HIGHER REPRESENTATIONS AND CORNERED HEEGAARD FLOER HOMOLOGY

ANDREW MANION AND RAPHAËL ROUQUIER

Abstract. We develop the 2-representation theory of the odd one-dimensional super Lie algebra $\mathfrak{gl}(1|1)^+$ and show it controls the Heegaard-Floer theory of surfaces of Lipshitz, Ozsváth and Thurston [LiOzTh1]. Our main tool is the construction of a tensor product for 2-representations. We show it corresponds to a gluing operation for surfaces, or the chord diagrams of arc decompositions. This provides an extension of Heegaard-Floer theory to dimension one, expanding the work of Douglas, Lipshitz and Manolescu [DouMa, DouLiMa].

Contents

1.	Introduction	2
2.	Differential and pointed structures	4
2.1.	Differential algebras and categories	4
2.2.	Bimodules	7
2.3.	Pointed sets and categories	9
2.4.	Symmetric powers	12
3.	Hecke algebras	12
3.1.	Differential graded nil Hecke algebras	13
3.2.	Extended affine symmetric groups	17
4.	2-representation theory	27
4.1.	Monoidal category	27
4.2.	Lax cocenter	29
4.3.	Diagonal action	30
4.4.	Dual diagonal action	38
4.5.	Tensor product and internal Hom	51
5.	Bimodule 2-representations	52
5.1.	Differential algebras	52
5.2.	Lax cocenter	53
5.3.	Diagonal action	53
5.4.	Dual diagonal action	57
5.5.	Differential categories	58
5.6.	Pointed categories	60

Date: September 20, 2020.

The first and second author thank the NSF for its support (grant DMS-1702305). The first author gratefully acknowledges support from the NSF (grant DMS-1502686). The second author gratefully acknowledges support from the NSF (grant DMS-1161999) and from the Simons Foundation (grant #376202). This material is based upon work supported by the NSF (Grant No. 1440140), while the second author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2018.

5.7. Douglas-Manolescu's algebra-modules 60
6. Hecke 2-representations 61
6.1. Regular 2-representations 61
6.2. Nil Hecke category 66
6.3. Positive and finite variants 72
7. Strand algebras 73
7.1. 1-dimensional spaces 73
7.2. Curves 81
7.3. Paths 88
7.4. Strands 99
8. 2-representations on strand algebras 117
8.1. Action on ends of curves 117
8.2. Gluing 129
8.3. Diagonal Action 137
Index of notations 143
Terms 145
References 145

1. Introduction

This article is the first step in a program to recast Heegaard-Floer theory in the setting of higher representation theory. It is also the first construction of tensor products in 2-representation theory, a mechanism that produces complicated categories out of simpler ones, an algebraic counterpart to the construction of moduli spaces in algebraic geometry.

Heegaard-Floer homology was introduced by Ozsváth and Szabó [OsSz1] and has become a very powerful tool in topology in dimensions 3 and 4. It is related to Seiberg-Witten theory and instanton Floer homology.

Heegaard-Floer theory associates bigraded vector spaces to links in S^3, hence two-variable invariants of links. Specializing one of the variables to -1, they coincide with the Alexander polynomial. The Alexander polynomial can be obtained from the category of representations of quantum $\mathfrak{gl}(1|1)$ (cf e.g. [KaSal, Sar]). It has been hoped that there is a 2-categorical version of that category ("2-representations of $\mathfrak{gl}(1|1)$") that would give rise to (a $(1,2,3,4)$-TQFT version of) Heegaard-Floer theory, in analogy with Crane-Frenkel's proposal [CrFr] to upgrade the 3-dimensional Witten-Reshetikhin-Turaev TQFT to dimension 4, in the case of ordinary simple Lie algebras. Very roughly, the theory would associate the monoidal 2-category of 2-representations of $\mathfrak{gl}(1|1)^+$ to an interval. A physical framework for the role played by Heegaard-Floer theory for 3-manifolds, and its more precise relation with a conjectural $\mathfrak{gl}(1|1)$-theory, is developed in [GuPuVa].

Bordered Heegaard-Floer theory is an extension down to dimension 2 introduced by Lipshitz, Ozsváth and Thurston [LiOzTh1]. Douglas and Manolescu initiated in [DouMa] an extension of this theory further down to dimension 1 (cf also [DouLiMa] for the analysis of 3-manifolds with codimension-2 corners).

In [Rou3], the second author constructs a monoidal structure for the 2-category of 2-representations of Kac-Moody algebras, as a step to obtain the sought-after 4-dimensional TQFT. This requires working in a suitable homotopical setting (A_∞- or ∞-categories), and this creates technical complications for the full construction of a braided monoidal 2-categorical structure.

A surprising discovery in our work is that those homotopical complications in the definition of the monoidal structure disappear for $\mathfrak{gl}(1|1)$, and one of the objectives of this article is to develop as much of the theory as possible without bringing in homotopical aspects. This essentially means we are probing the $4d$ theory only with respect to curves and surfaces. This also limits us to the consideration of the positive part of $\mathfrak{gl}(1|1)$. We will consider these homotopical aspects in future work.

Inspired by constructions in bordered Heegaard-Floer theory, Khovanov [Kh] introduced a monoidal category $\mathcal{U}$ whose Grothendieck group is the enveloping algebra of the positive part $\mathfrak{gl}(1|1)^+$ of the super Lie algebra $\mathfrak{gl}(1|1)$, namely the algebra $\mathbf{Z}[e]/e^2$. This monoidal category is built from the characteristic 2 nil Hecke algebras of symmetric groups, with a differential (that makes them acyclic in rank $\geqslant 2$), and the monoidal structure comes from induction and restriction functors.

Our 2-category of 2-representations of $\mathfrak{gl}(1|1)^+$ is defined as the 2-category of differential categories acted on by the monoidal differential category $\mathcal{U}$. Given two 2-representations, one has a category of lax morphisms of 2-representations. This generalizes to the case of a differential category with two (lax) commuting structures of 2-representations. In the particular case of a differential category obtained by tensor product from two 2-representations, this is a structure considered and studied in [DouMa]. Our new construction is that of a structure of 2-representation on a differential category with commuting structures of 2-representations. From a field theory perspective, we have a surface and two chosen boundary intervals, and we attach an open pair of pants to obtain a new surface which has now one fewer boundary circle.

Lipshitz-Ozsváth-Thurston [LiOzTh1] associate certain differential strand algebras to surfaces with extra structure. That structure can be encoded in a singular curve. An end of the curve gives rise to a structure of 2-representation. We show that gluing curves corresponds, at the level of strand categories, to the construction above. Since the category of differential modules over the strand algebra is equivalent to a partially wrapped Fukaya category of a symmetric power of the surface [Au1], our work can be viewed as providing an algebraic description of the Fukaya category of symmetric powers of a surface that is obtained by gluing simpler surfaces.

Let us now describe the structure of this article.

We gather in §2 a number of basic definitions and facts involving differential categories and bimodules. Most differential vector spaces we encounter come with bases, and we formalize this aspect in the notion of "differential pointed sets" and corresponding differential pointed categories.

We consider Hecke algebras in §3. We study in §3.1 the differential algebra structure on nil Hecke algebras of Coxeter groups over a field of characteristic 2 and we describe adjunctions for induction and restriction functors, in the case of finite Coxeter groups. An important fact is that those Hecke algebras are the graded algebras associated with the filtration of the group algebra with respect to the length function. The remainder of §3 is devoted to the case of symmetric groups and their affine versions. We introduce in §3.2.6 positive submonoids of the affine symmetric groups and we provide a description by generators and relations of their nil Hecke algebras.

Section §4 is devoted to the development of the 2-representation theory of $\mathfrak{gl}(1|1)^+$. We introduce the monoidal category $\mathcal{U}$. Our main construction is that of a tensor product product operation on 2-representations, and more generally, of a diagonal action given two (lax) commuting 2-representation structures. We also consider a more complicated "dual" construction in §4.4. In §5, we recast our functorial constructions into bimodule constructions. Note that our constructions work in the differential setting, but not in the usual differential graded setting.

In §6, we construct bimodules and 2-representations associated with nil Hecke algebras. In §6.1, we describe explicitly the structures of 2-representation coming from the left and the right action of the monoidal category $\mathcal{U}$ on itself and we show that the diagonal category arising from these commuting left and right actions corresponds to Hecke algebras of positive affine symmetric groups. It is a remarkable fact that those can be recovered from the Hecke algebras of the ordinary symmetric groups. We introduce in §6.2 a categorical version of affine symmetric groups and their Hecke algebras.

We develop in §7 an extension of Lipshitz-Ozsváth-Thurston [LiOzTh1] and Zarev's [Za] theory of strand algebras associated with matched circles and intervals. Instead of considering curves with matchings, we consider the corresponding quotient spaces, where the matched points are identified. We start in §7.1 with 1-dimensional spaces, which we define as complements of a finite set of points in a 1-dimensional finite CW-complex. In §7.2, we define our objects of interest, the singular curves. They are 1-dimensional spaces together with an additional structure at singular points, and a partially defined orientation. They arise as quotients of smooth curves, or, equivalently, as curves in $\mathbf{R}^n$ with transverse intersections of branches. This leads to a notion of admissible paths, those paths that lift to a smooth model for the curve (§7.3). We introduce in §7.4 the differential categories of strands associated to a curve. They are defined as graded categories associated with a filtered category, in a way similar to the constructions of §3.1. We show in §7.4.3 that strand categories on unoriented S^1 correspond to the categories built from nil Hecke algebras of affine symmetric groups.

The final section §8 shows that the strand category of a glued curve is obtained as a tensor (or more general diagonal) construction from the strand category of the original curve. This provides some sort of 1-dimensional field theory, which is really part of a 2-dimensional field theory for surfaces with extra structure. This gives a categorical mechanism by which strand categories can be computed by cutting the curve into basic building blocks. We start in §8.1 by constructing a structure of 2-representation associated with an unoriented "end" of a curve. We describe in §8.2 how the strand categories behave under the gluing of two ends of a curve. When the gluing operation does not create an S^1, we show in §8.3 that the resulting 2-representation is the one obtained from the diagonal action.

We thank Ciprian Manolescu for several useful conversations.

2. Differential and pointed structures

2.1. **Differential algebras and categories.**

2.1.1. *Categories.* Let $\mathcal{C}$ be a category. We denote by $\mathcal{C}^{\mathrm{opp}}$ the opposite category. We identify $\mathcal{C}$ with a full subcategory of $\mathrm{Hom}(\mathcal{C}^{\mathrm{opp}}, \mathrm{Sets})$ via the Yoneda embedding $c \mapsto \mathrm{Hom}(-, c)$.

Given (L, R) a pair of adjoint functors, we denote the unit of adjunction by $\eta_{L,R}$ and the counit by $\varepsilon_{L,R}$.

When $\mathcal{C}$ is enriched in abelian groups, we denote by $\mathrm{add}(\mathcal{C})$ the smallest full subcategory of $\mathrm{Hom}(\mathcal{C}^{\mathrm{opp}}, \mathrm{Sets})$ containing $\mathcal{C}$ and closed under finite coproducts and isomorphisms.

Let $\mathcal{X}$ be a 2-category. We denote by $\mathcal{X}^{\mathrm{opp}}$ the 2-category with same objects and $\mathcal{H}om(x, y) = \mathcal{H}om(x, y)^{\mathrm{opp}}$. We denote by $\mathcal{X}^{\mathrm{rev}}$ the 2-category with the same objects and with $\mathcal{H}om(x, y) = \mathcal{H}om(y, x)$ for x and y two objects of $\mathcal{X}$ (so that the composition of 1-arrows is reversed).

Let $\mathcal{C}at$ be the 2-category of categories. There is an equivalence $\mathcal{C}at \xrightarrow{\sim} \mathcal{C}at^{\mathrm{opp}}$ sending a category $\mathcal{C}$ to $\mathcal{C}^{\mathrm{opp}}$.

Let $\mathcal{C}at^r$ (resp. $\mathcal{C}at^l$) be the 2-full 2-subcategory of $\mathcal{C}at$ with 1-arrows those functors that admit a left (resp. right) adjoint. There is an equivalence of 2-categories $\mathcal{C}at^r \xrightarrow{\sim} (\mathcal{C}at^l)^{\mathrm{revopp}}$. It is the identity on objects and sends a functor to a left adjoint.

2.1.2. *Differential categories.* Let k be a field of characteristic 2. We write $\otimes$ for $\otimes_k$.

A *differential module* is a k-vector space M endowed with an endomorphism d satisfying $d^2 = 0$. We put $Z(M) = \ker d$. An element m of M is said to be *closed* when $d(m) = 0$. We define Hom-spaces in the category k-diff of differential modules by $\mathrm{Hom}_{k\text{-diff}}(M, M') = \mathrm{Hom}_{k\text{-Mod}}(M, M')$. That k-module has a differential given by $\mathrm{Hom}(d_M, M') + \mathrm{Hom}(M, d_{M'})$. We define the category $Z(k\text{-diff})$ as the subcategory of k-diff with same objects as k-diff and $\mathrm{Hom}_{Z(k\text{-diff})}(M, M') = Z(\mathrm{Hom}_{k\text{-diff}}(M, M'))$.

The tensor product of vector spaces and the permutation of factors equip k-diff and $Z(k\text{-diff})$ with a structure of symmetric monoidal category.

A *differential category* is a category enriched over $Z(k\text{-diff})$.

Let $\mathcal{V}$ and $\mathcal{V}'$ be two differential categories. We denote by $\mathrm{Hom}(\mathcal{V}, \mathcal{V}')$ the differential category of (k-linear) differential functors $\mathcal{V} \to \mathcal{V}'$. Its Hom spaces are k-linear natural transformations.

We denote by $\mathcal{V} \otimes \mathcal{V}'$ the differential category with set of objects $\mathrm{Obj}(\mathcal{V}) \times \mathrm{Obj}(\mathcal{V}')$ and with $\mathrm{Hom}_{\mathcal{V}\otimes\mathcal{V}'}((v_1, v_1'), (v_2, v_2')) = \mathrm{Hom}_{\mathcal{V}}(v_1, v_2) \otimes \mathrm{Hom}_{\mathcal{V}'}(v_1', v_2')$.

We denote by $\mathcal{V}\text{-diff} = \mathrm{Hom}(\mathcal{V}, k\text{-diff})$ the category of *$\mathcal{V}$-modules.* There is a fully faithful embedding $v \mapsto \mathrm{Hom}_{\mathcal{V}}(-, v) : \mathcal{V} \to \mathcal{V}^{\mathrm{opp}}\text{-diff}$ and we identify $\mathcal{V}$ with its image.

Note that $\mathrm{add}(\mathcal{V})$ identifies with the smallest full subcategory of $\mathcal{V}^{\mathrm{opp}}$-diff containing $\mathcal{V}$ and closed under finite direct sums and isomorphisms.

There is a differential functor $\otimes_{\mathcal{V}} : \mathcal{V}^{\mathrm{opp}}\text{-diff} \otimes \mathcal{V}\text{-diff} \to k\text{-diff}$. Given $M \in \mathcal{V}^{\mathrm{opp}}$-diff and $N \in \mathcal{V}$-diff, there is an exact sequence of differential k-modules

$$\bigoplus_{f \in \mathrm{Hom}_{\mathcal{V}}(v_1, v_2)} M(v_2) \otimes N(v_1) \xrightarrow[-a\otimes N(f)(b)]{a\otimes b \mapsto M(f)(a)\otimes b} \bigoplus_{v \in \mathcal{V}} M(v) \otimes N(v) \to M \otimes_{\mathcal{V}} N \to 0.$$

Given $v \in \mathcal{V}$, we have $\mathrm{Hom}(-, v) \otimes_{\mathcal{V}} N = N(v)$ and $M \otimes_{\mathcal{V}} \mathrm{Hom}(v, -) = M(v)$.

Recall that a category is *idempotent complete* if all idempotent maps have images.

We denote by $\mathcal{V}^i$ the *idempotent completion* of $\mathcal{V}$: this is the smallest full subcategory of $\mathcal{V}^{\mathrm{opp}}$-diff containing $\mathcal{V}$ and closed under direct summands and isomorphisms. The 2-functor $\mathcal{V} \mapsto \mathcal{V}^i$ is left adjoint to the embedding of idempotent-complete differential categories in differential categories.

2.1.3. *Objects.* Given v_1, v_2 two objects of $\mathcal{V}$ and given $f \in Z\operatorname{Hom}_{\mathcal{V}}(v_1, v_2)$, the *cone* of f is the object $\operatorname{cone}(\operatorname{Hom}_{\mathcal{V}}(-, f))$ of $\mathcal{V}^{\mathrm{opp}}$-diff denoted by $v_1 \overset{f}{\frown}\!\!\!\!\!\!\oplus v_2$. We say that $\mathcal{V}$ is *strongly pretriangulated* if the cone of any map of $\mathcal{V}$ is isomorphic to an object of $\mathcal{V}$. Note that $\mathcal{V}^{\mathrm{opp}}$-diff is strongly pretriangulated.

We denote by $\bar{\mathcal{V}}$ the smallest full strongly pretriangulated subcategory of $\mathcal{V}^{\mathrm{opp}}$-diff closed under taking isomorphic objects and containing $\mathcal{V}$. Note that $(\bar{\mathcal{V}})^i$ is strongly pretriangulated. Note also that if $\mathcal{V}$ is a full subcategory of a strongly pretriangulated $\mathcal{V}'$, then $\mathcal{V}$ is strongly pretriangulated if the cone in $\mathcal{V}'$ of a map between objects of $\mathcal{V}$ is isomorphic to an object of $\mathcal{V}$.

