Derived equivalences and finite dimensional algebras

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Abstract. We discuss the homological algebra of representation theory of finite dimensional algebras and finite groups. We present various methods for the construction and the study of equivalences of derived categories: local group theory, geometry and categorifications.

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1. Introduction

This paper discusses derived equivalences, their construction and their use, for finite dimensional algebras, with a special focus on finite group algebras.

In a first part, we discuss Broué's abelian defect group conjecture and its ramifications. This is one of the deepest problem in the representation theory of finite groups. It is part of local representation theory, which aims to relate characteristic p representations of a finite group with representations of local subgroups (normalizers of non-trivial p-subgroups). We have taken a more functorial viewpoint in the definition of classical concepts (defect groups, subpairs,...).

In § 2.1.4, we present Alperin's conjecture, which gives a prediction for the number of simple representations, and Broué's conjecture, which is a much more precise prediction for the derived category, but does apply only to certain blocks (those with abelian defect groups).

We discuss in § 2.2 various types of equivalences that arise and present the crucial problem of lifting stable equivalences to derived equivalences.

In § 2.3, we present some local methods. We give a stronger version of the abelian defect group conjecture that can be approached inductively and reduced to the problem explained above of lifting stable equivalences to derived equivalences. Roughly speaking, in a minimal counterexample to that refinement of the abelian defect conjecture, there is a stable equivalence. Work of Rickard suggested to impose conditions on the terms of the complexes: they should be direct summands of permutation modules. We explain that one needs also to put conditions on the maps, that make the complexes look like complexes of chains of simplicial complexes.

There is no understanding on how to construct candidates complexes who would provide the derived equivalences expected by the abelian defect group conjecture in general. For finite groups of Lie type (in non-describing characteristic), we explain

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(§ 2.4) Broué's idea that such complexes should arise as complexes of cohomology of Deligne–Lusztig varieties. We describe (§ 2.4.2) the Jordan decomposition of blocks (joint work with Bonnafé), as conjectured by Broué: Morita equivalences between blocks are constructed from the cohomology of Deligne–Lusztig varieties. For GL_n, every block is shown to be Morita equivalent to a unipotent block. This provides some counterpart to the Jordan decomposition of characters (Lusztig). In § 2.4.3 and 2.4.4, we explain the construction of complexes in the setting of the abelian defect conjecture. There are some delicate issues related to the choice of the Deligne–Lusztig variety and the extension of the action of the centralizer of a defect group to that of the normalizer. This brings braid groups and Hecke algebras of complex reflection groups.

In § 2.5, we explain how to view the problem of lifting stable equivalences to derived equivalences as a non-commutative version of the birational invariance of derived categories of Calabi–Yau varieties.

In § 2.6, we describe a class of derived equivalences which are filtered shifted Morita equivalences (joint work with Chuang). We believe these are the building bricks for most equivalences and the associated combinatorics should be interesting.

Part § 3 is devoted to some invariants of derived equivalences. In § 3.1, we explain a functorial approach to outer automorphism groups of finite dimensional algebras and deduce that their identity component is preserved under various equivalences. This functorial approach is similar to that of the Picard group of smooth projective schemes and we obtain also an invariance of the identity component of the product of the Picard group by the automorphism group, under derived equivalence.

In § 3.2, we explain how to transfer gradings through derived or stable equivalences. As a consequence, there should be very interesting gradings on blocks with abelian defect. This applies as well to Hecke algebras of type A in characteristic 0, where we obtain gradings which should be related to geometrical gradings.

Finally, in § 3.3, we explain the notion of dimension for triangulated categories, in particular for derived categories of algebras and schemes. This applies to answer a question of Auslander on the representation dimension and a question of Benson on Loewy length of group algebras.

Part § 4 is devoted to "categorifications". Such ideas have been advocated by I. Frenkel and have already shown their relevance in the work of Khovanov [57] on knot invariants. Our idea is that "classical" structures have natural higher counterparts. These act as symmetries of categories of representations or of sheaves.

In § 4.1, we explain the construction with Chuang of a categorification of $\mathfrak{sl}_2$ and we develop the associated "2-representation theory". There is an action on the sum of module categories of symmetric groups, and we deduce the existence of derived equivalences between blocks with isomorphic defect groups, using the general theory that provides a categorification of the adjoint action of the Weyl group. This applies as well to general linear groups, and gives a solution to the abelian defect group conjecture for symmetric and general linear groups.

In § 4.2, we define categorifications of braid groups. This is based on Soergel's bimodules.

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2. Broué's abelian defect group conjecture

2.1. Introduction

2.1.1. Blocks. Let ℓ be a prime number. Let $\mathcal{O}$ be the ring of integers of a finite extension K of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers and k its residue field.

Let G be a finite group. Modular representation theory is the study of the categories $\mathcal{O}G$-mod and kG-mod (finitely generated modules). The decomposition of Spec $Z(\mathcal{O}G)$ into connected components corresponds to the decomposition $Z(\mathcal{O}G) = \prod_b Z(\mathcal{O}G)b$, where b runs over the set of primitive idempotents of $Z(\mathcal{O}G)$ (the *block idempotents*). We have corresponding decompositions in *blocks* $\mathcal{O}G = \prod_b \mathcal{O}Gb$ and $\mathcal{O}G\text{-mod} = \bigoplus_b \mathcal{O}Gb\text{-mod}$.

Remark 2.1. One assumes usually that K is big enough so that KG is a product of matrix algebras over K (this will be the case if K contains the e-th roots of unity, where e is the exponent of G). Descent methods often allow a reduction to that case.

2.1.2. Defect groups. A *defect group* of a block $\mathcal{O}Gb$ is a minimal subgroup D of G such that $\operatorname{Res}_D^G = \mathcal{O}Gb \otimes_{\mathcal{O}Gb} - : D^b(\mathcal{O}Gb) \to D^b(\mathcal{O}D)$ is faithful (*i.e.*, injective on Hom's). Such a subgroup is an ℓ-subgroup and it is unique up to G-conjugacy.

The *principal block* $\mathcal{O}Gb_0$ is the one through which the trivial representation factors. Its defect groups are the Sylow ℓ-subgroups of G.

Defect groups measure the representation type of the block:

- kGb is simple if and only if $D = 1$.
- kGb-mod has finitely many indecomposable objects (up to isomorphism) if and only if the defect groups are cyclic.
- kGb is tame (*i.e.*, indecomposable modules are classifiable in a reasonable sense) if and only if the defect groups are cyclic or $\ell = 2$ and defect groups are dihedral, semi-dihedral or generalized quaternion groups.

2.1.3. Brauer correspondence. Let $\mathcal{O}Gb$ be a block and D a defect group. There is a unique block idempotent c of $\mathcal{O}N_G(D)$ such that the restriction functor $\operatorname{Res}_D^G = c\mathcal{O}Gb \otimes_{\mathcal{O}Gb} - : D^b(\mathcal{O}Gb) \to D^b(\mathcal{O}N_G(D)c)$ is faithful.

This correspondence provides a bijection between blocks of $\mathcal{O}G$ with defect group D and blocks of $\mathcal{O}N_G(D)$ with defect group D.

2.1.4. Conjectures. We have seen in § 2.1.3 that $D^b(\mathcal{O}G)$ embeds in $D^b(\mathcal{O}N_G(D)c)$. The *abelian defect conjecture* asserts that, when D is abelian, the categories are actually equivalent (via a different functor):

Conjecture 2.2 (Broué). If D is abelian, there is an equivalence $D^b(\mathcal{O}Gb) \xrightarrow{\sim} D^b(\mathcal{O}N_G(D)c)$.

A consequence of the conjecture is an isometry $K_0(KGb) \xrightarrow{\sim} K_0(KN_G(D)c)$ with good arithmetical properties (a *perfect isometry*). Note that the conjecture also carries homological information: if $\mathcal{O}Gb$ is the principal block and the equivalence sends the trivial module to the trivial module, we deduce that the cohomology rings of G and $N_G(D)$ are isomorphic, a classical and easy fact. It is unclear whether there should be some canonical equivalence in Conjecture 2.2.

Local representation theory is the study of the relation between modular representations and local structure of G. Alperin's conjecture asserts that the number of simple modules in a block can be computed in terms of local structure.

Conjecture 2.3 (Alperin). Assume $D \neq 1$. Then,

$$\operatorname{rank} K_0(kGb) = \sum_{\mathcal{S}} (-1)^{l(\mathcal{S})+1} \operatorname{rank} K_0(kN_G(\mathcal{S})c_{\mathcal{S}})$$

where $\mathcal{S}$ runs over the conjugacy classes of chains of subgroups $1 < Q_1 < Q_2 < \cdots < Q_n \leq_G D$, $l(\mathcal{S}) = n \geq 1$ and $c_{\mathcal{S}}$ is the sum of the block idempotents of $N_G(\mathcal{S})$ corresponding to b.

Remark 2.4. We have stated here Knörr–Robinson's reformulation of the conjecture [58]. Note that the conjecture is expected to be compatible with ℓ-local properties of character degrees, equivariance, rationality (Dade, Robinson, Isaacs, Navarro). When D is abelian, Alperin's conjecture (and its refinements) follows immediately from Broué's conjecture. It would be extremely interesting to find a common refinement of Alperin and Broué's conjectures. For principal blocks, it should contain the description of the cohomology ring as stable elements in the cohomology ring of a Sylow subgroup.

2.2. Various equivalences. Let A and B be two symmetric algebras over a noetherian commutative ring $\mathcal{O}$.

2.2.1. Definitions. Let M be a bounded complex of finitely generated (A, B)-bimodules which are projective as A-modules and as right B-modules. Assume there is an (A, A)-bimodule R and a (B, B)-bimodule S with

$$M \otimes_B M^* \simeq A \oplus R \text{ as complexes of } (A, A)\text{-bimodules,}$$
$$M^* \otimes_A M \simeq B \oplus S \text{ as complexes of } (B, B)\text{-bimodules.}$$

We say that M induces a

- *Morita equivalence* if M is concentrated in degree 0 and $R = S = 0$;
- *Rickard equivalence* if R and S are homotopy equivalent to 0 as complexes of bimodules;
- *derived equivalence* if R and S are acyclic;
- *stable equivalence* (of Morita type) if R and S are homotopy equivalent to bounded complexes of projective bimodules.

