker d_C^r$. If $\psi^r \neq 0$ then $\ker d_C^r \neq 0$. Since $C^r/D^r \simeq L_i$ is free of rank 1 and since C^r is free, if ψ^r is non-zero, then its restriction to L_i is injective (recall that R has no zero-divisor) and $\ker \psi^r = D^r$.

The composition $C \xrightarrow{\psi} C \xrightarrow{\text{can}} C^r[-r]$ is homotopic to 0 since $z(C^r[-r]) = 0$. So, there is $g : C^{r+1} \to C^r$ such that $\psi^r = g d_C^r$, hence $\ker d_C^r \subset \ker \psi^r$. Assume $z(C) \neq 0$. Then $\ker d_C^r \subset L_1 \oplus \cdots \oplus L_{i-1} \oplus L_{i+1} \oplus \cdots \oplus L_n$. This holds for all i, hence $\ker d_C^r = 0$. It follows that $\psi^r = 0$, hence $\psi = 0$ and finally $z(C) = 0$. □

Proposition 4.14. *Let X be an irreducible reduced quasi-projective variety. Then, the canonical morphism $\Gamma(X, \mathcal{O}_X) \to Z(X\text{-perf})$ is an isomorphism.*

Proof. Every object of X-perf is isomorphic to a bounded complex C whose terms are direct sums of line bundles. The proof is then the same as that of Proposition 4.13 with free modules of rank 1 replaced by line bundles. □

Remark 4.15. Let k be a field and let $\mathcal{C}$ be a k-linear category with finite dimensional Hom-spaces. Let P be an indecomposable object of $\mathcal{C}$ and let $\phi \in Z(\operatorname{End}(P))$ with the following properties:

- $\phi^2 = 0$
- given $x \in \operatorname{End}(P)$ such that $\phi x \neq 0$ or $x\phi \neq 0$, then x is invertible.

Given Q an indecomposable object of $\mathcal{C}$ not isomorphic to P, then

- $\operatorname{Hom}(P, Q)\phi \operatorname{Hom}(Q, P) = 0$
- for $f \in \operatorname{Hom}(P, Q)$ non zero, then, $\phi \in \operatorname{Hom}(Q, P) f$ and for $g \in \operatorname{Hom}(Q, P)$ non zero, then, $\phi \in g \operatorname{Hom}(P, Q)$.

Let $C = (0 \to P \xrightarrow{\phi} P \to 0)$, a complex with non-zero terms in degrees 0 and 1. Define an endomorphism ζ of C as ϕ in degree 1 and 0 elsewhere.

Then, one shows there is a unique element of the centre of the homotopy category $\mathcal{T}$ of complexes (all, bounded, bounded above or bounded below, ...) of objects of $\mathcal{C}$ with the following properties:

- it is 0 on indecomposable objects of $\mathcal{T}$ which are not isomorphic to $C[i]$ for some i.
- it is $\phi[i]$ on $C[i]$.

This applies to $\mathcal{C}$ the category of finitely generated projective A-modules when $A = k[x]/(x^2)$ (or $A = \mathbf{Z}/4\mathbf{Z}$ by slightly modifying the setting above): the centre of A-perf is larger that A.

Remark 4.16. Proposition 4.14 does not extend to Hochschild cohomology [Ca]. Let X be an elliptic curve. Then, $HH^2(X) = \mathrm{Ext}^2_{X\times X}(\mathcal{O}_{\Delta X}, \mathcal{O}_{\Delta X}) \neq 0$. On the other hand, X-coh is a hereditary category. In particular, $\mathrm{Hom}(\mathrm{Id}, \mathrm{Id}[2]) = 0$, where Id is the identity functor of $D^b(X\text{-coh})$.

4.2.4. Affine varieties

Let $\mathcal{L}$ be an ample line bundle on X. Then X-perf is generated by the powers $\mathcal{L}^{\otimes -i}$ for $i > 0$, as a thick subcategory (cf Lemma 3.9). It follows that a thick subcategory $\mathcal{I}$ is an ideal if for any $C \in \mathcal{I}$, we have $C \otimes \mathcal{L}^{-1} \in \mathcal{I}$.

We deduce that if X is affine, then every thick subcategory of X-perf is an ideal. We obtain a corollary to Theorem 4.11 (actually, Lemma 4.10 gives a direct proof in that case).