Let $v_1, \ldots, v_n$ be objects of $\mathcal{V}$ and $f_{ij} \in \operatorname{Hom}_{\mathcal{V}}(v_j, v_i)$ for $i < j$. Assume $d(f_{ij}) = \sum_{i<r<j} f_{ir} \circ f_{rj}$ for all $i < j$. We define the *twisted object* $\left[v_n \oplus \cdots \oplus v_1, \begin{pmatrix} 0 & & & \\ f_{n-1,n} & \ddots & & \\ \vdots & \ddots & 0 & \\ f_{1,n} & \ldots & f_{1,2} & 0 \end{pmatrix}\right]$ of $\bar{\mathcal{V}}$ inductively on n as the cone of

$$(f_{n-1,n}, \ldots, f_{1,n}) : v_n \to \left[v_{n-1} \oplus \cdots \oplus v_1, \begin{pmatrix} 0 & & & \\ f_{n-2,n-1} & \ddots & & \\ \vdots & \ddots & 0 & \\ f_{1,n-1} & \ldots & f_{1,2} & 0 \end{pmatrix}\right].$$

The objects of $\bar{\mathcal{V}}$ are the objects of $\mathcal{V}^{\mathrm{opp}}$-diff isomorphic to a twisted object of $\mathcal{V}$.

If $\mathcal{V}'$ is strongly pretriangulated, then the restriction functor $\operatorname{Hom}(\bar{\mathcal{V}}, \mathcal{V}') \to \operatorname{Hom}(\mathcal{V}, \mathcal{V}')$ is an equivalence. So, $\mathcal{V} \mapsto \bar{\mathcal{V}}$ is left adjoint to the embedding of strongly pretriangulated differential categories in differential categories.

2.1.4. *Algebras.* Let A be a differential algebra. We denote by A-diff the category of (left) differential A-modules. Note that $\operatorname{Hom}_{A\text{-diff}}(M, M')$ is the differential k-module of A-linear maps $M \to M'$. This is an idempotent-complete strongly pretriangulated differential category. We say that a differential A-module is *strictly perfect* if it is in $(\bar{A})^i$, where A denotes the full subcategory of A-diff with a unique object A.

A differential category $\mathcal{C}$ with one object c is the same as the data of a differential algebra $A = \operatorname{End}_{\mathcal{C}}(c)$. When $\mathcal{C}$ has a unique object c and $A = \operatorname{End}_{\mathcal{C}}(c)$, then there is an isomorphism $A\text{-diff} \xrightarrow{\sim} \mathcal{C}\text{-diff},\ M \mapsto (c \mapsto M)$.

More generally, a differential category $\mathcal{C}$ can be viewed as a "differential algebra with several objects". More precisely, there is an equivalence from the category of differential categories $\mathcal{C}$ with finitely many objects (arrows are differential functors) to the category of differential algebras A equiped with a finite set I of orthogonal idempotents with sum 1 (arrows $(A, I) \to (A', I')$ are non-unital morphisms of differential algebras $f : A \to A'$ such that $f(I) \subset I'$):

- to $\mathcal{C}$, we associate $A = \bigoplus_{c,c' \in \mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(c, c')$ and I the set of projectors on objects of $\mathcal{C}$;
- to (A, I), we associate the differential category $\mathcal{C}$ with set of objects I and $\operatorname{Hom}_{\mathcal{C}}(e, f) = fAe$.

2.1.5. *G-graded differential structures.* We define a **Z**-*monoid* G to be a monoid G endowed with an action of the group $\mathbf{Z}$, denoted by $g \mapsto g+n$ for $g \in G$ and $n \in \mathbf{Z}$, and such that $(g+n)(g'+n') = gg'+n+n'$. Note that $e_G + \mathbf{Z}$ is a central submonoid of G, where e_G denotes the unit of G. So, the data above is equivalent to the data of a morphism of monoids $\mathbf{Z} \to Z(G)$. This is itself determined by the image of 1, a central invertible element υ of G.

We define a *differential G-graded k-module* to be a G-graded k-module M together with a differential module structure such that $d(M_g) \subset M_{g+1}$ (cf [LiOzTh1, §2.5]).

Given $g \in G$, we define $M\langle g\rangle$ to be the differential G-graded k-module given by $(M\langle g\rangle)_h = M_{hg}$. Similarly, we define $\langle g\rangle M$ by $(\langle g\rangle M)_h = M_{gh}$.

We define similarly the notion of *differential G-graded algebra*, of *differential G-graded category*, etc.

When $G = \mathbf{Z}$ and $\upsilon = 1$, we recover the usual notion of differential graded k-module, etc.

Let G_1 and G_2 be two **Z**-monoids. We define $G_1 \times_{\mathbf{Z}} G_2$ as the quotient of $G_1 \times G_2$ by the equivalence relation $(g_1, g_2+n) \sim (g_1+n, g_2)$ for $g_1, g_2 \in G$ and $n \in \mathbf{Z}$. Denote by $p : G_1 \times G_2 \to G_1 \times_{\mathbf{Z}} G_2$ the quotient map, a morphism of monoids. There is a structure of **Z**-monoid on $G_1 \times_{\mathbf{Z}} G_2$ given by $p(g_1, g_2)+1 = p(g_1+1, g_2) = p(g_1, g_2+1)$.

Let M_i be a differential G_i-graded k-module for $i \in \{1,2\}$. We define a structure of differential $(G_1 \times_{\mathbf{Z}} G_2)$-module on the differential module $M_1 \otimes M_2$ by setting $(M_1 \otimes M_2)_g = \bigoplus_{(g_1,g_2)\in p^{-1}(g)} (M_1)_{g_1} \otimes (M_2)_{g_2}$.

2.2. **Bimodules.**

2.2.1. *Algebras.* Let Alg be the 2-category with objects the differential algebras, and $\mathrm{Hom}_{\mathrm{Alg}}(A, A')$ the category of (A', A)-bimodules. The composition of 1-arrows is the tensor product of differential bimodules.

Given M an (A', A)-bimodule, we put $M^\vee = \mathrm{Hom}_{A^{\mathrm{opp}}}(M, A)$, an (A, A')-bimodule.

There is a morphism of (A', A)-bimodules

$$M \to \mathrm{Hom}_A(M^\vee, A),\ m \mapsto (\zeta \mapsto \zeta(m)).$$

It is an isomorphism if M is finitely generated and projective as a (non-differential) A^{opp}-module.

There is a morphism of functors

$$\mathrm{Hom}_A(M^\vee, A) \otimes_A - \to \mathrm{Hom}_A(M^\vee, -),\ f \otimes r \mapsto (\zeta \mapsto f(\zeta)r).$$

It is an isomorphism if $M^\vee$ is finitely generated and projective as a (non-differential) A-module.

Combining those two morphisms, we obtain a morphism of functors

$$M \otimes_A - \to \mathrm{Hom}_A(M^\vee, -)$$

that is an isomorphism if M is finitely generated and projective as a (non-differential) A^{opp}-module. So, when this holds, we have an adjoint pair $(M^\vee \otimes_{A'} -, M \otimes_A -)$, with corresponding unit $\eta : A' \to M \otimes_A M^\vee$ and counit $\varepsilon : M^\vee \otimes_{A'} M \to A$. In other terms, the bimodule $M^\vee$ is a left dual of M .

Note conversely that given M such that $(M^\vee \otimes_{A'} -, M \otimes_A -)$ is an adjoint pair, then $M^\vee$ is a finitely generated projective A-module because $\mathrm{Hom}_A(M^\vee, -)$ is exact and commutes with direct sums, hence $M \simeq \mathrm{Hom}_A(M^\vee, A)$ is finitely generated and projective as an A^{opp}-module.

We say that M is *right finite* when it is finitely generated and projective as an A^{opp}-module. We say that M is *left finite* when it is finitely generated and projective as an A'-module.

Consider the 2-full subcategory Alg^r (resp. Alg^l) of Alg with same objects and 1-arrows the right (resp. left) finite bimodules. There is an equivalence of 2-categories $\mathrm{Alg}^r \xrightarrow{\sim} (\mathrm{Alg}^l)^{\mathrm{revopp}}$. It is the identity on objects and sends a bimodule M to $M^\vee$.

2.2.2. *Categories.* Let $\mathcal{C}$ and $\mathcal{C}'$ be differential categories. A $(\mathcal{C}, \mathcal{C}')$*-bimodule* is a differential functor $\mathcal{C} \otimes \mathcal{C}'^{\mathrm{opp}} \to k\text{-diff}$. There is a 2-category Bimod of differential categories and bimodules. Its objects are differential categories and $\mathcal{H}om_{\mathrm{Bimod}}(\mathcal{C}, \mathcal{C}')$ is the differential category of $(\mathcal{C}', \mathcal{C})$-bimodules. Composition is given by tensor product.

There is an equivalence of 2-categories $\mathrm{Bimod} \xrightarrow{\sim} \mathrm{Bimod}^{\mathrm{rev}}$ sending a differential category $\mathcal{C}$ to $\mathcal{C}^{\mathrm{opp}}$ and a $(\mathcal{C}, \mathcal{C}')$-bimodule to the same functor, viewed as a $(\mathcal{C}'^{\mathrm{opp}}, \mathcal{C}^{\mathrm{opp}})$-bimodule.

The bimodule $\mathrm{Hom} : \mathcal{C} \otimes \mathcal{C}^{\mathrm{opp}} \to k\text{-diff}$, $(c_1, c_2) \mapsto \mathrm{Hom}_{\mathcal{C}}(c_2, c_1)$ is an identity for the tensor product. The canonical isomorphism of $(\mathcal{C}, \mathcal{C})$-bimodules $\mathrm{Hom} \otimes_{\mathcal{C}} \mathrm{Hom} \xrightarrow{\sim} \mathrm{Hom}$ is given by

$$\mathrm{Hom}_{\mathcal{C}}(-, c_1) \otimes_{\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(c_2, -) \to \mathrm{Hom}(c_2, c_1),\ ((f : d \to c_1) \otimes (g : c_2 \to d) \mapsto f \circ g.$$

Let M be a $(\mathcal{C}', \mathcal{C})$-bimodule. We define the $(\mathcal{C}, \mathcal{C}')$-bimodule $M^\vee$ by

$$M^\vee(c, c') = \mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}\text{-diff}}(M(c', -), \mathrm{Hom}_{\mathcal{C}}(-, c)).$$

There is a morphism of $(\mathcal{C}, \mathcal{C})$-bimodules $\varepsilon_M : M^\vee \otimes_{\mathcal{C}'} M \to \mathrm{Hom}$ given by

$$\varepsilon_M(c_1, c_2) : M^\vee(c_1, -) \otimes_{\mathcal{C}'} M(-, c_2) \to \mathrm{Hom}(c_2, c_1)$$
$$(M(c', -) \xrightarrow{f} \mathrm{Hom}(-, c_1)) \otimes m \mapsto f(c_2)(m) \text{ for } m \in M(c', c_2).$$

Given $L \in \mathcal{C}\text{-diff}$ and $L' \in \mathcal{C}'\text{-diff}$, we have a morphism functorial in L and L'

$$\mathrm{Hom}(L', M \otimes_{\mathcal{C}} L) \xrightarrow{M^\vee \otimes -} \mathrm{Hom}(M^\vee \otimes_{\mathcal{C}'} L', M^\vee \otimes_{\mathcal{C}'} M \otimes_{\mathcal{C}} L) \xrightarrow{\mathrm{Hom}(M^\vee \otimes_{\mathcal{C}'} L', \varepsilon_M)} \mathrm{Hom}(M^\vee \otimes_{\mathcal{C}'} L', L).$$

We say that M is *right finite* if the morphism above is an isomorphism for all L and L'. When this holds, the functor $M^\vee \otimes_{\mathcal{C}'} -$ is left adjoint to $M \otimes_{\mathcal{C}} -$ and $M^\vee$ is left dual to M . We also write ${}^\vee N = M$ where $N = M^\vee$. We say that M is *left finite* if it is a right finite $(\mathcal{C}'^{\mathrm{opp}}, \mathcal{C}^{\mathrm{opp}})$-bimodule.

Let M be a $(\mathcal{C}, \mathcal{C})$-bimodule. We define the differential category $T_{\mathcal{C}}(M)$. Its objects are those of $\mathcal{C}$ and

$$\mathrm{Hom}_{T_{\mathcal{C}}(M)}(c_1, c_2) = \bigoplus_{i \geqslant 0} M^i(c_1, c_2).$$

2.2.3. *Bimodules and functors.* There is a 2-functor from Alg to Bimod: it sends A to the differential category $\mathcal{C}_A$ with one object c_A and $\mathrm{End}(c_A) = A$. It sends an (A', A)-bimodule M to the $(\mathcal{C}_{A'}, \mathcal{C}_A)$-bimodule $\mathcal{C}_M$ given by $\mathcal{C}_M(c_A, c_{A'}) = M$. This 2-functor provides isomorphisms of categories $\mathrm{Hom}_{\mathrm{Alg}}(A, A') \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Bimod}}(\mathcal{C}_A, \mathcal{C}_{A'})$.

There is a 2-fully faithful 2-functor from the 2-category of differential categories to $\mathrm{Bimod}^{\mathrm{rev}}$: it sends $\mathcal{C}$ to $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{C}'$ to the $(\mathcal{C}, \mathcal{C}')$-bimodule $(c, c') \mapsto \mathrm{Hom}(c', F(c))$.

There is a 2-fully faithful 2-functor from Bimod to the 2-category of differential categories: it sends $\mathcal{C}$ to $\mathcal{C}$-diff and M a $(\mathcal{C}', \mathcal{C})$-bimodule to $M \otimes_{\mathcal{C}} - : \mathcal{C}\text{-diff} \to \mathcal{C}'\text{-diff}$.

Composing the 2-functor $\mathrm{Alg} \to \mathrm{Bimod}$ and the 2-functor from Bimod to the 2-category of differential categories, we obtain a differential 2-functor from Alg to the 2-category of differential categories: it sends A to A-diff and it sends an (A', A)-bimodule M to the functor $M \otimes_A - : A\text{-diff} \to A'\text{-diff}$. Note that this 2-functor is 2-fully faithful.

2.3. **Pointed sets and categories.**

2.3.1. *Pointed sets.* A *pointed set* is a set with a distinguished element 0. The category $\mathrm{Sets}^{\bullet}$ of pointed sets has objects pointed sets and arrows those maps that preserve the distinguished element.

It has coproducts: $\bigvee S_i$ is the quotient of $\coprod S_i$ by the relation identifying the 0-objects of the S_i's.

We define $\bigwedge S_i$ as the quotient of $\prod S_i$ by the relation identifying an element with $(0)_i$ if one of its components is 0. There is a canonical isomorphism $S \wedge \{0, *\} \xrightarrow{\sim} S$. This provides the category of pointed sets with a structure of symmetric monoidal category (the tensor product of S_1 and S_2 is $S_1 \wedge S_2$) and there is a symmetric monoidal functor from the category of sets to the category of pointed sets $E \mapsto E_+ = E \sqcup \{0\}$.

Given S a pointed set and k a commutative ring, we denote by $k[S]$ the quotient of the free k-module with basis S by the k-submodule generated by the distinguished element of S. This gives a coproducts preserving monoidal functor from the category of pointed sets to the category of k-modules.

Assume k is finite. Let S and S' be two pointed sets. We say that a k-linear map $f : k[S] \to k[S']$ is *bounded* if there is $N > 0$ such that for all $s \in S$, the set of elements of S' that have a non-zero coefficient in $f(s)$ has less than N elements.

The functor $k[-]$ induces a bijection from $k[\mathrm{Hom}_{\mathrm{Sets}^{\bullet}}(S, S')]$ to the subspace of bounded maps in $\mathrm{Hom}_{k\text{-Mod}}(k[S], k[S'])$.

2.3.2. *Gradings and filtrations.* Let G be a set. A G-graded pointed set is a pointed set S together with pointed subsets S_g for $g \in G$ such that $S = \bigcup_{g \in G} S_g$ and $S_g \cap S_h = \{0\}$ for $g \neq h$.

Given a map $f : G \to G'$ and S a G-graded pointed set, we define a structure of G'-graded pointed set on S by setting $S_{g'} = \{0\} \cup \bigcup_{g \in f^{-1}(g')} S_g$.

Given G_1 and G_2 two sets and S_i a G_i-graded pointed set for $i \in \{1, 2\}$, then $S_1 \wedge S_2$ is a $(G_1 \times G_2)$-graded pointed set with $(S_1 \wedge S_2)_{(g_1, g_2)} = (S_1)_{g_1} \wedge (S_2)_{g_2}$.

Assume G is a monoid. Given two G-graded pointed sets S and T, there is a structure of $(G \times G)$-graded pointed set on $S \wedge T$. Via the multiplication map, we obtain a structure of G-graded pointed set on $S \wedge T$. This makes the category of G-graded pointed sets into a monoidal category with unit object the pointed set $S = \{0, *\}$ with $S_1 = S$ and $S_g = \{0\}$ for $g \neq 1$.

Let G be a poset. A G-filtered set (resp. pointed set) is a set (resp. a pointed set) S together with subsets (resp. pointed subsets) $S_{\geqslant g}$ for $g \in G$ such that $S_{\geqslant g} \subset S_{\geqslant g'}$ if $g > g'$ and such that given $s \in S$ (resp. $s \in S \backslash \{0\}$), the set $\{g \in G \mid s \in S_{\geqslant g}\}$ is non-empty and has a maximal element, which we denote by $\deg(s)$.

Note that a structure of G-filtered set on a set (resp. a pointed set) S is the same as the data of a map $S \to G$ (resp. a map $S \backslash \{0\} \to G$).

The associated G-graded pointed set is $\mathrm{gr} S = \{0\} \sqcup S$ (resp. $\mathrm{gr} S = S$) with

$$(\mathrm{gr} S)_g = \{0\} \sqcup \{s \in S \mid \deg(s) = g\} \text{ (resp. } (\mathrm{gr} S)_g = \{s \in S \backslash \{0\} \mid \deg(s) = g\}).$$

If G is a (partially) ordered monoid, then the category of G-filtered sets (resp. pointed sets) is a monoidal category with $(S \wedge T)_{\geqslant g}$ the image of $\coprod_{g_1, g_2 \in G, g_1 g_2 \geqslant g} (S_{\geqslant g_1} \times T_{\geqslant g_2})$ in $S \wedge T$. Its unit object is the set $S = \{*\}$ (resp. the pointed set $S = \{0, *\}$) with $S_{\geqslant g} = S$ if $1 \geqslant g$ and $S_{\geqslant g} = \varnothing$ (resp. $S_{\geqslant g} = \{0\}$) otherwise.