Note that Morita $\Rightarrow$ Rickard $\Rightarrow$ stable and Rickard $\Rightarrow$ derived. Note also that if there is a complex inducing a stable equivalence, then there is a bimodule inducing a stable equivalence. Finally, Rickard's theory says that if there is a complex inducing a derived equivalence, then there is a complex inducing a Rickard equivalence.

The definitions amount to requiring that $M \otimes_B -$ induces an equivalence

- (Morita) $B\text{-mod} \xrightarrow{\sim} A\text{-mod}$,
- (Rickard) $K^b(B\text{-mod}) \xrightarrow{\sim} K^b(A\text{-mod})$,
- (derived) $D^b(B) \xrightarrow{\sim} D^b(A)$,
- (stable) $B\text{-}\overline{\text{mod}} \xrightarrow{\sim} A\text{-}\overline{\text{mod}}$ (assuming $\mathcal{O}$ regular)

where $K^b(A\text{-mod})$ is the homotopy category of bounded complexes of objects of A-mod and $A\text{-}\overline{\text{mod}}$ is the stable category, additive quotient of A-mod by modules of the form $A \otimes_{\mathcal{O}} V$ with $V \in \mathcal{O}\text{-mod}$ (it is equivalent to $D^b(A)/A\text{-perf}$ when $\mathcal{O}$ is regular).

2.2.2. Stable equivalences. Stable equivalences arise fairly often in modular representation theory. For example, assume the Sylow ℓ-subgroups of G are TI, *i.e.*, given P a Sylow ℓ-subgroup, then $P \cap gPg^{-1} = 1$ for all $g \in G - N_G(P)$. Then, $M = \mathcal{O}G$ induces a stable equivalence between $\mathcal{O}G$ and $\mathcal{O}N_G(P)$, the corresponding functor is restriction (this is an immediate application of Mackey's formula). This restricts to a stable equivalence between principal blocks. Unfortunately, we do not know how to derive much numerical information from a stable equivalence.

A classical outstanding conjecture in representation theory of finite dimensional algebras is

Conjecture 2.5 (Alperin–Auslander). Assume $\mathcal{O}$ is an algebraically closed field. If A and B are stably equivalent, then they have the same number of isomorphism classes of simple non-projective modules.

A very strong generalization of Conjecture 2.5 is

Question 2.6. Let A and B be blocks with abelian defect groups and M a complex of (A, B)-bimodules inducing a stable equivalence. Assume K is big enough. Does there exist $\tilde{M}$ a complex of (A, B)-bimodules inducing a Rickard equivalence and such that M and $\tilde{M}$ are isomorphic in $(A \otimes B^{\text{opp}})\text{-}\overline{\text{mod}}$?

As will be explained in § 2.3.3, this is the key step for an inductive approach to Broué's conjecture.

Remark 2.7. There are examples of blocks with non abelian defect for which Question 2.6 has a negative answer, for example A the principal block of Suz(8), $\ell = 2$, and B the principal block of the normalizer of a Sylow 2-subgroup (TI case), cf. [17, §6]. A major problem with Question 2.6 and with Conjecture 2.2 is to understand the relevance of the assumption that the defect groups are abelian. Cf. § 3.2.2 for a possible idea.

2.3. Local theory. In an ideal situation, equivalences would arise from permutation modules or more generally, from chain complexes of simplicial complexes X acted on by the groups under consideration. Then, taking fixed points on X by an ℓ-subgroup Q would give rise to equivalences between blocks of the centralizers of Q. We would then have a compatible system of equivalences, corresponding to subgroups of the defect group. At the level of characters, Broué defined a corresponding notion of "isotypie" [17]: values of characters at ℓ-singular elements are related.

2.3.1. Subpairs. We explain here some classical facts.

A kG-module of the form $k\Omega$ where Ω is a G-set is a permutation module. An ℓ-permutation module is a direct summand of a permutation module and we denote by kG-lperm the corresponding full subcategory of kG-mod.

Suppose that Q is an ℓ-subgroup of G. We define the functor $\mathrm{Br}_Q\colon kG\text{-lperm} \to k(N_G(Q)/Q)\text{-lperm}$: $\mathrm{Br}_Q(M)$ is the image of M^Q in $M_Q = M/\sum_{x\in Q}(x-1)M$. If $M = k\Omega$, then $k(\Omega^Q) \xrightarrow{\sim} \mathrm{Br}_Q(M)$: the Brauer construction extends the fixed point construction on sets to ℓ-permutation modules. Note that this works only because Q is an ℓ-group and k has characteristic ℓ.

To deal with non principal blocks, we need to use Alperin–Broué's subpairs. A subpair of G is a pair (Q, e), where Q is an ℓ-subgroup of G and e a block idempotent of $kC_G(Q)$. If we restrict to the case where e is a principal block, we recover theory of ℓ-subgroups of G.

A maximal subpair is of the form (D, b_D), where D is a defect group of a block kGb and b_D is a block idempotent of $kC_G(D)$ such that $b_D c \neq 0$ (we say that (D, b_D) is a b-subpair). Fix such a maximal subpair. The $(kG, kN_G(D, b_D))$-bimodule $bkGb_D$ has, up to isomorphism, a unique indecomposable direct summand X with $\mathrm{Br}_{\Delta D}(X) \neq 0$. Here, we put $\Delta D = \{(x, x^{-1})\}_{x\in D} \leq D \times D^{\mathrm{opp}}$. More generally, given $\phi\colon Q \to R$, we put $\Delta_\phi(Q) = \{(x, \phi(x)^{-1})\}_{x\in Q} \leq Q \times R^{\mathrm{opp}}$.

We define the Brauer category $\mathcal{B}r(D, b_D)$: its objects are subpairs (Q, b_Q) with $Q \leq D$ and $b_Q \mathrm{Br}_{\Delta Q}(X) \neq 0$, and $\mathrm{Hom}((Q, b_Q), (R, b_R))$ is the set of $f \in \mathrm{Hom}(Q, R)$ such that there is $g \in G$ with $(Q^g, b_Q^g) \in \mathcal{B}r(D, b_D)$ and $f(x) = g^{-1}xg$ for all $x \in Q$.

Let $M \in kG$-lperm indecomposable. A vertex-subpair of M is a subpair (Q, b_Q) maximal such that $b_Q \mathrm{Br}_Q(M) \neq 0$ (such a subpair is unique up to conjugacy).

2.3.2. Splendid equivalences. Let G and H be two finite groups and b and b' two block idempotents of kG and kH.

The following Theorem [86], [92] shows that a stable equivalence corresponds to "local" Rickard equivalences, for complexes of ℓ-permutation modules.

Theorem 2.8. *Let M be an indecomposable complex of ℓ-permutation (kGb, kHb')-bimodules. Then M induces a stable equivalence between kGb and kHb' if and only if given (D, b_D) a maximal b-subpair, there is a maximal b'-subpair $(D', b'_{D'})$, an isomorphism $\phi\colon D \xrightarrow{\sim} D'$ inducing an isomorphism $\mathcal{B}r(D, b_D) \xrightarrow{\sim} \mathcal{B}r(D', b'_{D'})$ such that*

- *The indecomposable modules occurring in M have vertex-subpairs of the form $(\Delta_\phi(Q), b_Q \otimes b'_{\phi(Q)})$ for some $(Q, b_Q) \in \mathcal{B}r(D, b_D)$, with $(\phi(Q), b'_{\phi(Q)}) = \phi(Q, b_Q)$.*
- *For $1 \neq Q \leq D$, then $b_Q \cdot \mathrm{Br}_{\Delta_\phi Q} M \cdot b'_{\phi(Q)}$ induces a Rickard equivalence between $kC_G(Q)b_Q$ and $kC_H(Q)b'_{\phi(Q)}$, where $(Q, b_Q) \in \mathcal{B}r(D, b_D)$ and $(\phi(Q), b'_{\phi(Q)}) = \phi(Q, b_Q)$.*

Remark 2.9. In [83], Rickard introduced a notion of splendid equivalences for principal blocks (complexes of ℓ-permutation modules with diagonal vertices), later generalized by Harris [46] and Linckelmann [65]. Such equivalences were shown to induce equivalences for blocks of centralizers. In these approaches, an isomorphism between the defect groups of the two blocks involved was fixed a priori and vertex-subpairs were assumed to be "diagonal" with respect to the isomorphism. Theorem 2.8 shows it is actually easier and more natural to work with no a priori identification, and the property on vertex-subpairs is actually automatically satisfied.

The second part of the theorem (local Rickard equivalences $\Rightarrow$ stable equivalence) generalizes results of Alperin and Broué and is related to work of Bouc and Linckelmann.

Finally, a more general theory (terms need not be ℓ-permutation modules) has been constructed by Puig ("basic equivalences") [78].

Rickard proposed the following strengthening of Conjecture 2.2:

Conjecture 2.10. If D is abelian, there is a complex of ℓ-permutation modules inducing a Rickard equivalence between $\mathcal{O}Gb$ and $\mathcal{O}N_G(D)c$.

To the best of my knowledge, in all cases where Conjecture 2.2 is known to hold, then, Conjecture 2.10 is also known to hold.

Conjecture 2.10 is known to hold when D is cyclic [79], [62], [85]. In that case, one can construct a complex with length 2, but the longer complex originally constructed by Rickard might be more natural. The conjecture holds also when $D \simeq (\mathbb{Z}/2)^2$ [82], [63], [85]. In both cases, the representation type is tame. Note that there is no other ℓ-group P for which Conjecture 2.10 is known to hold for all $D \simeq P$.

Conjecture 2.10 holds when G is ℓ-solvable [35], [75], [47], when G is a symmetric group or a general linear group (cf. § 4.1; the describing characteristic case $G = \mathrm{SL}_2(\ell^n)$ is solved in [70]) and when G is a finite group of Lie type and $\ell \mid (q-1)$ (cf. § 2.4.3). There are many additional special groups for which the conjecture is known to hold (work of Gollan, Hida, Holloway, Koshitani, Kunugi, Linckelmann, Marcus, Miyachi, Okuyama, Rickard, Turner, Waki), cf. http://www.maths.bris.ac.uk/~majcr/adgc/adgc.html.