Corollary 4.17. *If X is affine and reduced, then the triangulated category* X-perf *determines X.*

4.2.5. (Anti-)ample canonical bundles

To study more interesting situations, let us introduce Serre functors, following Bondal and Kapranov.

Let $\mathcal{C}$ be a k-linear category. A *Serre functor* for $\mathcal{C}$ is an equivalence of categories $S : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ together with the data, for every $X, Y \in \mathcal{C}$, of bifunctorial isomorphisms

$$\mathrm{Hom}(X, Y)^* \xrightarrow{\sim} \mathrm{Hom}(Y, S(X)).$$

A Serre functor is unique up to unique isomorphism, when it exists.

Lemma 4.18. *Let X be a smooth projective variety of pure dimension n. Then $S = \omega_X[n] \otimes -$ is a Serre functor for* $D^b(X\text{-coh})$.

Proof. We can assume that X is irreducible. Given $C \in D^b(X\text{-coh})$, consider the Hom-pairing $\operatorname{Hom}(\mathcal{O}, C) \times \operatorname{Hom}(C, \omega_X[n]) \to H^n(X, \omega_X) \simeq k$. When the homology sheaves of C are concentrated in one degree, Serre's duality Theorem shows that this pairing is perfect. Since the thick subcategory of C's for which the pairing is perfect is thick, we deduce that the pairing is perfect for all C.

Via the canonical isomorphisms $\operatorname{Hom}(C, D) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{O}, R\mathcal{H}om(C, D))$ and $\operatorname{Hom}(D, C \otimes \omega_X[n]) \xrightarrow{\sim} \operatorname{Hom}(R\mathcal{H}om(C, D), \omega_X[n])$, we obtain a perfect pairing

$$\operatorname{Hom}(C, D) \times \operatorname{Hom}(D, C \otimes \omega_X[n]) \to k.$$

□

Theorem 4.19 (Bondal-Orlov). *Let X be a smooth projective variety such that ω_X or ω_X^{-1} is ample. Then the triangulated category $D^b(X\text{-coh})$ determines X.*

If Y is a smooth projective variety, then an equivalence of triangulated categories $D^b(X\text{-coh}) \xrightarrow{\sim} D^b(Y\text{-coh})$ gives rise to an isomorphism $X \xrightarrow{\sim} Y$.

Proof. The crucial point is the fact that the Serre functor is intrinsic to the category $D^b(X\text{-coh})$. Since the thick subcategories invariant by the Serre functor and its inverse are ideals (cf §4.2.4 and Lemma 4.18), we recover X from $D^b(X\text{-coh})$.

Consider now $F : D^b(X\text{-coh}) \xrightarrow{\sim} D^b(Y\text{-coh})$. Note that F commutes with the Serre functors : $FS_X \simeq S_Y F$.

Let Z be a closed subset of Y. Since $F^{-1}(D^b_Z(Y\text{-coh}))$ is a thick subcategory of $D^b(X\text{-coh})$ stable under S_X^i for all i, it is of the form $D^b_{\Phi(Z)}(X\text{-coh})$ and this provides an injection Φ from closed subsets of Y to closed subsets of X.

Assume Z is irreducible. Let V be an open affine subset of Y. Then F^{-1} induces an equivalence $D^b(V\text{-coh}) \xrightarrow{\sim} D^b((X - \Phi(Y - V))\text{-coh})$ that restricts to an equivalence $D^b_Z(V\text{-coh}) \xrightarrow{\sim} D^b_{\Phi(Z)}((X - \Phi(Y - V))\text{-coh})$. Since $D^b_Z(V\text{-coh})$ is an irreducible thick subcategory (§4.2.4 and Theorem 4.6), we deduce that $D^b_{\Phi(Z)}((X - \Phi(Y - V))\text{-coh})$ is an irreducible thick subcategory of $D^b((X - \Phi(Y - V))\text{-coh})$, hence $\Phi(Z) \cap (X - \Phi(Y - V))$ is irreducible. If $Y = V_1 \cup \cdots \cup V_r$ is a covering by open affine subsets, then the subsets $X - \Phi(Y - V_i)$ give an open affine covering of X, and it follows that $\Phi(Z)$ is irreducible.