There is a monoidal functor $S \mapsto \mathrm{gr}S$ from the monoidal category of G-filtered sets (resp. pointed sets) to the monoidal category of G-graded pointed sets. Given $f : S \to T$ a map between G-filtered sets (resp. pointed sets), the map $\mathrm{gr}f : \mathrm{gr}S \to \mathrm{gr}T$ is given for $s \in (\mathrm{gr}S)_g$ by $(\mathrm{gr}f)(s) = f(s)$ if $f(s) \in (\mathrm{gr}T)_g$ and $(\mathrm{gr}f)(s) = 0$ otherwise.

Note also that given a commutative ring k there is a monoidal functor $S \mapsto k[S]$ from the category of G-graded pointed sets to the category of G-graded k-modules.

2.3.3. *Pointed categories.* A *pointed category* is a category enriched in pointed sets. We define similarly G-graded pointed categories, etc. The monoidal functors $\mathcal{V}_1 \to \mathcal{V}_2$ defined above provide a construction from a category enriched in $\mathcal{V}_1$ of a category enriched in $\mathcal{V}_2$. Let us describe this more explicitly.

• Given a G-filtered category (or a G-filtered pointed category) $\mathcal{C}$, we have a G-graded pointed category $\mathrm{gr}\mathcal{C}$. Its objects are the same as those of $\mathcal{C}$ and $\mathrm{Hom}_{\mathrm{gr}\mathcal{C}}(c, c') = \mathrm{gr}\,\mathrm{Hom}_{\mathcal{C}}(c, c')$.

• Given a pointed category $\mathcal{C}$, we denote by $k[\mathcal{C}]$ the associated k-linear category: its objects are those of $\mathcal{C}$ and $\mathrm{Hom}_{k[\mathcal{C}]}(c, c') = k[\mathrm{Hom}_{\mathcal{C}}(c, c')]$. If $\mathcal{C}$ is a G-graded pointed category, then $k[\mathcal{C}]$ is a k-linear G-graded category.

• Given a category $\mathcal{C}$, the associated pointed category $\mathcal{C}_+$ has the same objects as $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}_+}(c, c') = \mathrm{Hom}_{\mathcal{C}}(c, c') \sqcup \{0\}$.

Consider a family $\{\mathcal{C}_i\}$ of pointed categories. We have a pointed category $\bigwedge \mathcal{C}_i$ with object set $\prod \mathrm{Obj}(\mathcal{C}_i)$ and $\mathrm{Hom}_{\bigwedge \mathcal{C}_i}((c_i), (c_i')) = \bigwedge \mathrm{Hom}_{\mathcal{C}_i}(c_i, c_i')$. Similarly, we have a pointed category $\bigvee \mathcal{C}_i$ with object set $\coprod \mathrm{Obj}(\mathcal{C}_i)$ and given $c \in \mathcal{C}_r$ and $c' \in \mathcal{C}_s$, we have

$$\mathrm{Hom}_{\bigwedge \mathcal{C}_i}(c, c') = \begin{cases} \mathrm{Hom}_{\mathcal{C}_r}(c, c') & \text{if } r = s \\ \{0\} & \text{otherwise.} \end{cases}$$

Note that the data of a structure of G-filtered pointed category on a pointed category $\mathcal{C}$ is the same as the data of a map deg from the set of non-zero maps of $\mathcal{C}$ to G such that $\deg(\beta \circ \alpha) \geqslant \deg(\beta)\deg(\alpha)$ for any two composable maps α and β such that $\beta \circ \alpha \neq 0$.

Given a G-filtered pointed category $\mathcal{C}$ with degree function deg and given a morphism of (partially) ordered monoids $f : G \to H$, we obtain a structure of H-filtered pointed category on $\mathcal{C}$ with degree function $f \circ \deg$.

Note that the category $\mathrm{Sets}^\bullet$ has a structure of pointed category: the distinguished map between two pointed sets is the map with image 0.

2.3.4. *Differential pointed categories.* We define a *differential pointed set* to be a pointed set S together with a bounded endomorphism d of $\mathbf{F}_2[S]$ satisfying $d^2 = 0$.

Given S and S' two differential pointed sets, then $S \vee S'$ and $S \wedge S'$ have structures of differential pointed sets coming from the canonical isomorphisms $\mathbf{F}_2[S \vee S'] \xrightarrow{\sim} \mathbf{F}_2[S] \oplus \mathbf{F}_2[S']$ and $\mathbf{F}_2[S \wedge S'] \xrightarrow{\sim} \mathbf{F}_2[S] \otimes \mathbf{F}_2[S']$.

We define the category diff of differential pointed sets: its objects are differential pointed sets and maps the maps of pointed sets. There is a functor $\mathbf{F}_2[-] : \mathrm{diff} \to \mathbf{F}_2\text{-diff}$. Let S and S' be two differential pointed sets. Because the differentials on $\mathbf{F}_2[S]$ and $\mathbf{F}_2[S']$ are bounded, the vector space $\mathbf{F}_2[\mathrm{Hom}_{\mathrm{Sets}^\bullet}(S,S')]$ identifies with a subspace of $\mathrm{Hom}_{\mathbf{F}_2\text{-Mod}}(\mathbf{F}_2[S],\mathbf{F}_2[S'])$ that is stable under the differential $\mathrm{Hom}(d_{\mathbf{F}_2[S]},-)+\mathrm{Hom}(-,d_{\mathbf{F}_2[S']})$.

We define $Z(\mathrm{diff})$ as the subcategory of diff with same objects as diff and with $\mathrm{Hom}_{Z(\mathrm{diff})}(S,S')$ the subset of maps in the kernel of d (where we view $\mathrm{Hom}_{\mathrm{diff}}(S,S')$ inside $\mathrm{Hom}_{\mathbf{F}_2\text{-Mod}}(\mathbf{F}_2[S],\mathbf{F}_2[S'])$). The categories diff and $Z(\mathrm{diff})$ have a structure of symmetric monoidal category coming from those on pointed sets and differential modules.

We define a *differential pointed category* to be a category enriched in $Z(\mathrm{diff})$. This is the same as a pointed category $\mathcal{V}$ together with a differential on $\mathbf{F}_2[\mathcal{V}]$ endowing it with a structure of differential category. The 2-functor $\mathcal{V}\mapsto\mathbf{F}_2[\mathcal{V}]$ from the 2-category of differential pointed categories to the 2-category of differential categories is 2-faithful and 2-conservative.

Note that the category diff is a differential pointed category:

All our constructions below for differential pointed categories are compatible with the corresponding constructions for differential categories, via the 2-functor $\mathbf{F}_2[?]$.

Given G a $\mathbf{Z}$-monoid, we will also consider *differential G-graded pointed sets*: these are differential pointed sets S with a structure of G-graded pointed set such that $d(S_g)]\subset\mathbf{F}_2[S_{g+1}]$ for $g\in G$. We have a corresponding notion of *differential G-graded pointed category.*

Let $\mathcal{V}$ be a differential pointed category. We say that a map of $\mathcal{V}$ is *closed* if its image in $\mathbf{F}_2[\mathcal{V}]$ is closed. Given $f:S\to S'$ a closed map of differential pointed sets, we define the cone $\mathrm{cone}(f)$ of f as the pointed set $S\vee S'$ with differential on $\mathbf{F}_2[S\vee S']=\mathbf{F}_2[S]\oplus\mathbf{F}_2[S']$ given by $\begin{pmatrix} d_{\mathbf{F}_2[S]} & 0\\ f & d_{\mathbf{F}_2[S']}\end{pmatrix}$.

We define a $\mathcal{V}$-module to be a differential pointed functor (*i.e.*, a functor enriched in $Z(\mathrm{diff})$) $\mathcal{V}\to\mathrm{diff}$. We denote by $\mathcal{V}$-diff the category of $\mathcal{V}$-modules.
Given $f:v_1\to v_2$ a closed map in $\mathcal{V}$, we define $\mathrm{cone}(f)=\mathrm{cone}(\mathrm{Hom}_{\mathcal{V}}(f,-))\in\mathcal{V}$-diff.

Let M be a $\mathcal{V}^{\mathrm{opp}}$-module and N a $\mathcal{V}$-module. We define the differential pointed set $M\wedge_{\mathcal{V}}N$ as the coequalizer of

$$\bigvee\nolimits_{f\in\mathrm{Hom}_{\mathcal{V}}(v_1,v_2)}(M(v_2)\wedge N(v_1)) \underset{a\wedge b\mapsto a\wedge N(f)(b)}{\overset{a\wedge b\mapsto M(f)(a)\wedge b}{\rightrightarrows}} \bigvee\nolimits_{v\in\mathcal{V}}(M(v)\wedge N(v)).$$

Given $\mathcal{V}'$ a differential pointed category, we define a $(\mathcal{V},\mathcal{V}')$-bimodule to be a differential pointed functor $\mathcal{V}\bigwedge\mathcal{V}'^{\mathrm{opp}}\to\mathrm{diff}$.
Given $\mathcal{V}''$ a differential pointed category, N a $(\mathcal{V},\mathcal{V}')$-bimodule and M a $(\mathcal{V}',\mathcal{V}'')$-bimodule, then $N\wedge_{\mathcal{V}'}M$ is a $(\mathcal{V},\mathcal{V}'')$-bimodule. This gives rise to a 2-category $\mathrm{Bimod}^\bullet$ of differential pointed categories and bimodules, with a 2-fully faithful functor to the 2-category of differential pointed categories and a 2-faithful functor $\mathbf{F}_2[-]$ to the 2-category Bimod.

Let M be a $(\mathcal{V},\mathcal{V})$-bimodule. We define a differential pointed category $T_{\mathcal{V}}(M)$. Its objects are those of $\mathcal{V}$ and

$$\mathrm{Hom}_{T_{\mathcal{V}}(M)}(v_1,v_2)=\bigvee_{i\geqslant 0}M^i(v_1,v_2).$$

2.3.5. *Pointed structures as $\mathbf{F}_2$-structures with a basis.* Let us reformulate the definitions of the previous sections in terms of $\mathbf{F}_2$-vector spaces with a basis.

The functor $\mathbf{F}_2[-]$ gives an equivalence from the category of pointed sets to the category with objects $\mathbf{F}_2$-vector spaces with a basis and where maps are $\mathbf{F}_2$-linear maps sending a basis element to a basis element or 0.

Under this equivalence, we have the following correspondences:

- a coproduct of pointed spaces corresponds to a direct sum with basis the union of bases
- a wedge product of pointed spaces corresponds to a tensor product with basis the product of bases
- a G-graded pointed set corresponds to a G-graded $\mathbf{F}_2$-vector space with a basis consisting of homogeneous elements
- a G-filtered pointed set corresponds to a G-filtered $\mathbf{F}_2$-vector space V, ie a family $\{V_{\geqslant g}\}_{g\in G}$ of subspaces of V with $V_{\geqslant g} \subset V_{\geqslant g'}$ if $g > g'$, with a basis B such that $B \cap V_{\geqslant g}$ is a basis of $V_{\geqslant g}$ for all $g \in G$ and such that given $v \in V\backslash\{0\}$, the set $\{g \in G \mid V_{\geqslant g} \neq 0\}$ is non-empty and has a maximal element
- a differential pointed set corresponds to an $\mathbf{F}_2$-vector space with a basis together with a bounded differential.

2.4. **Symmetric powers.** Let $\mathcal{C}$ be a pointed category. We define a pointed category $S(\mathcal{C})$. Its objects are finite families I of distinct objects of $\mathcal{C}$. We put

$$\mathrm{Hom}_{S(\mathcal{C})}(I,J) = \bigvee_{\phi} \bigwedge_{i\in I} \mathrm{Hom}_{\mathcal{C}}(i,\phi(i))$$

where ϕ runs over the set of bijections $I \xrightarrow{\sim} J$.

An element of $\mathrm{Hom}_{S(\mathcal{C})}(I,J)$ is a pair (ϕ, f) where $\phi : I \xrightarrow{\sim} J$ is a bijection and $f \in \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(i,\phi(i))$. All pairs with $f_i = 0$ for some i are identified, and they form the 0-element of $\mathrm{Hom}_{S(\mathcal{C})}(I,J)$. The composition is given by $(\psi, g) \circ (\phi, f) = (\psi\phi, (g_{\phi(i)} \circ f_i)_{i\in I})$.

Given a functor $F : \mathcal{C} \to \mathcal{C}'$ of pointed categories that is injective on the set of objects, we obtain a functor $S(F) : S(\mathcal{C}) \to S(\mathcal{C}')$ of pointed categories. If in addition F is faithful, then $S(F)$ is faithful.

Given a commutative ring k and a k-linear category $\mathcal{D}$, we define a k-linear category $S_k(\mathcal{D})$. Its objects are finite families I of distinct objects of $\mathcal{D}$. We put

$$\mathrm{Hom}_{S_k(\mathcal{D})}(I,J) = \bigoplus_{\phi: I \xrightarrow{\sim} J} \bigotimes_{i\in I} \mathrm{Hom}_{\mathcal{D}}(i,\phi(i)).$$

The composition is defined as in the case of pointed categories above.

Consider a functor $F : \mathcal{D} \to \mathcal{D}'$ of k-linear categories that is injective on the set of objects. We obtain a functor $S_k(F) : S_k(\mathcal{D}) \to S_k(\mathcal{D}')$ of pointed categories. If Hom-spaces in $\mathcal{D}$ and $\mathcal{D}'$ are flat over k and F is faithful, then $S_k(F)$ is faithful.

Given a pointed category $\mathcal{C}$, there is an isomorphism of k-linear categories $k[S(\mathcal{C})] \xrightarrow{\sim} S_k(k[\mathcal{C}])$.

3. Hecke algebras

In this section, we define and study variations of the nil affine Hecke algebra of GL_n. From §3.1.5 onwards, all additive structures will be defined over $k = \mathbf{F}_2$.

3.1. **Differential graded nil Hecke algebras.** We discuss here the case of general Coxeter groups. The results will be used only for types A_n and $\tilde{A}_n$.

3.1.1. *Coxeter groups.* We refer to [Hu, §5 and §7.1–7.3] for basic properties of Coxeter groups and Hecke algebras. Recall that a *Coxeter group* (W,S) is the data of a group W with a subset $S \subset W$ such that W has a presentation with generating set S and relations

$$s^2 = 1,\ \underbrace{sts\cdots}_{m_{st}\text{ terms}} = \underbrace{tst\cdots}_{m_{st}\text{ terms}} \text{ when } st \text{ has order } m_{st} \text{ for } s,t \in S.$$

A *reduced expression* of an element $w \in W$ is a decomposition $w = s_{i_1}\cdots s_{i_l}$ such that $s_{i_r} \in S$ for $r = 1,\ldots,l$ and such that l is minimal with this property. The integer l is the *length* $\ell(w)$ of w.

The Chevalley-Bruhat (partial) order on W is defined as follows. Let $w', w \in W$ and let $w = s_{i_1}\cdots s_{i_l}$ be a reduced decomposition. We say that $w' \leqslant w$ if there is $l' \leqslant l$ and an increasing injection $f : \{1,\ldots,l'\} \to \{1,\ldots,l\}$ such that $w' = s_{i_{f(1)}}\cdots s_{i_{f(l')}}$. This is independent of the choice of the reduced decomposition of w.

3.1.2. *Hecke algebras.* Let $R = \mathbf{Z}[\{a_s, b_s\}_{s\in S}]$ where a_s and b_s are indeterminates with $a_s = a_{s'}$ and $b_s = b_{s'}$ if s and s' are conjugate in W.

The Hecke algebra $H = H(W)$ of (W,S) is the R-algebra generated by $\{T_s\}_{s\in S}$ with relations

$$T_s^2 + a_sT_s + b_s = 0,\ \underbrace{T_sT_tT_s\cdots}_{m_{st}\text{ terms}} = \underbrace{T_tT_sT_t\cdots}_{m_{st}\text{ terms}} \text{ when } st \text{ has order } m_{st}.$$

Given a reduced decomposition $w = s_{i_1}\cdots s_{i_l}$, we put $T_w = T_{s_{i_1}}\cdots T_{s_{i_l}}$. This element is independent of the choice of the reduced decomposition of w. The set $\{T_w\}_{w\in W}$ is a basis of H.

Let $\iota : H \xrightarrow{\sim} H^{\mathrm{opp}}$ be the algebra automorphism defined by $T_s \mapsto T_s$ for $s \in S$.

Let I be a subset of S. We denote by W_I the subgroup of W generated by I. The group W_I, together with I, is a Coxeter group and the length function on W_I is the restriction of that on W [Hu, §1.10].

We put $R_I = \mathbf{Z}[\{a_{s,I}, b_{s,I}\}_{s\in I}]$ where $a_{s,I}$ and $b_{s,I}$ are indeterminates with $a_{s,I} = a_{s',I}$ and $b_{s,I} = b_{s',I}$ if s and s' are conjugate in W_I. There is a morphism of rings $R_I \to R,\ a_{s,I} \mapsto a_s,\ b_{s,I} \mapsto b_s$.

We denote by $H_I = H_I(W)$ the R-subalgebra of H generated by $\{T_s\}_{s\in I}$. There is an isomorphism of R-algebras $R \otimes_{R_I} H(W_I) \xrightarrow{\sim} H_I(W),\ T_w \mapsto T_w$.

We assume for the remainder of §3.1.2 that W is finite. In this case, there is a unique element w_S of W with maximal length [Hu, §1.8] and we denote by N its length. We have $w_S^2 = 1$ and $w_SSw_S = S$. There is an automorphism of algebras

$$\iota_S : H \xrightarrow{\sim} H,\ T_v \mapsto T_{w_S\cdot v\cdot w_S}.$$

We denote by w_I the longest element of W_I and by N_I its length. We denote by W^I (resp. ${}^IW$) the set of elements $v \in W$ such that v has minimal length in vW_I (resp. W_Iv). Note that $W^I \xrightarrow{\sim} W/W_I,\ v \mapsto vW_I$ [Hu, Proposition 1.10].

3.1.3. *Traces.* We assume in §3.1.3 that W is finite.