2.3.3. Gluing. Theorem 2.8 suggests an inductive approach to Conjecture 2.10: one should solve the conjecture for local subgroups (say, $C_G(Q)$, $1 \neq Q \leq D$) and glue the corresponding Rickard complexes. This would give rise to a complex inducing a stable equivalence, leaving us with the core problem of lifting a stable equivalence to a Rickard equivalence. Unfortunately, complexes are not rigid enough to allow gluing. This problem can be solved by using complexes endowed with some extra structure [86], [92]. The idea is to use complexes that have the properties of chain complexes of simplicial complexes: the key point is the existence of compatible splittings of the Brauer maps $M^Q \to M(Q)$. One can build an exact category of ℓ-permutation modules with compatible splittings of the Brauer maps. The subcategory of projective objects turns out to have a very simple description in terms of sets, and we use only this category. For simplicity, we restrict here to the case of principal blocks.

Let G be a finite group, ℓ a prime number, k an algebraically closed field of characteristic ℓ, b the principal block idempotent of kG, D a Sylow ℓ-subgroup of G and c the principal block idempotent of $H = N_G(D)$. We assume D is abelian. We denote by $Z_\ell(G)$ the Sylow ℓ-subgroup of $Z(G)$ and put $Z = \Delta Z_\ell(G)$.

Let G' be a finite group containing G as a normal subgroup, let $H' = N_{G'}(D)$ and $F = G'/G \xrightarrow{\sim} H'/H$. We assume F is an ℓ'-group, we put $N = \{(g,h) \in G' \times H'^{\mathrm{opp}} \mid (gG, hH^{\mathrm{opp}}) \in \Delta F\}$ and $\bar{N} = N/Z$.

Let $\mathcal{E}$ be the category of $\bar{N}$-sets whose point stabilizers are contained in $\Delta D/Z$. Let $\tilde{\mathcal{E}}$ be the Karoubian envelop of the linearization of $\mathcal{E}$ (objects are pairs (Ω, e) where Ω is a $\bar{N}$-set and e an idempotent of the monoid algebra of $\mathrm{End}_{\bar{N}}(\Omega)$). We have a faithful functor $\tilde{\mathcal{E}} \to k\bar{N}\text{-lperm}$, $(\Omega, e) \mapsto k(\Omega, e) := k\Omega e$.

We are now ready to state a further strengthening of Conjecture 2.2. For the inductive approach, it is important to take into account central ℓ-subgroups and ℓ'-automorphism groups.

Conjecture 2.11. There is a complex C of objects of $\tilde{\mathcal{E}}$ such that $\mathrm{Res}^{\bar{N}}_{G \times H^{\mathrm{opp}}} k(C)$ induces a Rickard equivalence between kGb and kHc.

We can also state a version of Question 2.6, for the pair (G', G):

Question 2.12. Let C be a complex of objects of $\tilde{\mathcal{E}}$ such that $\mathrm{Res}^{\bar{N}}_{G \times H^{\mathrm{opp}}} k(C)$ induces a stable equivalence between kGb and kHc. Is there a bounded complex

R of finitely generated projective $\bar{N}$-modules and a morphism $f\colon R \to k(C)$ such that $\operatorname{Res}^{\bar{N}}_{G\times H^{\mathrm{opp}}} \operatorname{cone}(f)$ induces a Rickard equivalence between kGb and kHc?

The following theorem reduces (a suitable version of) the abelian defect conjecture to (a suitable version of) the problem of lifting stable equivalences to Rickard equivalences.

Theorem 2.13. *Assume Question* 2.12 *has a positive answer for* $(N_{G'}(Q), C_G(Q))$ *for all non trivial subgroups* Q *of* D. *Then Conjecture* 2.11 *holds.*

The proof goes by building inductively (on the index of Q in D) a system of complexes for $N_{G'}(Q)$ and gluing them together. The key point is that, given a finite group Γ, the category of Γ-sets whose point stabilizers are non-trivial p-subgroups is locally determined. This allows us to manipulate objects of $\tilde{\mathcal{E}}$ as "sheaves".

2.4. Chevalley groups. We explain Broué's idea that complexes of cohomology of certain varieties should give rise to derived equivalences, for finite groups of Lie type.

2.4.1. Deligne–Lusztig varieties. Let $\boldsymbol{G}$ be a connected reductive algebraic group defined over a finite field and let F be an endomorphism of $\boldsymbol{G}$, a power F^d of which is a Frobenius endomorphism defining a structure over a finite field $\mathbb{F}_{q^d}$ for some $q \in \mathbb{R}_{>0}$. Let $G = \boldsymbol{G}^F$ be the associated finite group.

Let ℓ be a prime number with $\ell \nmid q$, K a finite extension of $\mathbb{Q}_l$, and $\mathcal{O}$ its ring of integers. We assume K is big enough.

Let $\boldsymbol{L}$ be an F-stable Levi subgroup of $\boldsymbol{G}$, $\boldsymbol{P}$ be a parabolic subgroup with Levi complement $\boldsymbol{L}$, and let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{P}$. We define the Deligne–Lusztig variety

$$Y_{\boldsymbol{U}} = \{g\boldsymbol{U} \in \boldsymbol{G}/\boldsymbol{U} \mid g^{-1}F(g) \in \boldsymbol{U} \cdot F(\boldsymbol{U})\},$$

a smooth affine variety with a left action of $\boldsymbol{G}^F$ and a right action of $\boldsymbol{L}^F$ by multiplication. The corresponding complex of cohomology $R\Gamma_c(Y_{\boldsymbol{U}}, \mathcal{O})$ induces the Deligne–Lusztig induction functor $R^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}}\colon D^b(\mathcal{O}\boldsymbol{L}^F) \to D^b(\mathcal{O}\boldsymbol{G}^F)$.

The effect of these functors on characters (*i.e.*, K_0's after extension to K) is a central tool for Deligne–Lusztig and Lusztig's construction of irreducible characters of G. It is important to also consider the finer invariant $\tilde{R}\Gamma_c(Y_{\boldsymbol{U}}, \mathcal{O})$, an object of $K^b(\mathcal{O}(\boldsymbol{G}^F \times (\boldsymbol{L}^F)^{\mathrm{opp}})\text{-lperm})$ which is quasi-isomorphic to $R\Gamma_c(Y_{\boldsymbol{U}}, \mathcal{O})$ [81], [87].

We put $X_{\boldsymbol{U}} = Y_{\boldsymbol{U}}/\boldsymbol{L}^F$ and denote by $\pi\colon Y_{\boldsymbol{U}} \to X_{\boldsymbol{U}}$ the quotient map.

Remark 2.14. One could use ordinary cohomology instead of the compact support version. One can conjecture that the two versions are interchanged by Alvis–Curtis duality: $(R\Gamma_c(Y_{\boldsymbol{U}}, \mathcal{O}) \otimes^{\boldsymbol{L}}_{\mathcal{O}\boldsymbol{L}^F} -) \circ D_{\boldsymbol{L}}$ and $D_{\boldsymbol{G}} \circ (R\Gamma(Y_{\boldsymbol{U}}, \mathcal{O})) \otimes^{\boldsymbol{L}}_{\mathcal{O}\boldsymbol{L}^F} -)$ should differ by a shift. This is known in the Harish-Chandra case, *i.e.*, when $\boldsymbol{P}$ is F-stable [24].

Let $\boldsymbol{T}_0 \subset \boldsymbol{B}_0$ be a pair consisting of an F-stable maximal torus and an F-stable Borel subgroup of $\boldsymbol{G}$. Let $\boldsymbol{U}_0$ be the unipotent radical of $\boldsymbol{B}_0$ and let $W = N_{\boldsymbol{G}}(\boldsymbol{T}_0)/\boldsymbol{T}_0$.

Let B^+ (resp. B) be the braid monoid (resp. group) of W. The canonical map $B^+ \to W$ has a unique section $w \mapsto \boldsymbol{w}$ that preserves lengths (it is not a group morphism!). We fix an F-equivariant morphism $\tau\colon B \to N_{\boldsymbol{G}}(\boldsymbol{T}_0)$ that lifts the canonical map $N_{\boldsymbol{G}}(\boldsymbol{T}_0) \to W$ [99]. Given $w \in W$, we put $\dot{w} = \tau(\boldsymbol{w})$. Let w_0 be the longest element of W and let $\pi = \boldsymbol{w}_0^2$, a central element of B.

Assume $\boldsymbol{L}$ above is a torus. We give a different model for $Y_{\boldsymbol{U}}$. Let $w \in W$ and $h \in \boldsymbol{G}$ such that $h^{-1}F(h) = \dot{w}$ and $\boldsymbol{U} = h\boldsymbol{U}_0h^{-1}$. Let

$$Y(w) = \{g\boldsymbol{U}_0 \in \boldsymbol{G}/\boldsymbol{U}_0 \mid g^{-1}F(g) \in \boldsymbol{U}_0\dot{w}\boldsymbol{U}_0\},$$

a variety with a left action of G and a right action of $\boldsymbol{T}_0^{wF}$ by multiplication. We have $\boldsymbol{L} = h\boldsymbol{T}_0h^{-1}$ and conjugation by h induces an isomorphism $\boldsymbol{L}^F \xrightarrow{\sim} \boldsymbol{T}_0^{wF}$. Right multiplication by h induces an isomorphism $Y_{\boldsymbol{U}} \xrightarrow{\sim} Y(w)$ compatible with the actions of G and $\boldsymbol{L}^F$. We have $\dim Y(w) = l(w)$. We write $Y_F(w)$ when the choice of F is important.

Given $w_1, \ldots, w_r \in W$, we put

$$\begin{aligned} Y(w_1, \ldots, w_r) = \{&(g_1\boldsymbol{U}_0, \ldots, g_r\boldsymbol{U}_0) \in (\boldsymbol{G}/\boldsymbol{U}_0)^r \mid \\ &g_1^{-1}g_2 \in \boldsymbol{U}_0\dot{w}_1\boldsymbol{U}_0, \ldots, g_{r-1}^{-1}g_r \in \boldsymbol{U}_0\dot{w}_{r-1}\boldsymbol{U}_0 \text{ and } g_r^{-1}F(g_1) \in \boldsymbol{U}_0\dot{w}_r\boldsymbol{U}_0\}. \end{aligned}$$

Up to a transitive system of canonical isomorphisms, $Y(w_1, \ldots, w_r)$ depends only on the product $b = \boldsymbol{w}_1 \cdots \boldsymbol{w}_r \in B^+$ and we denote that variety by $Y(b)$ [36], [22].

2.4.2. Jordan decomposition. As a first step in his classification of (complex) irreducible characters of finite groups of Lie type, Lusztig established a *Jordan decomposition* of characters.