We define an injection $\phi : Y \to X$ between points by $\overline{\phi(y)} = \Phi(\overline{y})$. If y is a closed point, then by Lemma 4.21 below the thick subcategory $D^b_{\{y\}}(Y\text{-coh})$ of $D^b(Y\text{-coh})$ is minimal as a non-zero thick subcategory. It follows that $F^{-1}(D^b_{\{y\}}(Y\text{-coh})) = D_{\overline{\phi(y)}}(X\text{-coh})$ is a minimal non-zero thick subcategory of $D^b(X\text{-coh})$ that is stable under S_X^i for all i. We deduce that $\phi(y)$ is a closed point.

Let x be a closed point of X that is not in the image of ϕ. Then $\mathrm{Hom}(\mathcal{O}_{\{x\}}, C[i]) = 0$ for all $i \in \mathbf{Z}$ and all $C \in D^b_{\{\phi(y)\}}(X\text{-coh})$. It follows that $\mathrm{Hom}(F(\mathcal{O}_{\{x\}}), \mathcal{O}_{\{y\}}[i]) = 0$ for all closed points y of Y and all $i \in \mathbf{Z}$. We deduce from Lemma 4.20 below that $F(\mathcal{O}_{\{x\}}) = 0$, a contradiction. So, ϕ is bijective.

Let Z be a closed subset of Y. A closed point y of Y is in Z if and only if $D^b_{\{y\}}(Y\text{-coh}) \subset D^b_Z(Y\text{-coh})$. We have $x \in \Phi(Z)$ if and only if $D^b_{\{x\}}(X\text{-coh}) \subset D^b_{\Phi(Z)}(X\text{-coh})$, hence $\phi(Z) = \Phi(Z)$ is closed in X. This shows that $\psi = \phi^{-1} : X \to Y$ is continuous.

Let $U = Y - Z$. We have a sequence of canonical isomorphisms (cf Lemma 4.10)

$$\Gamma(U) \xrightarrow{\sim} Z(D^b(U\text{-coh}))_{\mathrm{lred}} \xrightarrow[F^{-1}]{\sim} Z(D^b(\phi(U)\text{-coh}))_{\mathrm{lred}} \xrightarrow{\sim} \Gamma(\phi(U)).$$

This extends $\psi : X \to Y$ into an isomorphism of ringed spaces. □

Lemma 4.20. *Let X be a variety and let $C \in D^b(X\text{-coh})$ such that $\mathrm{Hom}(C, \mathcal{O}_{\{x\}}[i]) = 0$ for all closed points x in X and for all $i \in \mathbf{Z}$. Then $C = 0$.*

Proof. Consider i maximal such that $\mathcal{H}^i(C) \neq 0$ and let x be a closed point. The canonical map $\mathrm{Hom}(\mathcal{H}^i(C), \mathcal{O}_{\{x\}}) \to \mathrm{Hom}(C, \mathcal{O}_{\{x\}}[-i])$ is injective. Given x in the support of $\mathcal{H}^i(C)$, we have $\mathrm{Hom}(\mathcal{H}^i(C), \mathcal{O}_{\{x\}}) \neq 0$, hence $\mathrm{Hom}(C, \mathcal{O}_{\{x\}}[-i]) \neq 0$. □

Lemma 4.21. *Let X be an algebraic variety and let x be a closed point of X. Then X-perf$_{\{x\}}$ is a minimal non-zero thick subcategory of X-perf.*

Proof. Let U be an open affine subset of X containing x. The restriction functor $X\text{-perf}_{\{x\}} \to U\text{-perf}_{\{x\}}$ is fully faithful. Theorem 4.6 and §4.2.4 show that $U\text{-perf}_{\{x\}}$ is a minimal non-zero thick subcategory of U-perf. The lemma follows. □

Remark 4.22. Bondal and Orlov [BoOr] show that the structure of graded category of $D^b(X\text{-coh})$ (we forget the distinguished triangles) is enough to reconstruct X in Theorem 4.19.

Given $\alpha : X \xrightarrow{\sim} Y$ an isomorphism of varieties, we have an equivalence $\alpha_* : D^b(X\text{-coh}) \xrightarrow{\sim} D^b(Y\text{-coh})$. Given $\mathcal{L}$ a line bundle on X, we have a self-equivalence $\mathcal{L}\otimes ? : D^b(X\text{-coh}) \xrightarrow{\sim} D^b(X\text{-coh})$. Finally, given $n \in \mathbf{Z}$, we have the self-equivalence $[n]$.