Given $J \subset I$, we define an R-linear map

$$t_{I,J} : H_I \to H_J,\ T_v \mapsto \begin{cases} T_{w_J w_I v} & \text{if } v \in w_I \cdot W_J \\ 0 & \text{otherwise.} \end{cases}$$

The next proposition shows this is relative Frobenius form (cf eg [Rou1, §2.3.2]).

Proposition 3.1.1. *We have $t_{S,J} = t_{I,J} \circ t_{S,I}$.*

Given $h \in H$ and $x \in W_I$, we have

$$t_{S,I}(hT_x) = t_{S,I}(h)T_x,\ t_{S,I}(T_{w_S w_I \cdot x \cdot w_I w_S} h) = T_x t_{S,I}(h).$$

Given $h' \in H$ commuting with H_I, we have $t_{S,I}(hh') = t_{S,I}(\iota_S(h')h)$.

There is an isomorphism of R-modules

$$\hat{t}_{S,I} : H \xrightarrow{\sim} \operatorname{Hom}_{H_I^{\mathrm{opp}}}(H, H_I),\ h \mapsto (h' \mapsto t_{S,I}(hh'))$$

with

$$\hat{t}_{S,I}(T_{w_S w_I \cdot x \cdot w_I w_S} h T_y) = T_x \hat{t}_{S,I}(h) T_y \textit{ for } x \in W_I \textit{ and } y \in W.$$

Proof. Define $w^I = w_S w_I$ and ${}^I w = w_I w_S$, so that ${}^I w \cdot w^I = 1$. We have $w^I \in W^I$.

Let $v \in W$. There is a unique decomposition $v = v'v''$ where $\ell(v) = \ell(v') + \ell(v'')$, $v'' \in W_I$ and $v' \in W^I$ [Hu, Proposition 1.10]. Furthermore, $\ell(v') < \ell(w^I)$ unless $v' = w^I$. We have $t_{S,I}(T_v) = \delta_{v',w^I} T_{v''}$.

There is a unique decomposition $v'' = v_1 v_2$ with $\ell(v'') = \ell(v_1) + \ell(v_2)$, $v_2 \in W_J$ and v_1 has minimal length in $v''W_J$. We have $v = (v'v_1)v_2$ where $\ell(v) = \ell(v'v_1) + \ell(v_2)$ and $v'v_1$ has minimal length in vW_J. Furthermore, $v'v_1 = w^J$ if and only if $v' = w^I$ and $v_1 = w_I w_J$. It follows that

$$t_{I,J} \circ t_{S,I}(T_v) = \delta_{v',w^I} t_{I,J}(T_{v''}) = \delta_{v',w^I} \delta_{v_1, w_I w_J} T_{v_2} = t_{S,J}(T_v).$$

This shows the first statement of the lemma.

We have $T_{v''} T_x \in H_I$, hence

$$t_{S,I}(T_v T_x) = t_{S,I}(T_{v'}(T_{v''} T_x)) = \delta_{v',w^I} T_{v''} T_x = t_{S,I}(T_v) T_x.$$

This shows the second statement of the lemma.

Let $x' = w^I \cdot x \cdot {}^I w$. We have $\ell(w^I \cdot x \cdot {}^I w) = \ell(x)$. Since $T_{x'} T_v$ is a linear combination of elements T_{yz} with $y \leqslant x'$ and $z \leqslant v$, it follows that if $v' \neq w^I$, then $T_{x'} T_{v'}$ is a linear combination of elements $T_{w^I \cdot y \cdot {}^I w z}$ with $y \in W_I$ and $z \notin w^I W_I$, hence of elements T_u with $u \notin w^I W_I$. So, if $v' \neq w^I$, then $t_{S,I}(T_{x'} T_v) = 0$.

Assume now $v' = w^I$. We have $T_{x'} T_v = T_{w^I \cdot x \cdot {}^I w} T_{w^I} T_{v''} = T_{w^I \cdot x} T_{v''} = T_{w^I} T_x T_{v''}$ because $\ell(x' \cdot w^I) = \ell(w^I \cdot x) = \ell(w^I) + \ell(x) = \ell(x') + \ell(w^I)$. We deduce that $t_{S,I}(T_{x'} T_v) = T_x T_{v''} = T_x t_{S,I}(T_v)$. This shows the third statement of the lemma.

Let $v_0 \in W^I$. We have $\ell(w^I) = \ell(w^I v_0^{-1}) + \ell(v_0)$. Let $v \in W^I$. Note that $T_{w^I v_0^{-1}} T_v = T_{w^I v_0^{-1} v}$ or $T_{w^I v_0^{-1}} T_v$ is a linear combination of T_w's with $\ell(w) < \ell(w^I v_0^{-1}) + \ell(v)$. It follows that if $t_{S,I}(T_{w^I v_0^{-1}} T_v) = 0$ if $\ell(v) < \ell(v_0)$ or $\ell(v) = \ell(v_0)$ and $v \neq v_0$. We have also $t_{S,I}(T_{w^I v_0^{-1}} T_{v_0}) = 1$.

Since H is a free right H_I-module with basis $\{T_v\}_{v \in W^I}$, we deduce that $\hat{t}_{S,I}$ is surjective. Since $\hat{t}_{S,I}$ is an R-module morphism between free R-modules of the same finite rank, it follows that it is an isomorphism. This shows the fifth statement of the lemma.

Let $s \in S$ and $v \in W$. Let $s' = w_S \cdot s \cdot w_S \in S$. If $v \notin \{w_S, w_S \cdot s\}$, then $t_{S,\varnothing}(T_vT_s) = 0$ and $t_{S,\varnothing}(T_{s'}T_v) = 0$. If $v = w_S \cdot s$, then $T_vT_s = T_{w_S} = T_{s'}T_v$. If $v = w_S$, then $t_{S,\varnothing}(T_vT_s) = a_s = t_{S,\varnothing}(T_{s'}T_v)$. So, we have shown that $t_{S,\varnothing}(T_vT_s) = t_{S,\varnothing}(T_{s'}T_v)$. It follows by induction on $\ell(w)$ that $t_{S,\varnothing}(T_vT_w) = t_{S,\varnothing}(T_{w_S \cdot w \cdot w_S}T_v)$ for all $w \in W$.

Consider now $h' \in H$ commuting with H_I. Let $h'' \in H_I$. We have

$$t_{I,\varnothing}(t_{S,I}(hh')h'') = t_{I,\varnothing}(t_{S,I}(hh'h'')) = t_{S,\varnothing}(hh''h') = t_{S,\varnothing}(\iota_S(h')hh'') = \\ = t_{I,\varnothing}(t_{S,I}(\iota_S(h')hh'')) = t_{I,\varnothing}(t_{S,I}(\iota_S(h')h)h'').$$

It follows that $\hat{t}_{I,\varnothing}(t_{S,I}(hh')) = \hat{t}_{I,\varnothing}(t_{S,I}(\iota_S(h')h))$, hence $t_{S,I}(hh') = t_{S,I}(\iota_S(h')h)$. This completes the proof of the lemma. □

We put $t^+_{I,J} = t_{I,J}$. We define an R-linear map

$$t^-_{I,J} : H_I \to H_J,\ T_v \mapsto \begin{cases} T_{vw_Iw_J} & \text{if } v \in W_J \cdot w_I \\ 0 & \text{otherwise.} \end{cases}$$

We have $t^-_{I,J}(h) = \iota(t^+_{I,J}(\iota(h)))$.

We put $\hat{t}^+_{S,I} = \hat{t}_{S,I}$. We have an isomorphism of R-modules

$$\hat{t}^-_{S,I} : H \xrightarrow{\sim} \mathrm{Hom}_{H_I^{\mathrm{opp}}}(H, H_I),\ h \mapsto (h' \mapsto t^-_{S,I}(hh'))$$

with

$$\hat{t}^-_{S,I}(T_xhT_y) = T_x\hat{t}^-_{S,I}(h)T_y \text{ for } x \in W_I \text{ and } y \in W.$$

Consider $I, J \subset S$ with $I \subset J$ or $J \subset I$. We define an (H_I, H_J)-bimodule $L^\pm(I,J)$ with underlying R-module H. We put $a = 0$ if $\pm = +$ and $a = 1$ if $\pm = -$.

If $I \subset J$, then the right action of H_J is by right multiplication and the left action of $h \in H_I$ is by left multiplication by $(\iota_J\iota_I)^a(h)$.

If $J \subset I$, then the left action of H_I is by left multiplication and the right action of $h \in H_J$ is by right multiplication by $(\iota_I\iota_J)^a(h)$.

Note that $L^\pm(I,J)$ is free of finite rank as a left module and as a right module.

There is an isomorphism of (H, H_I)-bimodules

$$L^\pm(I,S)^\vee = \mathrm{Hom}_{H^{\mathrm{opp}}}(L^\pm(I,S), H) \xrightarrow{\sim} L^\pm(S,I),\ \zeta \mapsto \zeta(1).$$

The next result follows immediately from Proposition 3.1.1.

Corollary 3.1.2. *The map $\hat{t}^\pm_{S,I}$ is an isomorphism of (H_I, H)-bimodules*

$$L^\mp(I,S) \xrightarrow{\sim} L^\pm(S,I)^\vee = \mathrm{Hom}_{H_I^{\mathrm{opp}}}(L^\pm(S,I), H_I).$$

The results above can be formulated in terms of dual bases. Note that $\{T_w\}_{w \in W^I}$ is a basis of the free right H_I-module H, while $\{T_w\}_{w \in {}^IW}$ is a basis of the free left H_I-module H.

We have

$$t^+_{S,I}(T_{w_Sw_Iv^{-1}}T_w) = \delta_{v,w} \text{ and } t^-_{S,I}(T_{v'}T_{w'^{-1}w_Iw_S}) = \delta_{v',w'} \text{ for } v, w \in W^I \text{ and } v', w' \in {}^IW.$$

We deduce that the basis $(T_{w_Sw_Iw^{-1}})_{w \in W^I}$ when $\pm = +$ (resp. $(T_w)_{w \in {}^IW}$ when $\pm = -$) of the free left H_I-module $L^\mp(I,S)$ is dual to the basis $(T_w)_{w \in W^I}$ when $\pm = +$ (resp. $(T_{w^{-1}w_Iw_S})_{w \in {}^IW}$ when $\pm = -$) of the free right H_I-module $L^\pm(S,I)$, via the pairing providing the isomorphism of Corollary 3.1.2.

The counit of the adjoint pair $(L^{\pm}(S,I)\otimes_{H_I}-, L^{\mp}(I,S)\otimes_H -)$ is given by the morphism of (H_I,H_I)-bimodules

$$L^{\mp}(I,S)\otimes_H L^{\pm}(S,I)\to H_I,\ a\otimes b\mapsto t^{\pm}_{S,I}(ab)$$

while the unit is given by the morphism of (H,H)-bimodules

$$H\to L^{\pm}(S,I)\otimes_{H_I} L^{\mp}(I,S),\ 1\mapsto \begin{cases}\sum_{w\in W^I} T_w\otimes T_{w_S w_I w^{-1}} & \text{if } \pm=+\\ \sum_{w\in {}^I W} T_{w^{-1}w_I w_S}\otimes T_w & \text{if } \pm=-.\end{cases}$$

3.1.4. *Nil Hecke algebras.* We define the *nil Hecke algebra* $H^{\mathrm{nil}}_{\mathbf{Z}}(W)$ of (W,S) as the $\mathbf{Z}$-algebra $H(W)\otimes_R R/(a_s,b_s)_{s\in S}$. This is the $\mathbf{Z}$-algebra generated by $\{T_s\}_{s\in S}$ with relations

$$T_s^2=0,\ \underbrace{T_sT_tT_s\cdots}_{m_{st}\text{ terms}}=\underbrace{T_tT_sT_t\cdots}_{m_{st}\text{ terms}}\ \text{when } st \text{ has order } m_{st}.$$

This is a $\mathbf{Z}_{\leqslant 0}$-graded algebra with T_w in degree $-\ell(w)$ for $w\in W$.

The multiplication is given as follows:

$$T_wT_{w'}=\begin{cases}T_{ww'} & \text{if } \ell(ww')=\ell(w)+\ell(w')\\ 0 & \text{otherwise.}\end{cases} \tag{3.1.1}$$

Consider the filtration of the group algebra $\mathbf{Z}[W]$ where $\mathbf{Z}[W]^{\geqslant -i}$ is spanned by group elements $w\in W$ with $\ell(w)\leqslant i$, for $i\in\mathbf{Z}_{\geqslant 0}$. The associated $\mathbf{Z}_{\leqslant 0}$-graded algebra is $H^{\mathrm{nil}}_{\mathbf{Z}}(W)$ and T_w is the image of $w\in W$ in the degree $-\ell(w)$ homogeneous component of $H^{\mathrm{nil}}_{\mathbf{Z}}(W)$.

3.1.5. *Differential.* Let $H^{\mathrm{nil}}(W)=\mathbf{F}_2\otimes H^{\mathrm{nil}}_{\mathbf{Z}}(W)$. We define a linear map $d: H^{\mathrm{nil}}(W)\to H^{\mathrm{nil}}(W)$ by

$$d(T_w)=\sum_{w'<w,\ \ell(w')=\ell(w)-1} T_{w'}.$$

Proposition 3.1.3. *The map d defines a structure of differential graded algebra on $H^{\mathrm{nil}}(W)$.*

Proof. Let $w\in W$ and $s\in S$ with $ws>w$. We have $d(T_wT_s)=d(T_{ws})=\sum_{w'<ws,\ \ell(w')=\ell(w)}T_{w'}$. We have [Hu, Theorem 5.10]

$$\{w'\in W\ |w'<ws,\ \ell(w')=\ell(w)\}=\{w''s\mid w''<w,\ w''<w''s,\ \ell(w'')=\ell(w)-1\}\sqcup\{w\}.$$

It follows that $d(T_wT_s)=d(T_w)T_s+T_w=d(T_w)T_s+T_wd(T_s)$.

Consider now $v\in W$ and $s\in S$ with $vs<v$. We have $d(T_v)=d(T_{vs}T_s)=d(T_{vs})T_s+T_{vs}$ by the result above. It follows that $d(T_v)T_s+T_vd(T_s)=T_{vs}T_s+T_v=0=d(T_vT_s)$.

We deduce that $d(T_wT_{w'})=d(T_w)T_{w'}+T_wd(T_{w'})$ for all $w,w'\in W$.

Since $d^2(T_s)=0$ for $s\in S$, it follows that by induction that $d^2=0$. ☐

The following corollary shows that the computation of $d(T_w)$ can be done using the Leibniz rule, given a reduced decomposition of w. The terms that do not vanish are exactly the terms given in the original definition of $d(T_w)$.

Corollary 3.1.4. *Let $w=s_{i_1}\cdots s_{i_l}$ be a reduced expression of $w\in W$. We have*

$$d(T_w)=\sum_{r=1}^{l} T_{i_1}\cdots T_{i_{r-1}}T_{i_{r+1}}T_{i_l}.$$

We have $T_{i_1}\cdots T_{i_{r-1}}T_{i_{r+1}}T_{i_l}\neq 0$ *if and only if* $s_{i_1}\cdots s_{i_{r-1}}s_{i_{r+1}}\cdots s_{i_l}$ *is reduced,* i.e., *if and only if* $\ell(s_{i_1}\cdots s_{i_{r-1}}s_{i_{r+1}}\cdots s_{i_l})=\ell(w)-1$.

Given r,r' *with* $s_{i_1}\cdots s_{i_{r-1}}s_{i_{r+1}}\cdots s_{i_l}=s_{i_1}\cdots s_{i_{r'-1}}s_{i_{r'+1}}\cdots s_{i_l}$ *reduced, we have* $r=r'$.

Proof. The first statement follows from Proposition 3.1.3. The second statement is a property of the multiplication of T_w's.

For the third statement, let us assume $r<r'$. We have $s_{i_{r+1}}\cdots s_{i_{r'}}=s_{i_r}\cdots s_{i_{r'-1}}$ reduced, hence $s_{i_r}s_{i_{r+1}}\cdots s_{i_{r'}}$ is not reduced, a contradiction. □

Remark 3.1.5. Note that the algebra $H^{\mathrm{nil}}(W)$ is acyclic if $S\neq\varnothing$.

Note also that one can introduce a family of commuting differentials d_s for $s\in S$ modulo conjugacy by setting $d_s(T_t)=1$ if $t\in S$ is conjugate to s and $d_s(T_t)=0$ otherwise.

The specialization over $\mathbf{F}_2$ at $a_s=b_s=0$ of the bimodules $L^{\pm}(I,J)$ of §3.1.3 acquire a structure of differential graded bimodules, using the differential graded structure of $H^{\mathrm{nil}}(W)$. We keep the same notation for those differential graded specialized bimodules and for the maps t and $\hat{t}$.

Proposition 3.1.6. *If* W *is finite, then*

$$t_{S,I}:H^{\mathrm{nil}}(W)\to H^{\mathrm{nil}}(W_I)\langle N-N_I\rangle$$

is a morphism of differential graded $\mathbf{F}_2$*-modules and Corollary 3.1.2 provides an isomorphism of differential graded* $(H^{\mathrm{nil}}(W_I),H^{\mathrm{nil}}(W))$*-bimodules*

$$\hat{t}^{\pm}_{S,I}:L^{\mp}(I,S)\xrightarrow{\sim}L^{\pm}(S,I)^{\vee}\langle N-N_I\rangle.$$

Proof. Let $v\in W$. There is a unique decomposition $v=v'v''$ where $\ell(v)=\ell(v')+\ell(v'')$, $v''\in W_I$ and $v'\in W^I$.

We have $d(T_v)=d(T_{v'})T_{v''}+T_{v'}d(T_{v''})$. If $u\in W$ and $u<v'$, then $u\notin w_SW_I$. It follows that

$$t_{S,I}(d(T_v))=t_{S,I}(T_{v'}d(T_{v''}))=\delta_{v',w^I}d(T_{v''})=d(t_{S,I}(T_v)).$$

□

3.1.6. *Differential graded pointed Hecke monoid.* Let W^{nil} be the pointed $\mathbf{Z}_{\leqslant 0}$-graded monoid with underlying pointed set $\{T_w\}_{w\in W}\coprod\{0\}$ and multiplication given by (3.1.1). This is the pointed monoid gr$W$ associated to the filtration on W given by $W^{\geqslant -i}=\{w\in W\mid \ell(w)\leqslant i\}$ and there is an identification $\mathbf{F}_2[W^{\mathrm{nil}}]=H^{\mathrm{nil}}(W)$ making W^{nil} into a differential graded pointed monoid.