Let $(\boldsymbol{G}^*, F^*)$ be Langlands dual to $(\boldsymbol{G}, F)$. Then Lusztig defined a partition of the set $\mathrm{Irr}(G)$ of irreducible characters of G:

$$\mathrm{Irr}(G) = \coprod_{(s)} \mathrm{Irr}(G, (s))$$

where (s) runs over conjugacy classes of semi-simple elements of $(\boldsymbol{G}^*)^{F^*}$. The elements in $\mathrm{Irr}(G, 1)$ are the *unipotent characters*.

Furthermore, Lusztig constructed a bijection

$$\mathrm{Irr}(G, (s)) \xrightarrow{\sim} \mathrm{Irr}((C_{\boldsymbol{G}^*}(s)^*)^F, 1) \tag{1}$$

(assuming $C_{\boldsymbol{G}^*}(s)$ is connected). So, an irreducible character corresponds to a pair consisting of a semi-simple element in the dual and a unipotent character of the dual of the centralizer of that semi-simple element.

Broué and Michel [21] showed that the union of series corresponding to classes with a fixed ℓ'-part is a union of blocks: let t be a ℓ'-element of $(\boldsymbol{G}^*)^{F^*}$ and let

$$\mathrm{Irr}(G, (t))_\ell = \coprod_{(s)} \mathrm{Irr}(G, (s))$$

where (s) runs over conjugacy classes of semi-simple elements of $(\boldsymbol{G}^*)^{F^*}$ whose ℓ'-part is conjugate to t. Then $\mathrm{Irr}(G,(t))_\ell$ is a union of ℓ-blocks, and we denote by $B(G^F,(t))$ the corresponding factor of $\mathcal{O}G^F$.

Broué [18] conjectured that the decomposition (1) arises from a Morita equivalence (cf. also [48]). More, precisely, we have the following theorem [11, Theorem B'] obtained in joint work with C. Bonnafé (cf. also [23] for a detailed exposition). This was conjectured by Broué who gave a proof when t is regular [18].

Theorem 2.15 (Jordan decomposition of blocks). *Assume $C_{\boldsymbol{G}^*}(t)$ is contained in an F^*-stable Levi subgroup $\boldsymbol{L}^*$ of $\boldsymbol{G}^*$ with dual $\boldsymbol{L} \leq \boldsymbol{G}$. Let $\boldsymbol{P}$ be a parabolic subgroup of $\boldsymbol{G}$ with Levi complement $\boldsymbol{L}$ and unipotent radical $\boldsymbol{U}$. Let $d = \dim X_{\boldsymbol{U}}$ and let $\mathcal{F}_t = \pi_*\mathcal{O} \otimes_{\mathcal{O}\boldsymbol{L}^F} B(\boldsymbol{L}^F,(t))$.*

Then $H^i_c(X_{\boldsymbol{U}},\mathcal{F}_t) = 0$ for $i \neq d$ and $H^d_c(X_{\boldsymbol{U}},\mathcal{F}_t)$ induces a Morita equivalence between $B(G,(t))$ and $B(\boldsymbol{L}^F,(t))$.

The theorem reduces the study of blocks of finite groups of Lie type to the case of those associated to a quasi-isolated element t. When $\boldsymbol{L}^* = C_{\boldsymbol{G}^*}(t)$ is a Levi subgroup of $\boldsymbol{G}^*$, then $B(\boldsymbol{L}^F,(t))$ is isomorphic to $B(\boldsymbol{L}^F,1)$.

As shown by Broué, the key point is the statement about the vanishing of cohomology. When $\boldsymbol{L}$ is a torus, this is [37, Theorem 9.8]. For the general case, two difficulties arise: there are no known good smooth compactifications of the varieties $X_{\boldsymbol{U}}$ and the locally constant sheaf $\mathcal{F}_t$ has wild ramification. We solve these issues as follows. Let $\overline{X}$ be the closure of $X_{\boldsymbol{U}}$ in $\boldsymbol{G}/\boldsymbol{P}$. We construct new varieties of Deligne–Lusztig type and commutative diagrams

$$\begin{tikzcd} X_i \arrow[r, hook, "j_i"] \arrow[d, "f_i'"'] & Y_i \arrow[d, "f_i"] \\ X_{\boldsymbol{U}} \arrow[r, hook, "j"'] & \overline{X} \end{tikzcd}$$

where Y_i is smooth, $Y_i - X_i$ is a divisor with normal crossings, and f_i is proper. We also construct tamely ramified sheaves $\mathcal{F}_i$ on X_i with the following properties:

- $\mathcal{F}_t$ is in the thick subcategory of the derived category of constructible sheaves on $X_{\boldsymbol{U}}$ generated by the $Rf'_{i*}\mathcal{F}_i$
- $(Rj_{i*}\mathcal{F}_i)_{|f_i^{-1}(\overline{X}-X_{\boldsymbol{U}})} = 0$.

The first property follows from the following generation result of the derived category of a finite group of Lie type [11, Theorem A]:

Theorem 2.16. *The category of perfect complexes for $B(G,(t))$ is generated, as a thick subcategory, by the $R^{\boldsymbol{G}}_{\boldsymbol{T}\subset\boldsymbol{B}} B(\boldsymbol{T}^F,(t))$, where $\boldsymbol{T}$ runs over the F-stable maximal tori of $\boldsymbol{G}$ such that $t \in \boldsymbol{T}^*$ and $\boldsymbol{B}$ runs over the Borel subgroups of $\boldsymbol{G}$ containing $\boldsymbol{T}$.*

Remark 2.17. Note that the corresponding result for derived categories is true, under additional assumptions on $\boldsymbol{G}$ [13]: this is related to Quillen's Theorem, we need every elementary abelian ℓ-subgroup of G to be contained in an F-stable torus of $\boldsymbol{G}$.

Remark 2.18. Note that the Morita equivalence of Theorem 2.15 is not splendid in general. This issue is analyzed in [13].

Example 2.19. Let $\boldsymbol{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $F\colon (x_{ij})_{1\le i,j\le n} \mapsto (x_{ij}^q)_{i,j}$. We have $G = \mathrm{GL}_n(\mathbb{F}_q)$, $\boldsymbol{G} = \boldsymbol{G}^*$ and $F^* = F$. Centralizers of semi-simple elements are Levi subgroups, so Theorem 2.15 gives a Morita equivalence between any block of a general linear group over $\mathcal{O}$ and a unipotent block.

2.4.3. Abelian defect conjecture. Let b be a block idempotent of $\mathcal{O}G$. Let (D, b_D) be a maximal b-subpair, let $H = N_G(D, b_D)$ and let $\boldsymbol{L} = C_{\boldsymbol{G}}(D)$. We assume D is abelian and $\boldsymbol{L}$ is a Levi subgroup of $\boldsymbol{G}$ (these are satisfied if $\ell \nmid |W|$).

Broué conjectured that the sought-for complex in Conjecture 2.10 should arise from Deligne–Lusztig varieties ([17, p. 81], [20, §1], [19, §VI]):

Conjecture 2.20 (Broué). There is a parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}$ with Levi complement $\boldsymbol{L}$ and unipotent radical $\boldsymbol{U}$, and a complex C inducing a Rickard equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hb_D$ such that $\mathrm{Res}_{G\times(\boldsymbol{L}^F)^{\mathrm{opp}}} C$ is isomorphic to $\tilde{R}\Gamma_c(Y_{\boldsymbol{U}}, \mathcal{O})b_D$.

This conjecture 2.20 is known to hold [76] when there is a choice of an F-stable parabolic subgroup $\boldsymbol{P}$ (case $\ell \mid (q-1)$). Then $Y_{\boldsymbol{U}}$ is 0-dimensional and the Deligne–Lusztig induction is the Harish-Chandra induction. The key steps in the proof are:

- Produce an action of the reflection group $H/\boldsymbol{L}^F$ from a natural action of the associated Hecke algebra. One needs to show that certain obstructions vanish.
- Identify a 2-cocycle of $H/\boldsymbol{L}^F$ with values in $\mathcal{O}^\times$.
- Compute the dimension of the KG-endomorphism ring.

2.4.4. Regular elements. As a first step, one should make Conjecture 2.20 more precise by specifying $\boldsymbol{P}$ and by defining the extension of the action of $C_G(D)$ to an action of H on $\tilde{R}\Gamma_c(Y_{\boldsymbol{U}}, \mathcal{O})b_D$. These issues are partly solved and I will explain the best understood case where $\boldsymbol{L} = \boldsymbol{T}$ is a torus and $\mathcal{O}Gb$ is the principal block (cf. [22]). Assume as well $\ell \nmid (q-1)$. To simplify, assume further that F acts trivially on W ("split" case).

Note that $\boldsymbol{T}$ defines a conjugacy class $\mathcal{C}$ of W and the choice of $\boldsymbol{P}$ amounts to the choice of $w \in \mathcal{C}$ (defined from $\boldsymbol{P}$ as in § 2.4.1). Since $\boldsymbol{T} = C_{\boldsymbol{G}}(D)$, it follows that elements in $\mathcal{C}$ are Springer-regular. There is $w_d \in \mathcal{C}$ such that $(\boldsymbol{w}_d)^d = \pi$, where $d > 1$ is the order of w_d (a "good" regular element).

Given $w \in W$, we have a purely inseparable morphism

$$Y(w, w^{-1}w_0, w_0ww_0, w_0w^{-1}) \to Y(w^{-1}w_0, w_0ww_0, w_0w^{-1}, w)$$
$$(x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4, F(x_1)).$$

Via the canonical isomorphisms, this induces an endomorphism of $Y(\pi)$. This extends to an action of B^+ on $Y(\pi)$.

There is an embedding of $Y_F(w_d)$ as a closed subvariety of $Y_{F^d}(w_d,\dots,w_d)$ (d terms) given by

$$x \mapsto (x, F(x), \dots, F^{d-1}(x)).$$

The action of $C_{B^+}(\boldsymbol{w}_d)$ on $Y_{F^d}(\pi)$ restricts to an action on $Y_F(w_d)$. It induces an action of $C_B(\boldsymbol{w}^d)$ on $\tilde{R}\Gamma_c(Y(\boldsymbol{w}_d), \mathcal{O})$.

The group $H/C_G(D) \simeq C_W(w_d)$ is a complex reflection group and we denote by B_d its braid group. There is a morphism $B_d \to C_B(\boldsymbol{w}_d)$, uniquely defined up to conjugation by an element of the pure braid group of $C_W(w_d)$ (it is expected to be an isomorphism, and known to be such in a number of cases [8]).