This gives a injective morphism from $\mathbf{Z} \times (\mathrm{Pic}\, X \rtimes \mathrm{Aut}(X))$ to the group $\mathrm{Aut}(D^b(X\text{-coh}))$ of isomorphism classes of self-equivalences of $D^b(X\text{-coh})$.

We can now complete Theorem 4.19.

Theorem 4.23 (Bondal-Orlov). *Let X be a smooth connected projective variety with ω_X or ω_X^{-1} ample. Then the canonical map* $\mathbf{Z} \times (\operatorname{Pic} X \rtimes \operatorname{Aut}(X)) \xrightarrow{\sim} \operatorname{Aut}(D^b(X\text{-coh}))$ *is an isomorphism.*

Proof. Let F be a self-equivalence of $D^b(X\text{-coh})$. In the proof of Theorem 4.19, we have constructed an automorphism ψ of X such that $F(D_Z(X\text{-coh})) = D_{\psi(Z)}(X\text{-coh})$ for all closed subsets Z of X. Replacing F by $F\psi^*$, we can assume that $D_Z(X\text{-coh})$ is stable by F for all Z closed in X. Furthermore, F restricts to a self-equivalence F_x of $D^b(\mathcal{O}_x\text{-mod})$ for all closed points x of X.

Let $C = F(\mathcal{O}_X)$. Then $\operatorname{Hom}(C_x, C_x[i]) \simeq \operatorname{Hom}(\mathcal{O}_x, \mathcal{O}_x[i]) = 0$ for $i \neq 0$. Let D be a bounded complex of finitely generated projective $\mathcal{O}_x$-modules that is quasi-isomorphic to C_x and such that given i minimal (resp. maximal) with $d_D^i \neq 0$, then d_D^i is not a split injection (resp. a split surjection). Let r and s be those minimal and maximal integers. The canonical map $\operatorname{Hom}(D^r, D^{s+1}) \to \operatorname{Hom}(D, D[s-r])$ is non-zero: this is impossible. It follows that $H^i(C_x)$ is concentrated in a single degree $i = r_x$ and $H^{r_x}(C_x)$ is free. In addition, $\operatorname{End}(C_x) \simeq \operatorname{End}(\mathcal{O}_x) = \mathcal{O}_x$, hence $H^{r_x}(C_x)$ is free of rank 1. The set of closed points x with $H^i(C_x) \neq 0$ is closed in X. Since X is connected, we deduce that $r_x = r$ is constant. It follows that $C \simeq \mathcal{H}^r(C)[-r]$ and $\mathcal{L} = \mathcal{H}^r(C)$ is a line bundle on X. Replacing F by $(\mathcal{L}^{-1}[r] \otimes -) \circ F$, we can assume that $F(\mathcal{O}_X) \simeq \mathcal{O}_X$.

We want to prove now that F is isomorphic to the identity functor. By Orlov's representability Theorem [Or], there exists $K \in D^b((X \times X)\text{-coh})$ such that $F \simeq Rp_*(K \otimes^{\mathbf{L}} q^*(-))$, where p, q are the first and second projections $X \times X \to X$. Given x_1, x_2 closed points of X, we have $\operatorname{Hom}(K, \mathcal{O}_{\{(x_1,x_2)\}}[i]) \simeq \operatorname{Hom}(F(\mathcal{O}_{\{x_1\}}), \mathcal{O}_{\{x_2\}}[i]) \simeq \delta_{x_1,x_2}\delta_{0,i}k$. It follows that $K \simeq i_*\mathcal{M}$, where $\mathcal{M}$ is a line bundle on X and $i : X \to X \times X$ is the diagnal embedding. We deduce that $K \simeq i_*\mathcal{O}_X$ and $F \simeq \operatorname{Id}$. □

Remark 4.24. A proof similar to the one of Theorem 4.23 shows that $\operatorname{Pic}(X) \rtimes \operatorname{Aut}(X) \xrightarrow{\sim} \operatorname{Aut}(X\text{-coh})$ for any variety X.

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