3.2. **Extended affine symmetric groups.**

3.2.1. *Finite case.* Fix $n\geqslant 0$. The symmetric group $\mathfrak{S}_n$ is a Coxeter group with generating set $\{(1,2),\ldots,(n-1,n)\}$.

Its differential nil Hecke algebra H_n is the k-algebra generated by $T_1,\ldots,T_{n-1}$ with relations

$$T_i^2=0,\ T_iT_j=T_jT_i \text{ if } |i-j|>1 \text{ and } T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1} \tag{3.2.1}$$

and with differential given by $d(T_i)=1$.

The algebra H_n has a basis $(T_w)_{w\in\mathfrak{S}_n}$.

3.2.2. *Definition.* Let $n \geqslant 1$. We denote by $\hat{\mathfrak{S}}_n$ the extended affine symmetric group: this is the subgroup of the group of permutations of $\mathbf{Z}$ with elements those bijections $\sigma : \mathbf{Z} \xrightarrow{\sim} \mathbf{Z}$ such that $\sigma(n+r) = n + \sigma(r)$ for all $r \in \mathbf{Z}$.

Given $i, j \in \mathbf{Z}$ with $i - j \notin n\mathbf{Z}$, we denote by s_{ij} the element of $\hat{\mathfrak{S}}_n$ defined by

$$s_{ij}(r) = \begin{cases} j - i + r & \text{if } r = i \pmod n \\ i - j + r & \text{if } r = j \pmod n \\ r & \text{otherwise.} \end{cases}$$

Note that $s_{i+n,j+n} = s_{i,j}$, $s_{ij} = s_{ji}$ and $s_{ij}^2 = 1$.

The symmetric group $\mathfrak{S}_n$ identifies with the subgroup of $\hat{\mathfrak{S}}_n$ of permutations σ such that $\sigma(\{1, \ldots, n\}) = \{1, \ldots, n\}$. We have a surjective morphism $\hat{\mathfrak{S}}_n \to \mathfrak{S}_n$ sending σ to the induced permutation of $\mathbf{Z}/n$. We identify its kernel with $\mathbf{Z}^n$ via the injective morphism

$$\mathbf{Z}^n \to \hat{\mathfrak{S}}_n,\ (\lambda_1, \ldots, \lambda_n) \mapsto (\{1, \ldots, n\} \ni i \mapsto i + n\lambda_i).$$

We have $\hat{\mathfrak{S}}_n = \mathbf{Z}^n \rtimes \mathfrak{S}_n$.

Assume $n \geqslant 2$. Let W_n be the Coxeter group of type $\hat{A}_{n-1}$: it is generated by $\{s_a\}_{a \in \mathbf{Z}/n}$ with relations

$$s_a^2 = 1,\ s_a s_b = s_b s_a \text{ if } a \neq b \pm 1$$
$$s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1} \text{ (for } n > 2).$$

Consider the semi-direct product $W_n \rtimes \langle c \rangle$ of W_n by an infinite cyclic group generated by an element c, with relation $c s_a c^{-1} = s_{a+1}$.

Lemma 3.2.1. *There is an isomorphism of groups*

$$W_n \rtimes \langle c \rangle \xrightarrow{\sim} \hat{\mathfrak{S}}_n,\ c \mapsto (j \mapsto j+1),\ s_{i+n\mathbf{Z}} \mapsto s_{i,i+1} \textit{ for } i \in \{1, \ldots, n\}.$$

Proof. Denote by f the map of the lemma. By [Lus, §3.6] (cf also [BjBr, Proposition 8.3.3]), the restriction of f to W_n induces an isomorphism with the subgroup of $\hat{\mathfrak{S}}_n$ of elements σ such that $\sum_{i=1}^n (\sigma(i) - i) = 0$. It is immediate to check that f extends to a morphism of groups $W_n \rtimes \langle c \rangle \to \hat{\mathfrak{S}}_n$.

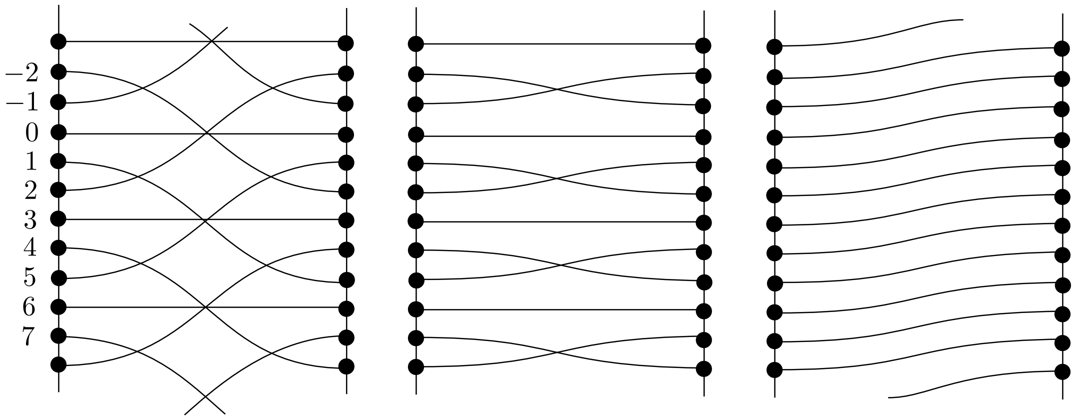
Consider $\sigma \in \hat{\mathfrak{S}}_n$ and let $N = \sum_{i=1}^n (\sigma(i) - i)$. Note that $n | N$. Put $\sigma' = \sigma f(c)^{-N/n}$. We have $\sigma' \in f(W_n)$, so f is surjective. Let $\sigma = f(wc^d)$. We have $\sum_{i=1}^n (\sigma(i) - i) = nd$. So, if $\sigma = 1$, then $d = 0$, hence $w \in \ker(f) \cap W_n = 1$. This shows that f is injective. □

We will identify $W_n \rtimes \langle c \rangle$ and $\hat{\mathfrak{S}}_n$ via the isomorphism of Lemma 3.2.1.

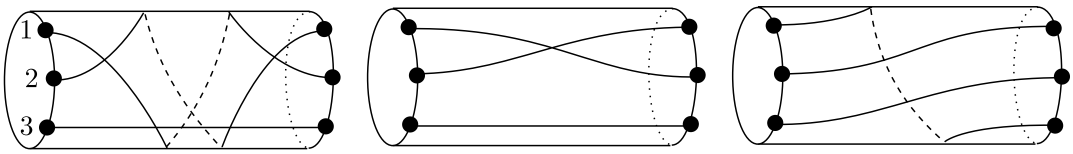
We put $W_1 = 1$, so that $\hat{\mathfrak{S}}_1 \simeq \langle c \rangle = W_1 \rtimes \langle c \rangle$. We also put $\hat{\mathfrak{S}}_0 = 1$.

3.2.3. *Diagrammatic representation.* The permutations of $\mathbf{Z}$ can be described as collections of strands in $[-1, 1] \times \mathbf{R}$ going leftwards from integers points on the vertical line $x = 1$ to integer points on the vertical line $x = -1$. Thanks to their n-periodicity, those permutations that are elements of $\hat{\mathfrak{S}}_n$ can also be encoded in a collection of strands drawn on a cylinder, going from right to left, by passing to the quotient of the vertical strip $[-1, 1] \times \mathbf{R}$ by the vertical action by translation of $n\mathbf{Z}$.

Here are some elements of $\hat{\mathfrak{S}}_3$:

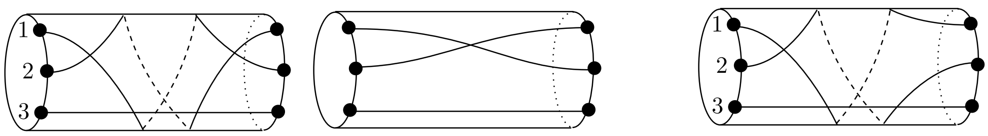


s_{15} s_{12} c

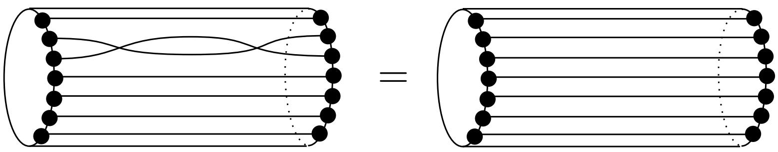


The multiplication $\sigma\sigma'$ of σ and σ' in $\hat{\mathfrak{S}}_n$ corresponds to the concatenation of the diagram of σ put to the left of the diagram of σ' as in the following example:

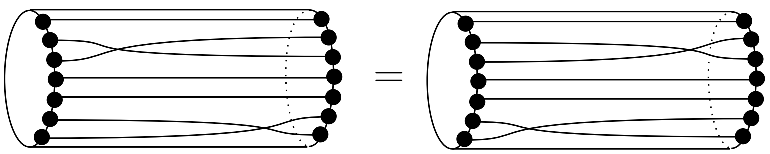
$$s_{15} \quad \cdot \quad s_{12} \quad = \quad s_{15}s_{12}$$



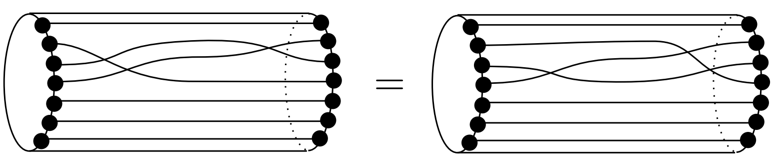
The defining relations for $\hat{\mathfrak{S}}_n$ are depicted as follows



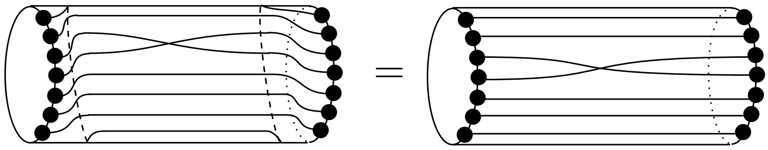
s_a^2 1



$s_a s_b$ $s_b s_a$

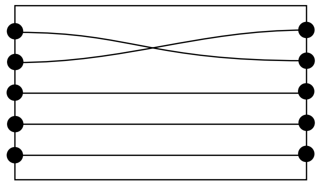


$s_a s_{a+1} s_a$ $s_{a+1} s_a s_{a+1}$



cs_ac^{-1} $\quad$ s_{a+1}

The elements of $\mathfrak{S}_n$ correspond to diagrams whose strands do not go in the back of the cylinder, hence can be drawn on a rectangle. For example, s_{12} above can be represented as follows:



s_{12}

3.2.4. *Length.* Assume now again that $n \geqslant 1$. We extend the length function on the Coxeter group W_n to one on $W_n \rtimes \langle c\rangle$ by setting $\ell(wc^d) = \ell(w)$ for $w \in W_n$ and $d \in \mathbf{Z}$. Note that the action of c on W_n preserves lengths. Similarly, we extend the Chevalley-Bruhat order on $W_n \rtimes \langle c\rangle$ by setting $w'c^{d'} < wc^d$ if $w' < w$ and $d' = d$ and we consider the corresponding order on $\hat{\mathfrak{S}}_n$. Note that the action of c on W_n preserves the order, hence $w'c^{d'} < wc^d$ if and only if $c^{d'}w' < c^d w$.

Lemma 3.2.2. *Let $\sigma', \sigma'' \in \hat{\mathfrak{S}}_n$ and $\sigma = \sigma'\sigma''$. Assume $\ell(\sigma) = \ell(\sigma') + \ell(\sigma'')$. Let $a \in \mathbf{Z}/n$ such that $\ell(\sigma s_a) < \ell(\sigma)$ and $\ell(\sigma'' s_a) > \ell(\sigma'')$.*

Let $\alpha'' = \sigma'' s_a$ and $\alpha' = \sigma'\sigma'' s_a \sigma''^{-1}$. We have $\sigma = \alpha'\alpha''$ and $\ell(\sigma) = \ell(\alpha') + \ell(\alpha'')$.

Proof. Multiplying if necessary σ' and σ'' by a power of c, we can assume σ, σ' and σ'' are in W_n.

Let $\sigma' = s_{a_1}\cdots s_{a_m}$ and $\sigma'' = s_{a_{m+1}}\cdots s_{a_d}$ be two reduced decompositions. The Exchange Lemma [Hu, Theorem 5.8] shows that there is i such that $\sigma s_a = s_{a_1}\cdots s_{a_{i-1}} s_{a_{i+1}}\cdots s_{a_d}$.

If $i > m$, then $\sigma'' s_a = s_{a_{m+1}}\cdots s_{a_{i-1}} s_{a_{i+1}}\cdots s_{a_d}$ and this contradicts $\ell(\sigma'' s_a) > \ell(\sigma'')$. So, $i \leqslant m$. We have $\sigma s_a = s_{a_1}\cdots s_{a_{i-1}} s_{a_{i+1}}\cdots s_{a_m}\sigma''$. We deduce that $\alpha' = s_{a_1}\cdots s_{a_{i-1}} s_{a_{i+1}}\cdots s_{a_m}$ has length $m-1$ and the lemma follows. ☐

Given $\sigma \in \hat{\mathfrak{S}}_n$, we put $L(\sigma) = \{(i,j) \in \mathbf{Z}\times\mathbf{Z} \mid i < j,\ \sigma(i) > \sigma(j)\}$. This set has a diagonal action of $n\mathbf{Z}$ by translation. We put $\tilde{L}(\sigma) = \{(i,j) \in L(\sigma) \mid 1 \leqslant i \leqslant n\}$. The canonical map $\tilde{L}(\sigma) \to L(\sigma)/n\mathbf{Z}$ is bijective.

The next lemma is a variation on classical results (cf [Sh, Lemma 4.2.2], [BjBr, Proposition 8.3.6] and [BjBr, §2.2]).

Lemma 3.2.3. *Let $\sigma \in \hat{\mathfrak{S}}_n$. We have $L(\sigma) = L(c^d\sigma)$ for all $d \in \mathbf{Z}$ and*

$$\ell(\sigma) = |\tilde{L}(\sigma)| = \sum_{0\leqslant i<j<n} \left|\left\lfloor \frac{\sigma(j)-\sigma(i)}{n}\right\rfloor\right|.$$

If $(i,j) \in L(\sigma)$, then $\sigma s_{ij} < \sigma$.

Assume $\sigma = c^d w$ and $w = s_{a_1} \cdots s_{a_l}$ is a reduced decomposition of $w \in W_n$. Given $1 \leqslant r \leqslant l$, let $i_r \in \{1, \ldots, n\}$ with $i_r + n\mathbf{Z} = a_r$.

The set $\{\big(s_{a_l} \cdots s_{a_{r+1}}(i_r), s_{a_l} \cdots s_{a_{r+1}}(i_r + 1)\big)\}_{1 \leqslant r \leqslant l}$ is a subset of $L(\sigma)$. This induces a bijection

$$\{\big((s_{a_l} \cdots s_{a_{r+1}}(i_r), s_{a_l} \cdots s_{a_{r+1}}(i_r + 1)\big)\}_{1 \leqslant r \leqslant l} \xrightarrow{\sim} L(\sigma)/n\mathbf{Z}.$$

Proof. Consider a pair $(i, j) \in L(\sigma)$ with $1 \leqslant i \leqslant n$ and such that $(i, j') \notin L(\sigma)$ and $(j', j) \notin L(\sigma)$ for $i < j' < j$. Given j' with $i < j' < j$, we have $\sigma(i) < \sigma(j') < \sigma(j)$, a contradiction. It follows that $j = i + 1$. We have

$$L(\sigma) = \big(\{(i, i + 1)\} + n\mathbf{Z}\big) \coprod (s_{i,i+1}, s_{i,i+1})(L(\sigma s_{i,i+1})).$$

We deduce by induction on $|\tilde{L}(\sigma)|$ that $\ell(\sigma) \leqslant |\tilde{L}(\sigma)|$.

We prove the statements on $\{\big(s_{a_l} \cdots s_{a_{r+1}}(i_r), s_{a_l} \cdots s_{a_{r+1}}(i_r + 1)\big)\}_{1 \leqslant r \leqslant l}$ by induction on $\ell(\sigma)$. By induction, the statements hold for $\sigma s_{a_l,a_l+1}$. In particular, $\ell(\sigma s_{a_l,a_l+1}) = |\tilde{L}(\sigma s_{a_l,a_l+1})|$. It follows that $\ell(\sigma) = \ell(\sigma s_{a_l,a_l+1}) + 1 > |\tilde{L}(\sigma s_{a_l,a_l+1})|$. Assume $(i_l, i_l + 1) \notin L(\sigma)$. It follows that $L(\sigma s_{a_l,a_l+1}) = s_{a_l,a_l+1}(L(\sigma)) \coprod \big(\{(i_l, i_l + 1)\} + n\mathbf{Z}\big)$, hence $|\tilde{L}(\sigma)| < |\tilde{L}(\sigma s_{a_l,a_l+1})| = \ell(\sigma s_{a_l,a_l+1}) = \ell(\sigma) - 1$, a contradiction. It follows that $(i_l, i_l + 1) \in L(\sigma)$, hence

$$L(\sigma) = s_{a_l,a_l+1}(L(\sigma s_{i_l,i_l+1})) \coprod \big(\{(i_l, i_l + 1)\} + n\mathbf{Z}\big).$$

The last statement of the lemma follows now by induction.