Now, the conjecture is that, up to homotopy, the action of $\mathcal{O}(\boldsymbol{T}_0^{w_d F} \rtimes B_d)$ on $\tilde{R}\Gamma_c(Y(w_d), \mathcal{O})b_D$ induces an action of the quotient algebra $\mathcal{O}Hc$ and the resulting object is a splendid Rickard complex:

Conjecture 2.21. There is a complex $C \in K^b((\mathcal{O}Gb) \otimes (\mathcal{O}Hb_D)^{\mathrm{opp}}\text{-lperm})$, unique up to isomorphism, with the following properties:

- There is a surjective morphism $f\colon \mathcal{O}\boldsymbol{T}_0^{w_d F} \rtimes B_d \to \mathcal{O}Hb_D$ extending the inclusion $\boldsymbol{T}_0^{w_d F} \subset H$ such that
  - f^*C and $\tilde{R}\Gamma_c(Y(w_d), \mathcal{O})b_D$ are isomorphic in $D^b((\mathcal{O}\boldsymbol{T}_0^{w_d F} \rtimes B_d) \otimes (\mathcal{O}H)^{\mathrm{opp}})$,
  - the map $kB_d \to kC_W(w_d)$ deduced from f by applying $k \otimes_{\mathcal{O}\boldsymbol{T}_0^{w_d F}} -$ is the canonical map.
- C is isomorphic to $\tilde{R}\Gamma_c(Y(w_d), \mathcal{O})b_D$ in $K^b((\mathcal{O}G) \otimes (\mathcal{O}C_G(D))^{\mathrm{opp}}\text{-lperm})$.

Furthermore, such a complex C induces a Rickard equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hb_D$.

The most crucial and difficult part in that conjecture is to show that we have no non-zero shifted endomorphisms of the complex ("disjunction property"), either for the action of G or for that H.

Conjecture 2.21 is known to hold when $l(w_d) = 1$ [87] and for GL_n and $d = n$ [12]. In the first case, we use good properties of cohomology of curves and prove disjunction for the action of G. In the second case, we study the variety $D(\boldsymbol{U}_0)^F \backslash Y(w_d)$ and prove disjunction for the action of H. This works only for GL_n, for we rely on the fact that induced Gelfand–Graev representations generate the category of projective modules.

Remark 2.22. When $\ell \mid (q-1)$ (case $d = 1$), one can formulate a version of Conjecture 2.21 using the variety $Y(\pi)$ [22, Conjectures 2.15].

Remark 2.23. The version "over K" of Conjecture 2.21 is open, even after restricting to unipotent representations (= applying the functor $K \otimes_{KT^{w_d F}} -$). The action of KB_d on $H_c^*(Y(w_d), K)$ should factor through an action of the Hecke algebra of

$C_W(w_d)$, for certain parameters. This is known in some cases: for $d = 1$ [22], [39], when $d = 2$ (work of Lusztig [67] and joint work with Digne and Michel [39]) and in some other cases [38]. The disjunction property is known for w_d a Coxeter element [66], for GL_n and $d = n - 1$ [38] and in most rank 2 groups [39].

2.5. Local representation theory as non-commutative birational geometry. It is expected that birational Calabi–Yau varieties should have equivalent derived categories (cf. [15]). We view Question 2.6 as a non-commutative version: one can expect that "sufficiently nice" Calabi–Yau triangulated categories are determined by (not too small) quotients. We explain here how this analogy can be made precise, in the setting of McKay's correspondence, via Koszul duality.

2.5.1. 2-elementary abelian defect groups. Let P be an elementary abelian 2-group. Let k be a field of characteristic 2 and $V = P \otimes_{\mathbb{F}_2} k$. Let E be a group of odd order of automorphisms of P. The algebras $kP \rtimes E$ and $\Lambda(V) \rtimes E$ are isomorphic.

Koszul duality (cf. eg [53]) gives an equivalence

$$D^b((\Lambda(V) \rtimes E)\text{-modgr}) \xrightarrow{\sim} D^b_{E \times \boldsymbol{G}_m}(V).$$

2.5.2. McKay's correspondence. Let V be a finite-dimensional vector space over k and E a finite subgroup of $\mathrm{GL}(V)$ of order invertible in k. Recall the following conjecture (independence of the crepant resolution):

Conjecture 2.24 (McKay's correspondence). If $X \to V/E$ is a crepant resolution, then $D^b(X) \simeq D^b_E(V)$.

The conjecture is known to hold when $\dim V = 3$ [16], [14] (in dimension 3, the Hilbert scheme of E-clusters on V is a crepant resolution). It is also known when V is a symplectic vector space and E respects the symplectic structure [9]. See [15, §2.2] for more details.

Examples in dimension > 3 where $E - \mathrm{Hilb}\, V$ is smooth are rare. An infinite family of examples is provided by the following theorem of Sebestean [95]:

Theorem 2.25. *Let $n \geq 2$, let k be a field containing a primitive $(2^n - 1)$-th root of unity ζ and let E be the subgroup of $\mathrm{SL}_n(k)$ generated by the diagonal matrix with entries $(\zeta, \zeta^2, \ldots, \zeta^{2^{n-1}})$. Assume $2^n - 1$ is invertible in k.*

Then $E - \mathrm{Hilb}(\boldsymbol{A}^n_k)$ is a smooth crepant resolution of $\boldsymbol{A}^n_k/E$ and there is an equivalence $D^b_E(\boldsymbol{A}^n_k) \xrightarrow{\sim} D^b(E - \mathrm{Hilb}(\boldsymbol{A}^n_k))$.

The diagonal action of $\boldsymbol{G}_m$ on $\boldsymbol{A}^n_k$ induces an action on $E - \mathrm{Hilb}(\boldsymbol{A}^n_k)$ and the equivalence is equivariant for these actions.

Let $G = \mathrm{SL}_2(2^n)$, let P be the subgroup of strict upper triangular matrices (a Sylow 2-subgroup), and let E be the subgroup of diagonal matrices. The action of E on $P \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2$ coincides with the one in Theorem 2.25. Combining the solution of

Conjecture 2.2 for G (Okuyama, [70]) and § 3.2.2, the Koszul duality equivalence, and Theorem 2.25, we deduce a geometric realization of modular representations of $\mathrm{SL}_2(2^n)$ in natural characteristic:

Corollary 2.26. *There is a grading on the principal* 2*-block* A *of* $\overline{\mathbb{F}}_2 G$ *and an equivalence* $D^b(A\text{-modgr}) \xrightarrow{\sim} D^b_{\boldsymbol{G}_m}(E - \mathrm{Hilb}\, \boldsymbol{A}^n_k)$.

Remark 2.27. It should be interesting to study homotopy categories of sheaves on singular varieties and their relation to derived categories of crepant resolutions.

2.6. Perverse Morita equivalences. In this part, we shall describe joint work with J. Chuang [30].

2.6.1. Definitions. Let $\mathcal{A}$, $\mathcal{A}'$ be two abelian categories. We assume every object has a finite composition series. Let $\mathcal{S}$ (resp. $\mathcal{S}'$) be the set of isomorphism classes of simple objects of $\mathcal{A}$ (resp. $\mathcal{A}'$).

Definition 2.28. An equivalence $F\colon D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is perverse if there is

- a filtration $\emptyset = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_r = \mathcal{S}$,
- a filtration $\emptyset = \mathcal{S}'_0 \subset \mathcal{S}'_1 \subset \cdots \subset \mathcal{S}'_r = \mathcal{S}'$,
- and a function $p\colon \{1, \ldots, r\} \to \mathbb{Z}$,

such that

- F restricts to equivalences $D^b_{\mathcal{A}_i}(\mathcal{A}) \xrightarrow{\sim} D^b_{\mathcal{A}'_i}(\mathcal{A}')$,
- $F[-p(i)]$ induces equivalences $\mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\sim} \mathcal{A}'_i/\mathcal{A}'_{i-1}$.

where $\mathcal{A}_i$ (resp. $\mathcal{A}'_i$) is the Serre subcategory of $\mathcal{A}$ (resp. $\mathcal{A}'$) generated by $\mathcal{S}_i$ (resp. $\mathcal{S}'_i$).

An important point is that $\mathcal{A}'$ is determined, up to equivalence, by $\mathcal{A}$, $\mathcal{S}_\bullet$ and p.

2.6.2. Symmetric algebras. Let A be a symmetric finite dimensional algebra and $\mathcal{A} = A$-mod.

We explain how to construct a perverse equivalence, given any $\mathcal{S}_\bullet$ and p (this cannot be done in general for a nonsymmetric algebra A).

Let I be a subset of $\mathcal{S}$. Given $V \in \mathcal{S}$, let P_V be a projective cover of V, let V_I be the largest quotient of P_V all of which composition factors are in I and let $Q_V \to \ker(P_V \to V_I)$ be a projective cover. We put $T_{A,V}(I) = P_V$ if $V \in S - I$, $T_{A,V}(I) = 0 \to Q_V \to P_V \to 0$ if $V \in I$ (where Q_V is in degree 0) and $T_A(I) = \bigoplus_V T_{A,V}(I)$, a tilting complex.

Let $\mathcal{T}$ be the set of isomorphism classes of families $(T_V)_{V\in S}$, where T_V is an indecomposable bounded complex of finitely generated projective A-modules and $\bigoplus_{V\in S} T_V$ is a tilting complex.

We denote by $\mathcal{P}(\mathcal{S})$ the set of subsets of $\mathcal{S}$. We define an action of $\mathrm{Free}(\mathcal{P}(\mathcal{S})) \rtimes \mathfrak{S}(\mathcal{S})$ on $\mathcal{T}$. The symmetric group acts by permutation of indices and $I \subset \mathcal{S}$ sends $(T_V)_V$ to $(T'_V)_V$ given by $T'_V = F^{-1}(T_{B,V}(I))$, where $B = \mathrm{End}_{D^b(A)}(\bigoplus_V T_V)$ and $F = R\,\mathrm{Hom}^\bullet_A(\bigoplus_V T_V, -)\colon D^b(A) \xrightarrow{\sim} D^b(B)$.

Fix now $\mathcal{S}_\bullet$ a filtration of $\mathcal{S}$ and $p\colon \mathcal{S} \to \mathbb{Z}$. We put

$$(T_V)_V = \mathcal{S}_r^{p(r)} \mathcal{S}_{r-1}^{p(r-1)-p(r)} \dots \mathcal{S}_1^{p(1)-p(2)}((P_V)_V),$$

$T = \bigoplus_V T_V$, $A' = \mathrm{End}_{D^b(A)(T)}$ and $F = R\,\mathrm{Hom}^\bullet_A(T, -)$. Then, F is perverse with respect to $\mathcal{S}_\bullet$ and p.

Remark 2.29. One might ask whether all derived equivalences between finite dimensional symmetric algebras are compositions of perverse equivalences, or at least, if two derived equivalent symmetric algebras can be related by a sequence of perverse equivalences. Many of the derived equivalences in block theory are known to be compositions of perverse equivalences and it would be interesting to see if this is also the case for those of [70].

Remark 2.30. One can expect the equivalences predicted in Conjecture 2.20 will be perverse. The filtration should be provided by Lusztig's a-function.

We expect the action of $\mathrm{Free}(\mathcal{P}(\mathcal{S})) \rtimes \mathfrak{S}(\mathcal{S})$ on $\mathcal{T}$ relates to Bridgeland's space of stability conditions [15, §4].

Remark 2.31. The considerations above are interesting for Calabi–Yau algebras of positive dimension. Given I a subset of $\mathcal{S}$, one obtains a torsion theory that needs not always come from a tilting complex. When $r = 2$ and $|\mathcal{S}_2 - \mathcal{S}_1| = 1$, tilting has been known in string theory as Seiberg duality.

3. Invariants

Invariants of triangulated categories and dg-categories are discussed in [55, §6]. We discuss here some more elementary invariants, used to study finite dimensional algebras.

3.1. Automorphisms of triangulated categories

3.1.1. Rings. Let k be a commutative ring and A be a k-algebra. We denote by $\mathrm{Pic}(A)$ the group of isomorphism classes of invertible (A, A)-bimodules and by $\mathrm{DPic}(A)$ the group of isomorphism classes of invertible objects of the derived category of (A, A)-bimodules: this is the part of the automorphism group of $D(A\text{-Mod})$ that comes from standard equivalences. By Rickard's Theorem, $\mathrm{DPic}(A)$ is invariant under derived equivalences.

The following Proposition has been observed by many people (Rickard, Roggenkamp–Zimmermann, [93, Proposition 3.3], [103, Proposition 3.4],…).

Proposition 3.1. *If A is local, then* $\mathrm{DPic}(A) = \mathrm{Pic}(A) \times \langle A[1] \rangle$.

Given R a flat commutative Z-algebra, there is a canonical morphism $\mathrm{DPic}(A) \to \mathrm{DPic}(A \otimes_Z R)$ (joint work with A. Zimmermann [93, §2.4]). If R is faithfully flat over Z, the kernel of that map is contained in $\mathrm{Pic}(A)$. This is the key point for the following (cf. [103, Proposition 3.5] and [93, Proposition 3.3]):

Theorem 3.2. *Assume A is commutative and indecomposable. Then* $\mathrm{DPic}(A) = \mathrm{Pic}(A) \times \langle A[1] \rangle$.

3.1.2. Invariance of automorphisms. Let A be a finite dimensional algebra over an algebraically closed field k. We denote by $\mathrm{Aut}(A)$ the group of automorphisms of A. This is an algebraic group and we denote by $\mathrm{Inn}(A)$ its closed subgroup of inner automorphisms. We put $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$. We have a morphism of groups $\mathrm{Aut}(A) \to \mathrm{Pic}(A)$, $\alpha \mapsto [A_\alpha]$, where $A_\alpha = A$ as a left A-module and the right action of $a \in A$ is given by right multiplication by $\alpha(a)$. It induces an injective morphism $\mathrm{Out}(A) \to \mathrm{Pic}(A)$.

The following result [91] gives a functorial interpretation of Out, to be compared with the functorial interpretation of $\mathrm{Pic}(X)$ for a smooth projective variety X.

Theorem 3.3. *The functor from the category of affine varieties over k to groups that sends X to the set of isomorphism classes of $(A \otimes A^{\mathrm{opp}} \otimes \mathcal{O}_X)$-modules that are locally free of rank* 1 *as $(A \otimes \mathcal{O}_X)$ and as $(A^{\mathrm{opp}} \otimes \mathcal{O}_X)$-modules is represented by* $\mathrm{Out}(A)$.

The following theorem [91] shows the invariance of Out^0, the identity component of Out, under certain equivalences. In the case of Morita equivalences, it goes back to Brauer, and for derived equivalences, it has been obtained independently by Huisgen-Zimmermann and Saorín [49]. In these cases, it follows easily from Theorem 3.3 while, for stable equivalences, some work is needed to get rid globally of projective direct summands.

Theorem 3.4. *Let B be a finite dimensional k-algebra and let C be a bounded complex of finitely generated (A, B)-bimodules inducing a derived equivalence or a stable equivalence* (*in which case we assume A and B are self-injective*). *Then there is a unique isomorphism of algebraic groups $\sigma :$ $\mathrm{Out}^0(A) \xrightarrow{\sim} \mathrm{Out}^0(B)$ such that $A_\alpha \otimes_A C \simeq C \otimes_B B_{\sigma(\alpha)}$ for all $\alpha \in$* $\mathrm{Out}^0(A)$.

Yekutieli [104] deduces that $\mathrm{DPic}(A)$ has a structure of a locally algebraic group, with connected component $\mathrm{Out}^0(A)$.

3.1.3. Coherent sheaves. The following result [91] is a variant of Theorem 3.4.

Theorem 3.5. *Let X and Y be two smooth projective schemes over an algebraically closed field k. An equivalence $D^b(X) \xrightarrow{\sim} D^b(Y)$ induces an isomorphism* $\mathrm{Pic}^0(X) \rtimes \mathrm{Aut}^0(X) \xrightarrow{\sim} \mathrm{Pic}^0(Y) \rtimes \mathrm{Aut}^0(Y)$.

This implies in particular that if A and B are derived equivalent abelian varieties, then there is a symplectic isomorphism $\hat{A} \times A \xrightarrow{\sim} \hat{B} \times B$ (and the converse holds as well [71], [74]).

3.1.4. Automorphisms of stable categories and endo-trivial modules. Let A be a finite dimensional self-injective algebra over an algebraically closed field k. We denote by $\mathrm{StPic}(A)$ the group of isomorphism classes of invertible objects of $(A \otimes A^{\mathrm{opp}})\text{-}\overline{\mathrm{mod}}$.

Let P be an ℓ-group and k a field of characteristic ℓ. A finitely generated kP-module L is an *endo-trivial* module if $L \otimes_k L^* \simeq k$ in $kP\text{-}\overline{\mathrm{mod}}$ or equivalently, if $\mathrm{End}_{kP\text{-}\overline{\mathrm{mod}}}(L) = k$ [25]. Note that the classification of endo-trivial modules has been recently completed [27] (the case where P is abelian goes back to [34]).

Let $\mathcal{T}(kP)$ be the group of isomorphism classes of indecomposable endo-trivial modules. We have an injective morphism of groups

$$\mathcal{T}(kP) \to \mathrm{StPic}(kP),\ [L] \mapsto [\mathrm{Ind}_{\Delta P}^{P \times P^{\mathrm{opp}}} L].$$

This extends to an isomorphism $\mathcal{T}(kP) \times \mathrm{Out}(kP) \xrightarrow{\sim} \mathrm{StPic}(kP)$ ([64, §3] and [26, §2]).

Let Q be an ℓ-group. A stable equivalence of Morita type $kP\text{-}\overline{\mathrm{mod}} \xrightarrow{\sim} kQ\text{-}\overline{\mathrm{mod}}$ induces an isomorphism $\mathcal{T}(kP) \xrightarrow{\sim} \mathcal{T}(kQ)$. It actually forces the algebras kP and kQ to be isomorphic ([64, §3], [26, Corollary 2.4]). It is an open question whether this implies that P and Q are isomorphic.

Theorem 3.6 ([26, Theorem 3.2]). *Let P be an abelian ℓ-group and E a cyclic ℓ'-group acting freely on P. We put $G = P \rtimes E$. Then* $\mathrm{StPic}(kG) = \mathrm{Pic}(kG) \cdot \langle \Omega \rangle$. *In particular, the canonical morphism* $\mathrm{TrPic}(kG) \to \mathrm{StPic}(kG)$ *is surjective.*

Remark 3.7. Let A be a block over k of a finite group, with defect group isomorphic to P and $N_G(P)/P$ acting as E on P. From Theorem 3.6, one deduces [26, Corollary 4.4] via a construction of Puig [77], that a stable equivalence of Morita type between A and kG lifts to a Rickard equivalence if and only if A and kG are Rickard equivalent if and only if they are splendidly Rickard equivalent. In particular, for blocks with abelian defect group D such that $N_G(D, b_D)/C_G(D)$ is cyclic, then Conjecture 2.2 implies Conjecture 2.10.

3.2. Gradings. In this section, we describe results of [91].

3.2.1. Transfer of gradings. We assume we are in the situation of Theorem 3.4. Assume A is graded, *i.e.*, there is a morphism $\boldsymbol{G}_m \to \mathrm{Aut}(A)$. The induced morphism $\boldsymbol{G}_m \to \mathrm{Out}^0(A)$ induces a morphism $\boldsymbol{G}_m \to \mathrm{Out}^0(B)$. There exists a lift to a morphism $\boldsymbol{G}_m \to \mathrm{Aut}^0(B)$, and this corresponds to a grading on B. There is a grading on (an object isomorphic to) C that makes it into a complex of graded (A, B)-bimodules and it induces an equivalence between the appropriate graded categories.

Let A be a self-injective indecomposable graded algebra, let n be the largest integer such that $A_n \neq 0$, and let $C \in \mathbb{Z}[q, q^{-1}]$ be the graded Cartan matrix of A.

If A is non-negatively graded and the Cartan matrix of A_0 has non-zero determinant, then $\deg \det(C) = nr$, where r is the number of simple A-modules. As a consequence, one gets a positive solution of a "non-negatively graded" version of Conjecture 2.5:

Proposition 3.8. *Let A and B be two indecomposable self-injective non-negatively graded algebras. Assume A_0 has finite global dimension and there is a graded stable equivalence of Morita type between A and B. Then A and B have the same number of simple modules.*

Remark 3.9. Let A be a non-negatively graded indecomposable self-injective algebra with A_0 of finite global dimension. Let B be a stably equivalent self-injective algebra. One could hope that there is a compatible grading on B that is non-negative, but this is not possible in general. It would be still be very interesting to see if this can be achieved if the grading on A is "tight" in the sense of Cline–Parshall–Scott, *i.e.*, if $\bigoplus_{j \leq i} A_j = (JA)^i$ (cf. the gradings in § 3.2.2).