Consider now $(i, j) \in L(\sigma)$. Up to translating (i, j) diagonally by $n\mathbf{Z}$, we can assume there is r such that $i = s_{a_l} \cdots s_{a_{r+1}}(i_r)$ and $j = s_{a_l} \cdots s_{a_{r+1}}(i_r + 1)$. So $\sigma s_{i,j} = c^d s_{a_1} \cdots s_{a_{r-1}} s_{a_{r+1}} \cdots s_{a_l}$, hence $\sigma s_{i,j} < \sigma$. The lemma follows. ☐

Lemma 3.2.4. *Given $\sigma, \sigma' \in \hat{\mathfrak{S}}_n$, we have $\sigma' < \sigma$ and $\ell(\sigma') = \ell(\sigma) - 1$ if and only if there is $(j_1, j_2) \in L(\sigma)$ such that $\sigma' = \sigma s_{j_1,j_2}$ and*

- *$j_2 - j_1 < n$ or $\sigma(j_1) - \sigma(j_2) < n$ and*
- *given $i \in \mathbf{Z}$ with $j_1 < i < j_2$, we have $\sigma(j_1) < \sigma(i)$ or $\sigma(i) < \sigma(j_2)$.*

Proof. Consider $(j_1, j_2) \in L(\sigma)$ and let $s = s_{j_1,j_2}$. Consider integers $i < j$ with $i - j \notin n\mathbf{Z}$.

If $s(i) < s(j)$, then $(i, j) \in L(\sigma)$ if and only if $s(i, j) = (s(i), s(j)) \in L(\sigma s)$.

Assume now $s(i) > s(j)$. We have three possibilities:

• $i - j_1 \in n\mathbf{Z}$, $j - j_2 \notin n\mathbf{Z}$: we have $(i, j) \in L(\sigma)$ if and only if $(i, j) \in L(\sigma s)$ or $\sigma(i) > \sigma(j) > \sigma s(i)$ (and then $(i, j) \notin L(\sigma s)$)

• $i - j_1 \notin n\mathbf{Z}$, $j - j_2 \in n\mathbf{Z}$: we have $(i, j) \in L(\sigma)$ if and only if $(i, j) \in L(\sigma s)$ or $\sigma s(j) > \sigma(i) > \sigma(j)$ (and then $(i, j) \notin L(\sigma s)$).

• $i = j_1 + nr$, $j = j_2 + nr'$ with $r, r' \in \mathbf{Z}$: we have $(i, j) \in L(\sigma)$ if and only if $(i, j) \in L(\sigma s)$ or $\sigma(j_1) - \sigma(j_2) > n(r' - r) > \sigma(j_2) - \sigma(j_1)$ (and then $(i, j) \notin L(\sigma s)$).

We deduce there is an injective map $a : L(\sigma s) \to L(\sigma)$ given by

$$a((i, j)) = \begin{cases} (i, j) & \text{if } s(i) > s(j) \\ s(i, j) & \text{otherwise} \end{cases}$$

and

$$L(\sigma) = a(L(\sigma s)) \sqcup \coprod_{|r|<\min\left(\frac{j_2-j_1}{n},\frac{\sigma(j_1)-\sigma(j_2)}{n}\right)} \left((j_1+nr,j_2)+n\mathbf{Z}\right)\sqcup$$

$$\coprod_{\substack{j_1<i<j_2\\ \sigma(j_1)>\sigma(i)>\sigma(j_2)}} \Big(\left((j_1,i)+n\mathbf{Z}\right)\sqcup\left((i,j_2)+n\mathbf{Z}\right)\Big).$$

Note that $a(L(\sigma s))\sqcup((j_1,j_2)+n\mathbf{Z})\subset L(\sigma)$.

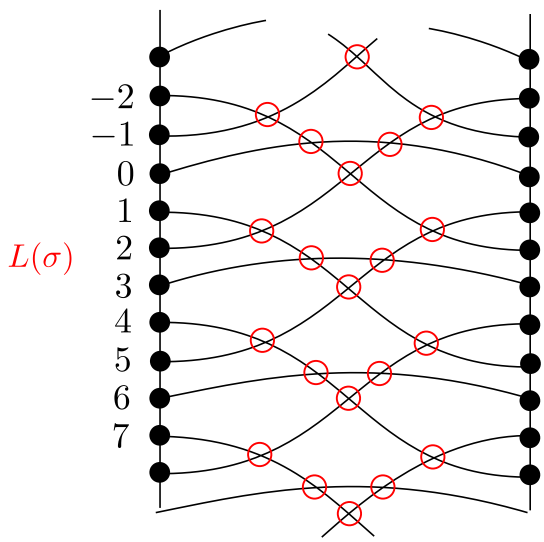
Let us now prove the lemma. We have $\sigma = c^d w$ and $\sigma' = c^{d'}w' \in \hat{\mathfrak{S}}_n$ for some $w, w' \in W_n$. Assume $\sigma' < \sigma$ and $\ell(\sigma') = \ell(\sigma)-1$. We have $d = d'$, $w' < w$ and $\ell(w') = \ell(w)-1$. It follows that there is a reduced decomposition $w = s_{a_1}\cdots s_{a_l}$ and $r\in\{1,\ldots,l\}$ such that $w' = s_{a_1}\cdots s_{a_{r-1}}s_{a_{r+1}}\cdots s_{a_l}$. Let $j_1 = s_{a_l}\cdots s_{a_{r+1}}(i_r)$ and $j_2 = s_{a_l}\cdots s_{a_{r+1}}(i_r+1)$. We have $(j_1,j_2)\in L(\sigma)$ and $\sigma' = \sigma s_{j_1,j_2}$ (Lemma 3.2.3).

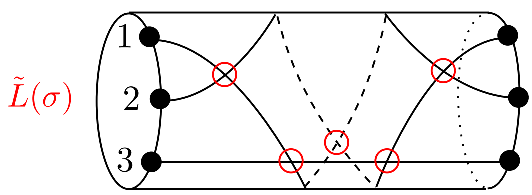
The discussion above shows that $\{i\in\mathbf{Z}\mid j_1<i<j_2,\ \sigma(j_1)>\sigma(i)>\sigma(j_2)\} = \varnothing$ and $\min\left(\frac{j_2-j_1}{n},\frac{\sigma(j_1)-\sigma(j_2)}{n}\right)<1$. The lemma follows. □

Example 3.2.5. The elements of $\tilde{L}(\sigma)$ are in bijection with intersection points between strands of a "good diagram" representing σ. Here, we define a strand diagram to be good if no more than two strands intersect at a given point and if the diagram minimizes the total number of intersection points. Similarly, the elements of $L(\sigma)$ correspond to intersections in an unfolded good strand diagram.

These descriptions can be deduced from Lemma 6.2.3 below, that shows those statements hold for pairs of strands. Now, the intersection point set for a good diagram is the disjoint union over intersection sets between pairs of strands, and a good diagram minimizes the intersection number among good diagrams if and only of each pair of strands minimizes its intersection number.

For example:





3.2.5. *Extended affine Hecke algebra.* We let c act on the differential graded algebra $H^{\mathrm{nil}}(W_n)$ by $c(T_a) = T_{a+1}$. Let $\hat{H}_n = H^{\mathrm{nil}}(W_n)\rtimes\langle c\rangle$. For $n\geqslant 2$, it is the differential graded $\mathbf{F}_2$-algebra generated by $\{T_a\}_{a\in\mathbf{Z}/n}$ and $c^{\pm 1}$ with relations

$$T_a^2 = 0,\ cT_a = T_{a+1}c,\ T_aT_b = T_bT_a \text{ if } a \neq b \pm 1$$

$$T_aT_{a+1}T_a = T_{a+1}T_aT_{a+1} \text{ (for } n > 2 \text{)}$$

and differential $d(T_a) = 1$, $d(c) = 0$. The element c has degree 0, while T_a has degree -1. Note that $\hat{H}_1 = \mathbf{F}_2[\hat{\mathfrak{S}}_1] = \mathbf{F}_2\langle c\rangle$, a differential graded algebra in degree 0 with $d = 0$.

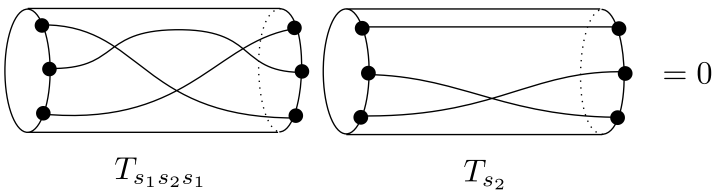
Let $w \in W_n$, $d \in \mathbf{Z}$ and $w' = wc^d$. We put $T_{w'} = T_wc^d$. We also put $T_\sigma = T_wc^d$ for $\sigma = wc^d$. The set $\{T_\sigma\}_{\sigma\in\hat{\mathfrak{S}}_n}$ is a basis of $\hat{H}_n$.

Remark 3.2.6. Define a filtration on $\mathbf{F}_2[\hat{\mathfrak{S}}_n]$ with $(\mathbf{F}_2[\hat{\mathfrak{S}}_n])^{\geqslant -i}$ the subspace spanned by group elements $w \in \hat{\mathfrak{S}}_n$ with $\ell(w) \leqslant i$. The associated graded algebra is $\hat{H}_n$.

We put $\hat{H}_0 = \mathbf{F}_2$.

Remark 3.2.7. The group $\hat{\mathfrak{S}}_n$ is more classically described as a semi-direct product $\mathbf{Z}^n \rtimes \mathfrak{S}_n$ (cf §3.2.2) coming from its description as the extended affine Weyl group of GL_n. The nil affine Hecke algebra of GL_n associated with this description (cf e.g. [Rou2, §2.2.2]) is *not* isomorphic to $\hat{H}_n$. When considering invertible (instead of 0) parameters, the two algebras are isomorphic.

Example 3.2.8. An element T_σ of $\hat{H}_n$ will be representated by a good strand diagram for σ. The multiplication of T_σ and $T_{\sigma'}$ is obtained by concatenating the diagrams of σ and σ' (as in the multiplication of σ and σ'). If the corresponding diagram is good, then $T_\sigma T_{\sigma'} = T_{\sigma''}$, where σ'' is represented by the concatenated diagram. Otherwise, $T_\sigma T_{\sigma'} = 0$. For example:



3.2.6. *Positive versions.* Let $\hat{\mathfrak{S}}_n^+$ be the submonoid of $\hat{\mathfrak{S}}_n$ of permutations σ such that $\sigma(\mathbf{Z}_{>0}) \subset \mathbf{Z}_{>0}$. Note that $\hat{\mathfrak{S}}_n^+$ is stable under left and right multiplication by $\mathfrak{S}_n$.

There is a decomposition $\hat{\mathfrak{S}}_n^+ = (\mathbf{Z}_{\geqslant 0})^n \rtimes \mathfrak{S}_n$.

We have $s_{r-1}s_{r-2}\cdots s_1cs_{n-1}s_{n-2}\cdots s_r = (\underbrace{0,\ldots,0,1,0,\ldots,0}_{\text{pos.}r}) \in (\mathbf{Z}_{\geqslant 0})^n$ for $r \in \{1,\ldots,n\}$, hence v restricts to an isomorphism from the submonoid of $W_n \rtimes \langle c\rangle$ generated by $s_1,\ldots,s_{n-1},c$ to $\hat{\mathfrak{S}}_n^+$.

Let $\hat{H}_n^+ = \bigoplus_{w\in\hat{\mathfrak{S}}_n^+} \mathbf{F}_2T_w$, an $\mathbf{F}_2$-subspace of $\hat{H}_n$ containing H_n.

Proposition 3.2.9. *$\hat{H}_n^+$ is a differential graded subalgebra of $\hat{H}_n$.*

The algebra $\hat{H}_n^+$ has a presentation with generators $T_1,\ldots,T_{n-1},c$ and relations

$$T_i^2 = 0,\ T_iT_j = T_jT_i \text{ if } |i-j| > 1,\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} (\text{ if } n > 2 \text{ })$$

$$cT_i = T_{i+1}c \text{ for } 1 \leqslant i < n-1 \text{ and } c^2T_{n-1} = T_1c^2.$$

The remainder of §3.2.6 will be devoted to the proof of Proposition 3.2.9.

Let A_n be the k-algebra with generators $t_1, \ldots, t_{n-1}, b$ and relations

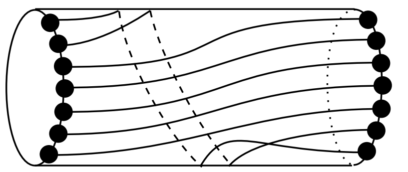
$$t_i^2 = 0,\ t_i t_j = t_j t_i \text{ if } |i-j| > 1,\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} (\text{ if } n > 2\ )$$

$$b t_i = t_{i+1} b \text{ for } 1 \leqslant i < n-1 \text{ and } b^2 t_{n-1} = t_1 b^2.$$

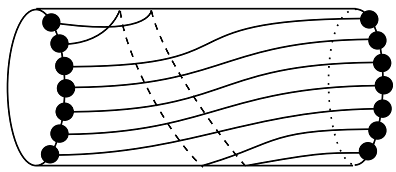
Given $i \in \{1, \ldots, n\}$, we put $\beta_i = b t_{n-1} \cdots t_i$. Given $I \subset \{1, \ldots, n\}$ non-empty with elements $1 \leqslant i_1 < \cdots < i_r \leqslant n$, we put $\gamma_I = \beta_{i_1+r-1} \beta_{i_2+r-2} \cdots \beta_{i_r}$. Note that $\gamma_{\{1,\ldots,n\}} = b^n$.

There is a morphism of algebras $H_n \to A_n,\ T_i \mapsto t_i$ and we denote by t_w the image of T_w for $w \in \mathfrak{S}_n$.

Example 3.2.10. The elements of $\hat{\mathfrak{S}}_n^+$ correspond to strand diagrams where the strands wind positively around the cylinder. The relation $c^2 T_{n-1} = T_1 c^2$ is illustrated below:

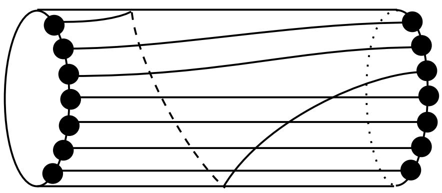


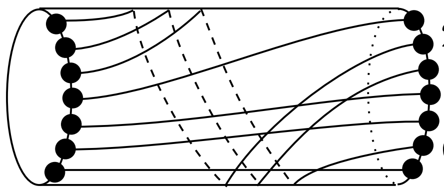
=



$c^2 T_{n-1}$ $\qquad$ $T_1 c^2$

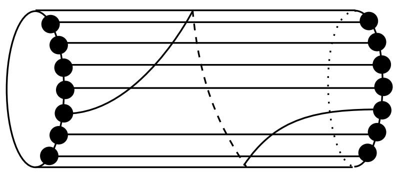
We describe some elements $w \in \hat{\mathfrak{S}}_7^+$ and the image of T_w in A_7:





β_3 $\qquad$ $\gamma_{\{2,3,6\}}$

The element $(0,0,0,0,1,0,0) \in (\mathbf{Z}_{\geqslant 0})^7$ corresponds to the following element of $\mathfrak{S}_7^+$:



$s_4 s_3 s_2 s_1 c s_6 s_5$

Lemma 3.2.11. *The set* $\{t_w \gamma_{I_m} \cdots \gamma_{I_1}\}$ *with* $w \in \mathfrak{S}_n$, $m \geqslant 0$ *and* $I_1 \subset \{1, \ldots, n\}$, $I_r \subset \{1, \ldots, |I_{r-1}|\}$ *for* $1 < r \leqslant m$ *generates* A_n *as a* k*-vector space.*

Proof. Let $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n-1\}$. We have

$$\beta_i t_j = \begin{cases} t_{j+1}\beta_i & \text{if } j < i-1 \\ \beta_{i-1} & \text{if } j = i-1 \\ 0 & \text{if } j = i \\ t_j \beta_i & \text{if } j > i. \end{cases}$$

Consider $I \subset \{1, \ldots, n\}$ non-empty with elements $1 \leqslant i_1 < \cdots < i_r \leqslant n$. We put $i_0 = 0$ and $i_{r+1} = n+1$.

Consider $j \in \{1, \ldots, n-1\}$. Fix $k \in \{0, \ldots, r\}$ such that $i_k \leqslant j < i_{k+1}$. Let us show that

$$\gamma_I t_j = \begin{cases} t_{j+r-k}\gamma_I & \text{if } i_k < j < i_{k+1}-1 \\ 0 & \text{if } i_k = j < i_{k+1}-1 \\ \gamma_{\{i_1<\cdots<i_k<i_{k+1}-1<i_{k+2}<\cdots<i_r\}} & \text{if } i_k < j = i_{k+1}-1 \\ t_k\gamma_I & \text{if } i_k = j = i_{k+1}-1. \end{cases} \tag{3.2.2}$$

We have

$$\gamma_I t_j = \beta_{i_1+r-1}\cdots\beta_{i_{k+1}+r-k-1}t_{j+r-k-1}\beta_{i_{k+2}+r-k-2}\cdots\beta_{i_r}.$$

If $j < i_{k+1}-1$, then $\beta_{i_{k+1}+r-k-1}t_{j+r-k-1} = t_{j+r-k}\beta_{i_{k+1}+r-k-1}$ and we deduce the first two equalities in (3.2.2). Assume now $j = i_{k+1}-1$. We have $\beta_{i_{k+1}+r-k-1}t_{j+r-k-1} = \beta_{i_{k+1}+r-k-2}$ and the third equality in (3.2.2) follows. The last equality from the fact that given $i \in \{1, \ldots, n-1\}$, we have

$$\begin{aligned}\beta_{i+1}\beta_i &= b^2t_{n-2}\cdots t_it_{n-1}\cdots t_i = b^2t_{n-1}\cdots t_it_{n-1}\cdots t_{i+1} = t_1b^2t_{n-2}\cdots t_it_{n-1}\cdots t_{i+1} \\ &= t_1\beta_{i+1}^2.\end{aligned}$$

We deduce that $\gamma_I t_j = u\gamma_{I'}$ for some $I' \subset \{1, \ldots, n\}$ with $|I'| = |I|$ and $\max(I') \leqslant \max(I)$ and $u \in \{0, 1, t_1, \ldots, t_{n-1}\}$.