3.2.2. Blocks with abelian defect. Let P be an abelian ℓ-group and k an algebraically closed field of characteristic ℓ. The algebra kP is (non-canonically) isomorphic to the graded algebra associated to the radical filtration of kP. Fixing such an isomorphism provides a grading on kP. Let E be an ℓ'-group of automorphisms of P. Then the isomorphism above can be made E-equivariant and we obtain a structure of graded algebra on $kP \rtimes E$ extending the grading on kP and with kE in degree 0. Given a central extension of E by $k^\times$, this construction applies as well to the twisted group algebra $k_*P \rtimes E$.

Let A be a block of a finite group over k with defect group D. Then there is E and a central extension as above such that the corresponding block of $N_G(D)$ is Morita equivalent to $k_*D \rtimes E$ [60]. So, Conjecture 2.2 predicts there are interesting gradings on A. In the inductive approach to Conjecture 2.11, there is a stable equivalence of Morita type between A and $k_*D \rtimes E$, and we can provide A with a grading compatible with the equivalence (but we do not know if the grading can be chosen to be non-negative).

Remark 3.10. The gradings on blocks with abelian defect should satisfy some Koszulity properties (cf. [73], as well as work of Chuang). Turner [101] expects that gradings will even exist for blocks of symmetric groups with non abelian defect.

Remark 3.11. Using the equivalences in § 4.1, we obtain gradings on blocks of abelian defect of symmetric groups and on blocks of Hecke algebras over $\mathbb{C}$. One can expect the corresponding graded Cartan matrices to be given in terms of Kazhdan–Lusztig polynomials. So, the equivalences carry some "geometric meaning".

3.3. Dimensions

3.3.1. Definition and bounds. Let us explain how to associate a dimension to a triangulated category $\mathcal{T}$ (cf. [88]). For the derived category of a finite dimensional algebra, this is related to the Loewy length and to the global dimension, none of which are invariant under derived equivalences.

Given $\mathcal{I}_1$ and $\mathcal{I}_2$ two subcategories of $\mathcal{T}$, we denote by $\mathcal{I}_1 * \mathcal{I}_2$ the smallest full subcategory of $\mathcal{T}$ closed under direct summands and containing the objects M such that there is a distinguished triangle

$$M_1 \to M \to M_2 \rightsquigarrow$$

with $M_i \in \mathcal{I}_i$. Given $M \in \mathcal{T}$, we denote by $\langle M \rangle$ the smallest full subcategory of $\mathcal{T}$ containing M and closed under direct summands, direct sums, and shifts. Finally, we put $\langle M \rangle_0 = 0$ and define inductively $\langle M \rangle_i = \langle M \rangle_{i-1} * \langle M \rangle$.

The dimension of $\mathcal{T}$ is defined to be the smallest integer $d \geq 0$ such that there is $M \in \mathcal{T}$ with $\mathcal{T} = \langle M \rangle_{d+1}$ (we set $\dim \mathcal{T} = \infty$ if there is no such d). The notion of finite-dimensionality corresponds to Bondal–Van den Bergh's property of being strongly finitely generated [10].

Given a right coherent ring A, then $\dim D^b(A) \leq \operatorname{gldim} A$ (cf. [59, Proposition 2.6] and [88, Propositions 7.4 and 7.24]).

Let A be a finite dimensional algebra over a field k. Denote by $J(A)$ the Jacobson radical of A. The Loewy length of A is the smallest integer $d \geq 1$ such that $J(A)^d = 0$. We have $\dim D^b(A) < \text{Loewy length}(A)$.

Let X be a separated scheme of finite type over a perfect field k.

Theorem 3.12. *We have* $\dim D^b(X) < \infty$.

- *If X is reduced, then* $\dim D^b(X) \geq \dim X$.
- *If X is smooth and quasi-projective, then* $\dim D^b(X) \leq 2 \dim X$.
- *If X is smooth and affine, then* $\dim D^b(X) = \dim X$.

There does not seem to be any known example of a smooth projective variety X with $\dim D^b(X) > \dim X$, although this is expected to happen, for example when X is an elliptic curve (note nevertheless that $\dim D^b(\boldsymbol{P}^n) = n$).

Note that a triangulated category with finitely many indecomposable objects up to isomorphism has dimension 0. This applies to $D^b(kQ)$, where Q is a quiver of type ADE. This applies also to the orbit categories constructed by Keller (cf. [55, §4.9], [54, §8.4]). They depend on a positive integer d, and they are Calabi–Yau of dimension d.

When $\mathcal{T}$ is compactly generated, the property for $\mathcal{T}^c$ to be finite-dimensional can be viewed as a counterpart of having "finite global dimension".

3.3.2. Representation dimension. Auslander [5] introduced a measure for how far an algebra is from being representation finite. The example of exterior algebras below shows that this notion is pertinent. Let A be a finite dimensional algebra. The representation dimension of A is $\inf\{\mathrm{gldim}(A \oplus A^* \oplus M)\}_{M \in A\text{-mod}}$. This is known to be finite [50].

In [89], we show that this notion is related to the notion of dimension for associated triangulated categories. For example, $\dim D^b(A) \leq \mathrm{repdim}\, A$.

Let A be a non semi-simple self-injective k-algebra. We have

$$2 + \dim A\text{-}\overline{\mathrm{mod}} \leq \mathrm{repdim}\, A \leq \text{Loewy length}(A)$$

(the second inequality comes from [5, §III.5, Proposition]).

The following theorem is obtained by computing $\dim \Lambda(k^n)\text{-}\overline{\mathrm{mod}}$ via Koszul duality. It gives the first examples of algebras with representation dimension > 3.

Theorem 3.13. *Let n be a positive integer. We have* $\mathrm{repdim}\, \Lambda(k^n) = n + 1$.

Remark 3.14. One can actually show more quickly [59] that the algebra with quiver

$$\begin{tikzcd} 0 \arrow[r, bend left, "x_1"] \arrow[r, bend right, "x_{n+1}"'] \arrow[r, phantom, "\vdots"] & 1 & \cdots & n-1 \arrow[r, bend left, "x_1"] \arrow[r, bend right, "x_{n+1}"'] \arrow[r, phantom, "\vdots"] & n \end{tikzcd}$$

and relations $x_i x_j = x_j x_i$ has representation dimension $\geq n$, using that its derived category is equivalent to $D^b(\boldsymbol{P}^n)$ [6].

Using the inequality above, one obtains the following theorem, which solves the prime 2 case of a conjecture of Benson.

Theorem 3.15. *Let G be a finite group and k a field of characteristic* 2*. If G has a subgroup isomorphic to* $(\mathbb{Z}/2)^n$, *then* $n <$ Loewy length(kG).

4. Categorifications

This chapter discusses the categorifications of two structures, which are related to derived equivalences. We hope these categorifications will eventually lead to the construction of four-dimensional quantum field theories (as advocated in [33]), via the construction of appropriate tensor structures.

4.1. $\mathfrak{sl}_2$

4.1.1. Abelian defect conjecture for symmetric and general linear groups. Let G be a symmetric group and B an ℓ-block of kG with defect group D. Assume D is abelian and let $w = \log_\ell |D|$. In 1992, a three steps strategy was proposed for Conjecture 2.10 (inspired by the simpler character-theoretic part [84]):

- Rickard equivalence between $k(\mathbb{Z}/\ell \rtimes \mathbb{Z}/(\ell-1)) \wr \mathfrak{S}_w$ and the principal block of $k\mathfrak{S}_\ell \wr \mathfrak{S}_w$;
- Morita equivalence between the principal block of $k\mathfrak{S}_\ell \wr \mathfrak{S}_w$ and B_w;
- Rickard equivalence between B_w and B.

Here, B_w is a certain ℓ-block of symmetric groups (a "good block"). Scopes [94] has constructed a number of Morita equivalences between blocks of symmetric groups. For fixed w, there are only finitely many classes of blocks of symmetric groups up to Scopes equivalence, and B_w is defined to be the largest block that is not Scopes equivalent to a smaller block.

The first equivalence is deduced from an equivalence between the principal blocks of $\mathfrak{S}_\ell$ and $\mathbb{Z}/\ell \rtimes \mathbb{Z}/(\ell-1)$ via Clifford theory [68].

The second equivalence was established by Chuang and Kessar [28], the functor used is a direct summand of the induction functor.

The third equivalence is part of the general problem, raised by Broué, of constructing Rickard equivalences between two blocks of symmetric groups with isomorphic defect groups (equivalently, with same local structure). Rickard [80] constructed complexes of bimodules that he conjectured would solve that problem, generalizing Scopes construction (case where the complex has only one non-zero term). Rickard proved the invertibility of his complexes when they have two non-zero terms. The general case has proven difficult to handle directly.

Remark 4.1. The same strategy applies for general linear groups (in non-describing characteristic). Theorem 2.15 reduces the study to unipotent blocks. Step 2 above was handled in [69], [100]. As pointed out by H. Miyachi, this generalizes Puig's result [76] ($\mathrm{GL}_n(q)$, $\ell \mid (q-1)$).

Remark 4.2. "Good" blocks of symmetric groups have "good" properties. After the Morita equivalence theorem of [28], their properties were first analyzed by Miyachi [69], in the more complicated case of general linear groups: decomposition matrices and radical series of Specht modules were determined in the abelian defect case, by a direct analysis of the wreath product. As a consequence, decomposition matrices were known for good blocks of Hecke algebras in characteristic zero. For good blocks of symmetric groups with abelian defect, as well as for Hecke algebras in characteristic zero, a direct computation of the decomposition numbers is given in [52] (cf. also [51] for earlier results in that direction) and another approach is the determination of the relevant part of the canonical/global crystal basis [31], [32], [61].

For blocks of symmetric groups with non abelian defect, the decomposition matrices can be described in terms of decomposition matrices of smaller symmetric groups and remarkable structural properties are conjectured by Turner [101], [102], [72]. Good blocks have also been used by Fayers for the classification of irreducible Specht modules [40] and to show that blocks of weight 3 have decomposition numbers 0 or 1 (for $\ell > 3$) [41].