Fix $s \in \{1, \ldots, n\}$ with $s \geqslant \max(I)$. We have

$$\gamma_I\beta_s = \begin{cases} \beta_r\gamma_{\{i_2-1,\ldots,i_r-1,s\}} & \text{if } 1 \in I \\ \gamma_{(I-1)\cup\{s\}} & \text{otherwise.} \end{cases}$$

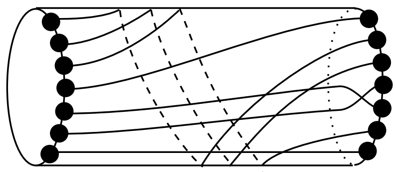
Consider $I_1, \ldots, I_m$ as in the lemma. Let k be minimal such that $1 \notin I_k$. We put $k = m+1$ if there is no such k. Define $u = \gamma_{\{|I_m|\}}$ if $k = m+1$ and $u = 1$ otherwise. Put $I_0 = \{1, \ldots, n\}$. Recall that $b = \beta_n$. We have

$$\gamma_{I_m}\cdots\gamma_{I_1}b = u\gamma_{I'_m}\cdots\gamma_{I'_1}$$

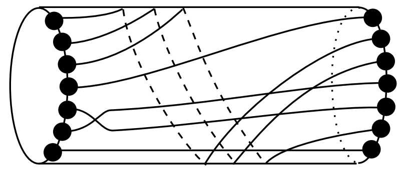
where $I'_r = \{i-1 | i \in I_r \backslash \{1\}\} \cup \{|I_{r-1}|\}$ for $1 \leqslant r < k$, $I'_k = \{i-1 | i \in I_k\} \cup \{|I_{k-1}|\}$ and $I'_r = I_r$ for $r > k$.

We deduce that the set $B = \{t_w\gamma_{I_m}\cdots\gamma_{I_1}\}$ of the lemma is stable under right multiplication by t_j for $j \in \{1, \ldots, n-1\}$ and by b. Since B contains 1, it follows that B is a generating family for A_n as an $\mathbf{F}_2$-vector space. □

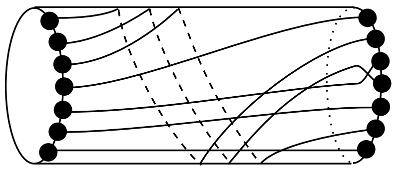
Remark 3.2.12. An example of the description of $\gamma_I t_j$ in the proof of Lemma 3.2.11 is given below:



$=$

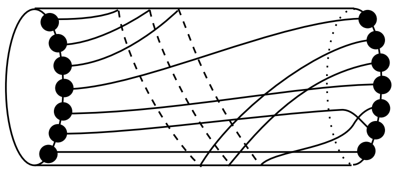


$\gamma_{\{2,3,6\}}t_4$ $\qquad$ $t_5\gamma_{\{2,3,6\}}$

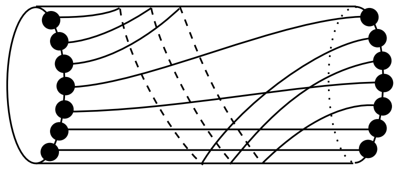


$= 0$

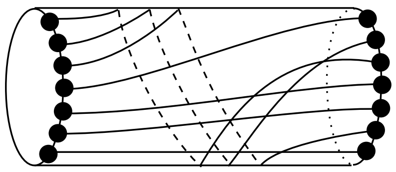
$\gamma_{\{2,3,6\}}t_3$



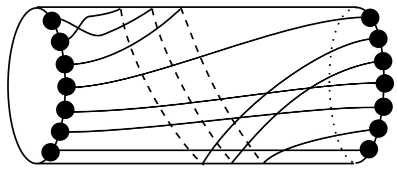
$=$



$\gamma_{\{2,3,6\}}t_5$ $\qquad$ $\gamma_{\{2,3,5\}}$



$=$



$\gamma_{\{2,3,6\}}t_2$ $\qquad$ $t_1\gamma_{\{2,3,6\}}$

Proof of Proposition 3.2.9. Let H be the subalgebra of $\hat{H}_n$ generated by $T_1, \ldots, T_{n-1}, c$. This is a differential graded subalgebra of $\hat{H}_n$. Given $w \in \hat{\mathfrak{S}}_n$, let $|w| = \sum_{i=1}^n w(i)$. Let $w \in \hat{\mathfrak{S}}_n^+$, $w \neq 1$. We show by induction on $\ell(w) + |w|$ that $T_w \in H$.

Assume $\ell(ws_i) < \ell(w)$ for some $i \in \{1, \ldots, n-1\}$. We have $ws_i \in \hat{\mathfrak{S}}_n^+$ and $|ws_i| = |w|$, hence by induction $T_{ws_i} \in H$. We deduce that $T_w = T_{ws_i}T_i \in H$.

Otherwise, we have $0 < w(1) < \cdots < w(n)$, hence $w(n) > n$ since $w \neq 1$. It follows that $wc^{-1} \in \hat{\mathfrak{S}}_n^+$ and $|wc^{-1}| < |w|$, hence $T_{wc^{-1}} \in H$ by induction. So $T_w = T_{wc^{-1}}T_c \in H$.

We have shown that $\hat{H}_n^+ \subset H$. Since $\hat{H}_n^+$ is stable under right multiplication by T_c and by T_i for $i \in \{1, \ldots, n-1\}$, it follows that $H = \hat{H}_n^+$.

There is a surjective morphism of algebras $\rho : A_n \to \hat{H}_n^+,\ t_i \mapsto T_i,\ b \mapsto c$. Given $I = \{i_1 < \cdots < i_r\}$ a non-empty subset of $\{1, \ldots, n\}$, we put

$$c_I = (cs_{n-1} \cdots s_{i_1+r-1})(cs_{n-1} \cdots s_{i_2+r-2}) \cdots (cs_{n-1} \cdots s_{i_r}) \in \hat{\mathfrak{S}}_n.$$

We have $c_I(i_l) = n + l$ for $1 \leqslant l \leqslant r$ and $c_I(j) = j + r - k$ if $i_k < j < i_{k+1}$ (where we put $i_0 = 0$ and $i_{r+1} = n + 1$).

Let E be the set of families $(I_1, \ldots, I_m)$ where $m \geqslant 0$, $I_1 \subset \{1, \ldots, n\}$ and $I_r \subset \{1, \ldots, |I_{r-1}|\}$ for $1 < r \leqslant m$.

Given $w \in \mathfrak{S}_n$ and $(I_1, \ldots, I_m) \in E$, we have $\rho(t_w\gamma_{I_1} \cdots \gamma_{I_m}) = T_wT_{c_{I_1}} \cdots T_{c_{I_m}}$ and that element is either $T_{wc_{I_1} \cdots c_{I_m}}$ or 0.

We define a map $\phi : (\mathbf{Z}_{\geqslant 0})^n \to E$. Let $a \in (\mathbf{Z}_{\geqslant 0})^n$. Let $m = \max\{a(i)\}_{1\leqslant i\leqslant n}$. We put $I_1 = a^{-1}(\mathbf{Z}_{\geqslant 1})$ and we define inductively I_r for $2 \leqslant r \leqslant m$ by $I_r = c_{I_{r-1}} \cdots c_{I_1}(a^{-1}(\mathbf{Z}_{\geqslant r}))$. We put $\phi(a) = (I_1, \ldots, I_m)$. We have

$$c_{I_m} \cdots c_{I_1}(i) = na(i) + |a^{-1}(\mathbf{Z}_{>a(i)})| + (\text{position of } i \text{ in } a^{-1}(a(i))).$$

We define a map $\psi : E \to (\mathbf{Z}_{\geqslant 0})^n$. Let $(I_1, \ldots, I_m) \in E$. We define $a \in (\mathbf{Z}_{\geqslant 0})^n$ by $a(i) = \lfloor \frac{c_{I_m}\cdots c_{I_1}(i)-1}{n} \rfloor$ and we put $\psi(I_1, \ldots, I_m) = a$. The maps ψ and ϕ are inverse bijections. We deduce that the map $E \to (\mathfrak{S}_n \backslash \hat{\mathfrak{S}}_n^+)$ sending $(I_1, \ldots, I_m)$ to the class of $c_{I_m} \cdots c_{I_1}$ is bijective. It follows that the map $\mathfrak{S}_n \times E \to \hat{\mathfrak{S}}_n^+,\ (w, (I_1, \ldots, I_m)) \mapsto wc_{I_m} \cdots c_{I_1}$ is bijective.

If $\rho(t_w\gamma_{I_m} \cdots \gamma_{I_1}) = T_{wc_{I_m}\cdots c_{I_1}} = 0$ for some $w \in \mathfrak{S}_n$ and $(I_1, \ldots, I_m) \in E$, then the bijectivty of the map above shows that the image of ρ is the span of a proper subset of a basis of $\hat{H}_n^+$, contradicting the surjectivity of ρ.

This shows that the elements $\rho(t_w\gamma_{I_m} \cdots \gamma_{I_1})$ are distinct basis elements of $\hat{H}_n^+$, hence ρ is an isomorphism. □

Remark 3.2.13. The same method as the one used in the proof of Proposition 3.2.9 shows that $\hat{\mathfrak{S}}_n^+$ is the free $(\mathfrak{S}_n, \mathfrak{S}_n)$-monoid on a generator c with relations $c \cdot s_r = s_{r+1} \cdot c$ for $r \in \{1, \ldots, n-1\}$ and $c^2 \cdot s_{n-1} = s_1 \cdot c^2$.

3.2.7. *Pointed versions.* Given $n \geqslant 0$, we put $H_n^\bullet = (\mathfrak{S}_n)^{\mathrm{nil}}$. This is the quotient of the free pointed monoid generated by $T_1, \ldots, T_{n-1}$ by the relations (3.2.1). The differential is given by $d(T_i) = 1$. Note that $k[H_n^\bullet] = H_n$ and $H_n^\bullet = \{0\} \cup \{T_w\}_{w\in\mathfrak{S}_n}$.

We define $\hat{\mathfrak{S}}_n^{\mathrm{nil}}$ to be the differential graded pointed monoid with underlying differential pointed set $\{T_\sigma\}_{\sigma\in\hat{\mathfrak{S}}_n} \coprod \{0\}$ and multiplication, grading and differential that of $\hat{H}_n$.

We define $\hat{\mathfrak{S}}_n^{+,\mathrm{nil}}$ to be its differential graded pointed submonoid with non-zero elements those that stabilize $\mathbf{Z}_{>0}$.

4. 2-representation theory

We recall that k is a field of characteristic 2.

4.1. **Monoidal category.**

4.1.1. *Definition.* Let $\mathcal{U}$ be the differential strict monoidal category generated by an object e and a map $\tau : e^2 \to e^2$ subject to the relations

$$d(\tau) = 1,\ \tau^2 = 0 \text{ and } e\tau \circ \tau e \circ e\tau = \tau e \circ e\tau \circ \tau e. \tag{4.1.1}$$

There are isomorphisms of differential monoidal categories $\mathrm{opp} : \mathcal{U} \xrightarrow{\sim} \mathcal{U}^{\mathrm{opp}}$ and $\mathrm{rev} : \mathcal{U} \xrightarrow{\sim} \mathcal{U}^{\mathrm{rev}}$ given on generators by $e \mapsto e$ and $\tau \mapsto \tau$.

The following result is clear.

Proposition 4.1.1. *The objects of the category $\mathcal{U}$ are the e^n, $n \geqslant 0$. We have* $\mathrm{Hom}(e^n, e^m) = 0$ *if $n \neq m$ and there is an isomorphism of differential algebras*

$$H_n \xrightarrow{\sim} \mathrm{End}(e^n),\ T_i \mapsto e^{i-1}\tau e^{n-i-1}.$$

There is a commutative diagram

$$\begin{tikzcd} H_m \otimes H_n \arrow[r, "T_i\otimes T_j \mapsto T_i T_{m+j}"] \arrow[d, "\mathrm{can}"', "\sim"] & H_{m+n} \arrow[d, "\sim"', "\mathrm{can}"] \\ \mathrm{End}(E^m)\otimes\mathrm{End}(E^n) \arrow[r, "\otimes"'] & \mathrm{End}(E^{m+n}) \end{tikzcd}$$

The isomorphism $\mathrm{opp}:\mathcal{U}\xrightarrow{\sim}\mathcal{U}^{\mathrm{opp}}$ gives rise to the isomorphism of differential algebras

$$\mathrm{opp}: H_n \xrightarrow{\sim} H_n^{\mathrm{opp}},\ T_i \mapsto T_i.$$

The isomorphism $\mathrm{rev}:\mathcal{U}\xrightarrow{\sim}\mathcal{U}^{\mathrm{rev}}$ gives rise to the isomorphism of differential algebras

$$\iota_n : H_n \xrightarrow{\sim} H_n,\ T_i \mapsto T_{n-i}.$$

The functor $-\otimes E^n$ induces an injective morphism of differential algebras $H_r = \mathrm{End}(E^r) \to H_{r+n} = \mathrm{End}(E^{r+n}),\ T_i \mapsto T_i$ and we will identify H_r with a subalgebra of H_{r+n} via this morphism.

The functor $E^n \otimes -$ induces a morphism of differential algebras

$$f_n : H_r = \mathrm{End}(E^r) \to H_{n+r} = \mathrm{End}(E^{n+r}),\ T_i \mapsto T_{n+i}.$$

Note that H_n commutes with $f_n(H_r)$ and that $f_n = \iota_{n+r}\circ\iota_r$.

4.1.2. *2-representations.* Let $\mathcal{V}$ be a differential category.

Definition 4.1.2. *A* 2-representation *on $\mathcal{V}$ is the data of a strict monoidal differential functor $\mathcal{U}\to\mathrm{End}(\mathcal{V})$.*

The data of a 2-representation on $\mathcal{V}$ is the same as the data of a differential endofunctor E of $\mathcal{V}$ and of $\tau\in\mathrm{End}(E^2)$ satisfying (4.1.1).

Note that a 2-representation on $\mathcal{V}$ extends to a 2-representation on $\bar{\mathcal{V}}$ and on $\mathcal{V}^i$ (uniquely up to an equivalence unique up to isomorphism).

A *morphism of 2-representations* $(\mathcal{V},E,\tau)\to(\mathcal{V}',E',\tau)$ is the data of a differential functor $\Phi:\mathcal{V}\to\mathcal{V}'$ and of an isomorphism of functors $\varphi:\Phi E\xrightarrow{\sim}E'\Phi$ (with $d(\varphi)=0$) such that $\tau'\Phi\circ E'\varphi\circ\varphi E = E'\varphi\circ\varphi E\circ\Phi\tau : \Phi E^2\to E'^2\Phi$.

Example 4.1.3. Let $\mathcal{V}=k$-diff and $E=\tau=0$. This is the "trivial" 2-representation.

Let $\mathcal{V}$ be a 2-representation. The *opposite* 2-representation is $(\mathcal{V}\text{-diff},E',\tau')$, where $E'(\zeta)=\zeta E$ and $\tau'(\zeta)=\zeta\tau\in\mathrm{End}(E'^2(\zeta))$ for $\zeta\in\mathcal{V}$-diff. Note that the canonical functor $\mathcal{V}\to(\mathcal{V}\text{-diff})\text{-diff},\ v\mapsto(\zeta\mapsto\zeta(v))$ is a fully faithful morphism of 2-representations.

Assume E has a left adjoint $E^\vee$. We still denote by τ the endomorphism of $(E^\vee)^2$ corresponding to τ (cf §2.1.1). The pair $(E^\vee,\tau)$ defines the *left dual 2-representation* of (E,τ). Similarly, if E has a right adjoint ${}^\vee E$, we obtain a *right dual 2-representation* $({}^\vee E,\tau)$ of (E,τ).

Remark 4.1.4. One can also consider a *lax 2-representation* on $\mathcal{V}$: this is the data of a lax monoidal differential functor $\mathcal{U}\to\mathrm{End}(\mathcal{V})$.

Remark 4.1.5. The category $\mathcal{U}$ has a structure of differential graded monoidal category with τ in degree -1 and one can consider (lax) 2-representations on differential graded categories.

4.1.3. *Pointed case.* We denote by $\mathcal{U}^\bullet$ the strict monoidal differential pointed category generated by an object e and a map $\tau \in \mathrm{End}(e^2)$ subject to the relations (4.1.1). Its objects are the e^n, $n \geqslant 0$, $\mathrm{Hom}(e^n, e^m) = 0$ for $m \neq n$ and $\mathrm{End}(e^n) = H_n^\bullet$.

Let $\mathcal{V}$ be a differential pointed category.

A 2-*representation* on $\mathcal{V}$ is the data of a strict monoidal differential pointed functor $\mathcal{U}^\bullet \to \mathrm{End}(\mathcal{V})$. This is equivalent to the data of an endofunctor E of the differential pointed category $\mathcal{V}$ and $\tau \in \mathrm{End}(E^2)$ such that (E, τ) induce a 2-representation on $k[\mathcal{V}]$.

4.2. **Lax cocenter.**

4.2.1. *Lax bi-2-representations.* A *lax bi-2-representation* on $\mathcal{V}$ is a lax monoidal differential functor $E : \mathcal{U} \otimes \mathcal{U} \to \mathrm{End}(\mathcal{V})$. It corresponds to the data of

- differential endofunctors $E_{i,j} = E(e^i \otimes e^j)$ of $\mathcal{V}$
- morphisms of differential algebras $H_i \otimes H_j \to \mathrm{End}(E_{i,j})$
- morphisms of differential functors $\mu_{(i,j),(i',j')} : E_{i,j}E_{i',j'} \to E_{i+i',j+j'}$

such that

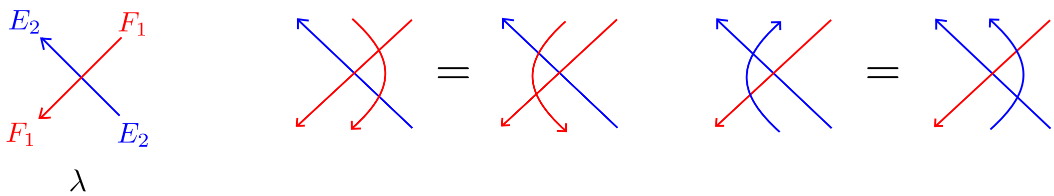
(1) $\mu_{(i,j),(i',j')}$ is equivariant for the action of $(H_i \otimes H_j) \otimes (H_{i'} \otimes H_{j'})$, where the action on $E_{i+i',j+j'}$ is the restriction of the action of $H_{i+i'} \otimes H_{j+j'}$ via the morphism $(a \otimes b) \otimes (a' \otimes b') \mapsto af_i(a') \otimes bf_j(b')$

(2) $\mu_{(i+i',j+j'),(i'',j'')} \circ (\mu_{(i,j),(i',j')}E_{i'',j''}) = \mu_{(i,j),(i'+i'',j'+j'')} \circ (E_{i,j}\mu_{(i',j'),(i'',j'')})$.