4.1.2. Fock spaces. Let us recall the Lie algebra setting for symmetric group representations (cf. e.g. [4]). Let $M = \bigoplus_{n\geq 0} \mathbb{Q} \otimes_{\mathbb{Z}} K_0(\mathbb{C}\mathfrak{S}_n\text{-mod})$. The complex irreducible representations of the symmetric group $\mathfrak{S}_n$ are parametrized by partitions of n and we obtain a basis of M parametrized by all partitions. We view M as a Fock space, with an action of $\hat{\mathfrak{sl}}_\ell$ and we recall a construction of this action, for the generators e_a and f_a (where $a \in \mathbb{F}_\ell$).

We have a decomposition

$$\operatorname{Res}^{\mathbb{F}_\ell\mathfrak{S}_n}_{\mathbb{F}_\ell\mathfrak{S}_{n-1}} = \bigoplus_{a\in\mathbb{F}_\ell} F_a,$$

where $F_a(M)$ is the generalized a-eigenspace of $X_n = (1,n)+(2,n)+\cdots+(n-1,n)$. Taking classes in K_0 and summing over all n, we obtain endomorphisms f_a of

$$V = \bigoplus_{n\geq 0} \mathbb{Q} \otimes_{\mathbb{Z}} K_0(\mathbb{F}_\ell\mathfrak{S}_n\text{-mod}).$$

Using induction, we obtain similarly endomorphisms e_a (adjoint to the f_a). The decomposition lifts to a decomposition of $\operatorname{Res}^{\mathbb{Z}_\ell\mathfrak{S}_n}_{\mathbb{Z}_\ell\mathfrak{S}_{n-1}}$ and we obtain endomorphisms e_a and f_a of M. The decomposition map $M \to V$ and the Cartan map $\bigoplus_{n\geq 0} \mathbb{Q} \otimes_{\mathbb{Z}} K_0(\mathbb{F}_\ell\mathfrak{S}_n\text{-proj}) \to M$ are morphisms of $\hat{\mathfrak{sl}}_\ell$-modules. The image of the Cartan map is the irreducible highest weight submodule L of M generated by $[\emptyset]$.

Let us note two important properties relating the module structure of V and the modular representation theory of symmetric groups:

- The decomposition of V into weight spaces corresponds to the block decomposition.
- Two blocks have isomorphic defect groups if and only if they are in the same orbit under the adjoint action of the affine Weyl group $\tilde{A}_{\ell-1}$.

In order to prove that two blocks of symmetric groups with isomorphic defect groups are derived equivalent, it is enough to consider a block and its image by a simple reflection s_a of $\tilde{A}_{\ell-1}$ (this involves only the $\mathfrak{sl}_2$-subalgebra generated by e_a and f_a). This is the situation in which Rickard constructed his complexes Θ_a.

Remark 4.3. These constructions extend to Hecke algebras of symmetric groups over $\mathbb{C}$, at an ℓ-th root of unity (here, $\ell \geq 2$ can be an arbitrary integer). In that situation, the classes of the indecomposable projective modules form the canonical/global crystal basis of L (Lascoux–Leclerc–Thibon's conjecture, proven by Ariki [3], cf. also [43]).

4.1.3. $\mathfrak{sl}_2$-categorifications. We describe here joint work with J. Chuang [29] (cf. also [90] for a survey and [44], [45], [7], [42] for related work). This is the special case of a more general theory under construction for Kac–Moody algebras.

Let k be an algebraically closed field and $\mathcal{A}$ a k-linear abelian category all of whose objects have finite composition series.

An $\mathfrak{sl}_2$-categorification on $\mathcal{A}$ is the data of

- (E, F) a pair of adjoint exact functors $\mathcal{A} \to \mathcal{A}$,
- $X \in \mathrm{End}(E)$, $T \in \mathrm{End}(E^2)$, $q \in k^\times$, and $a \in k$ (with $a \neq 0$ if $q \neq 1$)

satisfying the following properties:

- $[E]$ and $[F]$ give rise to a locally finite representation of $\mathfrak{sl}_2$ on $K_0(\mathcal{A})$,
- for S a simple object of $\mathcal{A}$, $[S]$ is a weight vector,
- F is isomorphic to a left adjoint of E,
- $(T\mathbf{1}_E) \circ (\mathbf{1}_E T) \circ (T\mathbf{1}_E) = (\mathbf{1}_E T) \circ (T\mathbf{1}_E) \circ (\mathbf{1}_E T)$,
- $(T + \mathbf{1}_{E^2}) \circ (T - q\mathbf{1}_{E^2}) = 0$,
- $T \circ (\mathbf{1}_E X) \circ T = \begin{cases} q(X\mathbf{1}_E) & \text{if } q \neq 1, \\ X\mathbf{1}_E - T & \text{if } q = 1, \end{cases}$
- $X - a\mathbf{1}_E$ is locally nilpotent.

From that data, we define two truncated powers $E^{(n,\pm)}$ (non-canonically isomorphic), using an affine Hecke algebra action on E^n. Following Rickard, we construct a complex Θ with terms $E^{(i,-)}F^{(j,+)}$.

The following theorem is proved by reduction to the case of "minimal categorifications", which are naturally associated to simple representations of $\mathfrak{sl}_2$.

Theorem 4.4. *Θ gives rise to self-equivalences of $K^b(\mathcal{A})$ and $D^b(\mathcal{A})$. This categorifies the action of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on $K_0(\mathcal{A})$.*

Remark 4.5. The self-equivalence Θ is perverse (cf. § 2.6), and this is a crucial point in the proof.

The construction of § 4.1.2 provides a structure of $\mathfrak{sl}_2$-categorification on $\mathcal{A} = \bigoplus_{n\geq 0} \overline{\mathbb{F}}_\ell \mathfrak{S}_n$-mod (for a given $a \in \mathbb{F}_\ell$). From the previous theorem, we deduce

Corollary 4.6. *Two blocks of symmetric groups with isomorphic defect groups are splendidly Rickard equivalent.*

Conjecture 2.10 *holds for blocks of symmetric groups.*

This corollary has a counterpart for $\mathrm{GL}_n(\mathbb{F}_q)$ and $\ell \nmid q$.

Remark 4.7. In general, there is a decomposition $\mathcal{A} = \bigoplus_\lambda \mathcal{A}_\lambda$ coming from the weight space decomposition of $K_0(\mathcal{A})$. There is a categorification of $[e, f] = h$ in the form of isomorphisms $EF_{|\mathcal{A}_\lambda} \xrightarrow{\sim} FE_{|\mathcal{A}_\lambda} \oplus \mathrm{Id}_{\mathcal{A}_\lambda}^{\bigoplus \lambda}$ (for $\lambda \geq 0$).

Remark 4.8. One can give a definition of $\mathfrak{sl}_2$-categorifications for triangulated categories and the definition above becomes a theorem that says that there is an induced categorification on $K^b(\mathcal{A})$ (and on $D^b(\mathcal{A})$).

Remark 4.9. One can also construct $\mathfrak{sl}_2$-categorifications on category $\mathcal{O}$ for $\mathfrak{gl}_n(\mathbb{C})$ and for rational representations of $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$. One deduces from Theorem 4.4 that blocks with the same stabilizers under the affine Weyl groups are derived equivalent (a conjecture of Rickard).

Remark 4.10. The endomorphism X has different incarnations: Jucys–Murphy element, Casimir,….

Remark 4.11. It is expected that the functors Θ_a constructed for $a \in \mathbb{F}_\ell$ provide an action of the affine braid group $B_{\tilde{A}_{\ell-1}}$ on $\bigoplus_n D^b(\mathbb{F}_\ell \mathfrak{S}_n)$.

4.2. Braid groups

4.2.1. Definition. We present here a categorification of braid groups associated to Coxeter groups, following [90]. This should be useful for the study of categories of representations of semi-simple Lie algebras, affine Lie algebras, simple algebraic groups over an algebraically closed field,... On the other hand, work of Khovanov [56] shows its relevance for invariants of links (type A), cf. also [98].

Let (W, S) be a Coxeter group, with S finite. Let V be its reflection representation over $\mathbb{C}$ and let B_W be the braid group of W. Let $A = \mathbb{C}[V]$. Given $s \in S$, let $F_s = 0 \to A \otimes_{A^s} A \xrightarrow{\text{mult}} A \to 0$, where A is in degree 1. This is an invertible object of $K^b(A \otimes A)$. Given two decompositions of an element of B_W in a product of the generators and their inverses, we construct a canonical isomorphism between the corresponding products of F_s. The system of isomorphism coming from the various decompositions of an element $b \in B_W$ is transitive and, taking its limit, we obtain an element $F_b \in K^b(A \otimes A)$. The full subcategory of $K^b(A \otimes A)$ with objects the F_b's defines a strict monoidal category $\mathcal{B}_W$.

We expect that there is a simple presentation of $\mathcal{B}_W$ by generator and relations (or rather of a related 2-category involving subsets of S). This should be related to the vanishing of certain Hom-spaces, for example $\mathrm{Hom}_{K^b(A\otimes A)}(F_b, F_{b'}^{-1}[i])$ should be 0 when b and b' are the canonical lifts of distinct elements of W.

Remark 4.12. The bimodules obtained by tensoring the $A \otimes_{A^s} A$ are Soergel's bimodules. Soergel showed they categorify the Hecke algebra of W. He also conjectured that the indecomposable objects correspond to the Kazhdan–Lusztig basis of W [96], [97].

Remark 4.13. When W is finite, one can expect that there is a construction of $\mathcal{B}_W$ that does not depend on the choice of S. Such a construction might then make sense for complex reflection groups.

4.2.2. Representations and geometry. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra with Weyl group W and let $\mathcal{O}_0$ be the principal block of its category $\mathcal{O}$. It has been widely noticed that there is a weak action of B_W on $D^b(\mathcal{O})$, using wall-crossing functors. We show that there is a genuine action of B_W on $D^b(\mathcal{O}_0)$ and there is a much more precise statement: there is a monoidal functor from $\mathcal{B}_W$ to the category of self-equivalences of $D^b(\mathcal{O}_0)$. This has a counterpart for the derived category of B-equivariant sheaves on the flag variety (in which case the genuine action of the braid group goes back to [36]). These actions are compatible with Beilinson–Bernstein's equivalence. Conversely, a suitable presentation of $\mathcal{B}_W$ by generators and relations should provide a quick proof of that equivalence (and of affine counterparts), in the spirit of Soergel's construction. The representation-theoretic and the geometrical categories should be viewed as two realizations of the same "2-representation" of $\mathcal{B}_W$. Also, this approach should give a new proof of the results of [2] comparing quantum groups at roots of unity and algebraic groups in characteristic p.

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