Consider two actions of $\mathcal{U}$ given by (F_1, τ_1) and (E_2, τ_2) on $\mathcal{V}$ and a closed morphism of functors $\lambda : F_1E_2 \to E_2F_1$ such that the following diagrams commute:

(4.2.1)
$$\begin{tikzcd} F_1^2E_2 \arrow[r, "F_1\lambda"] \arrow[d, "\tau_1E_2"'] & F_1E_2F_1 \arrow[r, "\lambda F_1"] & E_2F_1^2 \arrow[d, "E_2\tau_1"] \\ F_1^2E_2 \arrow[r, "F_1\lambda"'] & F_1E_2F_1 \arrow[r, "\lambda F_1"'] & E_2F_1^2 \end{tikzcd} \qquad \begin{tikzcd} F_1E_2^2 \arrow[r, "\lambda E_2"] \arrow[d, "F_1\tau_2"'] & E_2F_1E_2 \arrow[r, "E_2\lambda"] & E_2^2F_1 \arrow[d, "\tau_2F_1"] \\ F_1E_2^2 \arrow[r, "\lambda E_2"'] & E_2F_1E_2 \arrow[r, "E_2\lambda"'] & E_2^2F_1 \end{tikzcd}$$

Remark 4.2.1. The data of λ and the required relations are described graphically as:



Define morphisms

$$\lambda_{i,1} = (\lambda F_1^{i-1}) \circ \cdots \circ (F_1^{i-2}\lambda F_1) \circ (F_1^{i-1}\lambda) : F_1^iE_2 \to E_2F_1^i$$

and

$$\lambda_{i,j} = (E_2^{j-1}\lambda_{i,1}) \circ \cdots \circ (E_2\lambda_{i,1}E_2^{j-2}) \circ (\lambda_{i,1}E_2^{j-1}) : F_1^iE_2^j \to E_2^jF_1^i.$$

We define a lax bi-2-representation on $\mathcal{V}$ by $E_{i,j} = E_2^iF_1^j$. The actions of H_i on E_2^i and H_j on F_1^j provide an action of $H_i \otimes H_j$ on $E_{i,j}$ and $\mu_{(i,j),(i',j')} = E_2^i\lambda_{j,i'}F_1^{j'}$:

$$\mu_{(i,j),(i',j')} : E_2^iF_1^jE_2^{i'}F_1^{j'} \xrightarrow{E_2^i\lambda_{j,i'}F_1^{j'}} E_2^iE_2^{i'}F_1^jF_1^{j'} = E_2^{i+i'}F_1^{j+j'}.$$

Remark 4.2.2. One can also consider the notion of colax 2-representation. A colax 2-representation on $\mathcal{V}$ is the same data as a lax 2-representation on $\mathcal{V}^{\mathrm{opp}}$.

4.2.2. *Category.* Let $\mathcal{W}$ be a differential category endowed with a lax action $(E_{i,j})$ of $\mathcal{U}^2$. We define a differential category $\Delta_E\mathcal{W}$.

- The objects of $\Delta_E\mathcal{W}$ are pairs (m,ς) where $m\in\overline{\mathcal{W}}^i$ and $\varsigma\in Z\operatorname{Hom}_{\overline{\mathcal{W}}^i}(E_{0,1}E_{1,0}(m),m)$ such that for all $i\geqslant 1$, there exists $\varsigma_i\in Z\operatorname{Hom}_{\overline{\mathcal{W}}^i}(E_{i,i}(m),m)$ such that the composition b_i

$$b_i:(E_{0,1}E_{1,0})^i(m)\xrightarrow{(E_{0,1}E_{1,0})^{i-1}\varsigma}(E_{0,1}E_{1,0})^{i-1}(m)\xrightarrow{(E_{0,1}E_{1,0})^{i-2}\varsigma}\cdots\to E_{0,1}E_{1,0}(m)\xrightarrow{\varsigma}m \tag{4.2.2}$$

is equal to

$$(E_{0,1}E_{1,0})^i(m)\xrightarrow{\mathrm{can}}E_{i,i}(m)\xrightarrow{\varsigma_i}m$$

and $\varsigma_i\circ(T_r\otimes 1)=\varsigma_i\circ(1\otimes T_r)$ for $1\leqslant r<i$.

- $\operatorname{Hom}_{\Delta_E\mathcal{W}}((m,\varsigma),(m',\varsigma'(m))$ is the differential submodule of $\operatorname{Hom}_{\overline{\mathcal{W}}^i}(m,m')$ of elements f such that the following diagram commutes

$$\begin{tikzcd}
E_{0,1}E_{1,0}(m) \arrow[r, "\varsigma"] \arrow[d, "E_{0,1}E_{1,0}f"'] & m \arrow[d, "f"] \\
E_{0,1}E_{1,0}(m') \arrow[r, "\varsigma"] & m'
\end{tikzcd}$$

The composition of maps is defined by restricting that of $\overline{\mathcal{W}}^i$. So, we have a faithful forgetful functor $\omega:\Delta_E\mathcal{W}\to\overline{\mathcal{W}}^i$, $(m,\varsigma)\mapsto m$. Note that $\Delta_E\mathcal{W}$ is strongly pretriangulated and idempotent-complete.

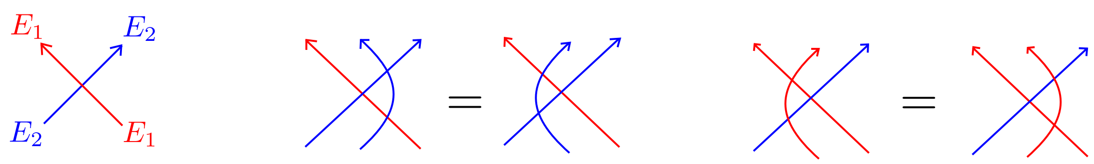
Remark 4.2.3. Note that applying the self-equivalence $(a,b)\mapsto(b,a)$ of $\mathcal{U}^2$ provides another lax action E' of $\mathcal{U}^2$ on $\mathcal{W}$. The corresponding differential category $\Delta_{E'}\mathcal{W}$ is not equivalent to $\Delta_E\mathcal{W}$ in general.

4.3. **Diagonal action.**

4.3.1. *Category.* Consider a differential category $\mathcal{W}$ endowed with two actions of $\mathcal{U}$ given by (E_1,τ_1) and (E_2,τ_2) and a closed morphism of functors $\sigma:E_2E_1\to E_1E_2$ such that the following diagrams commute:

$$\begin{tikzcd}
E_2^2E_1 \arrow[r, "E_2\sigma"] \arrow[d, "\tau_2E_1"'] & E_2E_1E_2 \arrow[r, "\sigma E_2"] & E_1E_2^2 \arrow[d, "E_1\tau_2"] \\
E_2^2E_1 \arrow[r, "E_2\sigma"'] & E_2E_1E_2 \arrow[r, "\sigma E_2"'] & E_1E_2^2
\end{tikzcd}
\qquad
\begin{tikzcd}
E_2E_1^2 \arrow[r, "\sigma E_1"] \arrow[d, "E_2\tau_1"'] & E_1E_2E_1 \arrow[r, "E_1\sigma"] & E_1^2E_2 \arrow[d, "\tau_1E_2"] \\
E_2E_1^2 \arrow[r, "\sigma E_1"'] & E_1E_2E_1 \arrow[r, "E_1\sigma"'] & E_1^2E_2
\end{tikzcd} \tag{4.3.1}$$

Remark 4.3.1. The data of σ and the relations can be described graphically as follows:



σ

We define a differential category $\mathcal{V} = \Delta_\sigma \mathcal{W}$.

- The objects of $\mathcal{V}$ are pairs (m, π) where $m \in \overline{\mathcal{W}}^i$ and $\pi \in Z\operatorname{Hom}_{\overline{\mathcal{W}}^i}(E_2(m), E_1(m))$ such that the following diagram commutes

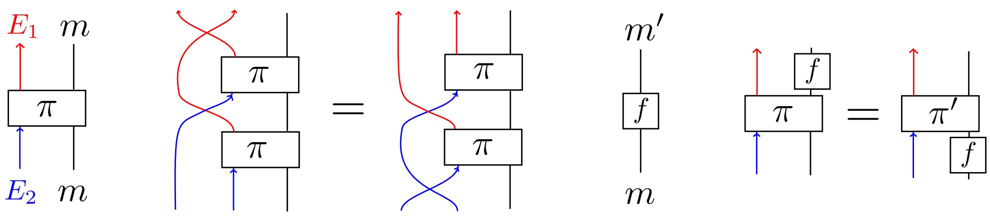
$$\begin{tikzcd}
E_2^2(m) \arrow[r, "E_2\pi"] \arrow[d, "\tau_2"'] & E_2E_1(m) \arrow[r, "\sigma"] & E_1E_2(m) \arrow[r, "E_1\pi"] & E_1^2(m) \arrow[d, "\tau_1"] \\
E_2^2(m) \arrow[r, "E_2\pi"'] & E_2E_1(m) \arrow[r, "\sigma"'] & E_1E_2(m) \arrow[r, "E_1\pi"'] & E_1^2(m)
\end{tikzcd}$$

- $\operatorname{Hom}_{\mathcal{V}}((m,\pi),(m',\pi'))$ is the differential submodule of $\operatorname{Hom}_{\overline{\mathcal{W}}^i}(m, m')$ of elements f such that the following diagram commutes

$$\begin{tikzcd}
E_2(m) \arrow[r, "\pi"] \arrow[d, "E_2 f"'] & E_1(m) \arrow[d, "E_1 f"] \\
E_2(m') \arrow[r, "\pi'"'] & E_1(m')
\end{tikzcd}$$

The composition of maps is defined by restricting that of $\overline{\mathcal{W}}^i$. So, we have a faithful forgetful functor $\omega : \mathcal{V} \to \overline{\mathcal{W}}^i$, $(m, \pi) \mapsto m$. Note that $\mathcal{V}$ is strongly pretriangulated and idempotent-complete.

Remark 4.3.2. The structure of objects and maps in $\mathcal{V}$ can be described graphically as follows:



Remark 4.3.3. Assume E_1 admits a left adjoint F_1. The data of the map $\sigma : E_2E_1 \to E_1E_2$ corresponds by adjunction to the data of a map

$$\lambda : F_1E_2 \xrightarrow{\bullet\eta_1} F_1E_2E_1F_1 \xrightarrow{F_1\sigma F_1} F_1E_1E_2F_1 \xrightarrow{\varepsilon_1\bullet} E_2F_1.$$

The commutativity of the diagrams (4.3.1) is equivalent to the commutativity of the diagrams (4.2.1). Assume the diagrams commute. We obtain a lax bi-2-representation $(E_{i,j})$ on $\mathcal{W}$ (cf §4.2.1).

Let $(m, \varsigma) \in \Delta_E \mathcal{W}$. We have an adjunction isomorphism

$$\phi : \operatorname{Hom}(E_2(m), E_1(m)) \xrightarrow{\sim} \operatorname{Hom}(F_1E_2(m), m).$$

Let $\pi = \phi^{-1}(\varsigma) \in Z\operatorname{Hom}(E_2(m), E_1(m))$. The object (m, π) is in $\Delta_\sigma\mathcal{W}$ and $(m, \varsigma) \mapsto (m, \pi)$ defines a fully faithful functor of differential categories $\Delta_E\mathcal{W} \to \Delta_\sigma\mathcal{W}$.

Assume now λ is invertible. The canonical map $f_i : (E_{0,1}E_{1,0})^i \to E_{i,i}$ is invertible. Let $\varsigma_i = b_i \circ f_i^{-1}$. Consider $r \in \{1, \ldots, i-1\}$. We have

$$\begin{aligned}
\varsigma_i \circ (T_r \otimes 1) &= b_{r-1} \circ (E_{01}E_{10})^{r-1}(\varsigma_2 \circ (T_1 \otimes 1) \circ f_2) \circ (E_{01}E_{10})^{r+1} b_{i-r-1} \circ f_i^{-1} \\
&= b_{r-1} \circ (E_{01}E_{10})^{r-1}(\varsigma_2 \circ (1 \otimes T_1) \circ f_2) \circ (E_{01}E_{10})^{r+1} b_{i-r-1} \circ f_i^{-1} \\
&= \varsigma_i \circ (1 \otimes T_r)
\end{aligned}$$

As a consequence, the functor above is an isomorphism of differential categories $\Delta_E \mathcal{W} \xrightarrow{\sim} \Delta_\sigma \mathcal{W}$.

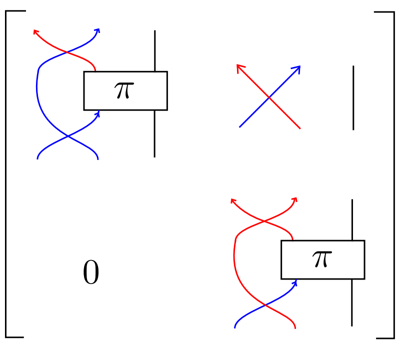
4.3.2. 1-*arrows.* We define now a differential functor $E : \mathcal{V} \to \mathcal{V}$.

- Let $(m, \pi) \in \mathcal{V}$. Let $m' = E_2(m) \oplus E_1(m)$. We define

$$\pi' = \begin{pmatrix} \sigma \circ E_2\pi \circ \tau_2 & \sigma \\ 0 & \tau_1 \circ E_1\pi \circ \sigma \end{pmatrix} : E_2(m') \to E_1(m')$$

$$\pi' : \quad \begin{tikzcd}
E_2^2(m) \arrow[rr, bend left, "E_2\pi"] \arrow[d, "\sigma\circ E_2\pi\circ\tau_2"'] & \oplus & E_2E_1(m) \arrow[lld, "\sigma"'] \arrow[d, "\tau_1\circ E_1\pi\circ\sigma"] \\
E_1E_2(m) \arrow[rr, bend right, "E_1\pi"'] & \oplus & E_1^2(m)
\end{tikzcd}$$

Remark 4.3.4. The graphical description of π' is the following:



Lemma 4.3.5. *(m', π') is an object of $\mathcal{V}$.*

Proof. Note that $d(\pi') = 0$.

Let $a = \tau_1 \circ E_1\pi' \circ \sigma(m') \circ E_2\pi'$ and $b = E_1\pi' \circ \sigma(m') \circ E_2\pi' \circ \tau_2$. We have

$$\begin{aligned}
a_{11} &= \tau_1 E_2 \circ E_1\sigma \circ E_1E_2\pi \circ E_1\tau_2 \circ \sigma E_2 \circ E_2\sigma \circ E_2^2\pi \circ E_2\tau_2 \\
&= \tau_1 E_2 \circ E_1\sigma \circ E_1E_2\pi \circ \sigma E_2 \circ E_2\sigma \circ \tau_2 E_1 \circ E_2^2\pi \circ E_2\tau_2 \\
&= \tau_1 E_2 \circ E_1\sigma \circ \sigma E_1 \circ E_2E_1\pi \circ E_2\sigma \circ E_2^2\pi \circ \tau_2 E_2 \circ E_2\tau_2 \\
&= E_1\sigma \circ \sigma E_1 \circ E_2\tau_1 \circ E_2E_1\pi \circ E_2\sigma \circ E_2^2\pi \circ \tau_2 E_2 \circ E_2\tau_2 \\
&= E_1\sigma \circ \sigma E_1 \circ E_2E_1\pi \circ E_2\sigma \circ E_2^2\pi \circ E_2\tau_2 \circ \tau_2 E_2 \circ E_2\tau_2 \\
&= E_1\sigma \circ E_1E_2\pi \circ \sigma E_2 \circ E_2\sigma \circ E_2^2\pi \circ \tau_2 E_2 \circ E_2\tau_2 \circ \tau_2 E_2 \\
&= E_1\sigma \circ E_1E_2\pi \circ \sigma E_2 \circ E_2\sigma \circ \tau_2 E_1 \circ E_2^2\pi \circ E_2\tau_2 \circ \tau_2 E_2 \\
&= E_1\sigma \circ E_1E_2\pi \circ E_1\tau_2 \circ \sigma E_2 \circ E_2\sigma \circ E_2^2\pi \circ E_2\tau_2 \circ \tau_2 E_2 = b_{11},
\end{aligned}$$

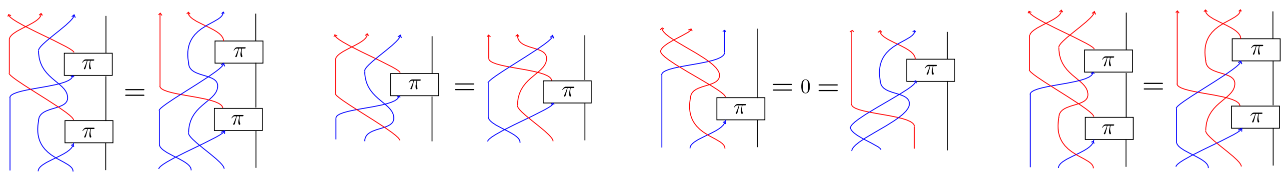
$$\begin{aligned}
a_{12} &= \tau_1 E_2 \circ E_1\sigma \circ E_1E_2\pi \circ E_1\tau_2 \circ \sigma E_2 \circ E_2\sigma + \tau_1 E_2 \circ E_1\sigma \circ \sigma E_1 \circ E_2\tau_1 \circ E_2E_1\pi \circ E_2\sigma \\
&= \tau_1 E_2 \circ E_1\sigma \circ E_1E_2\pi \circ \sigma E_2 \circ E_2\sigma \circ \tau_2 E_1 + \tau_1^2 E_2 \circ E_1\sigma \circ \sigma E_1 \circ E_2E_1\pi \circ E_2\sigma \\
&= \tau_1 E_2 \circ E_1\sigma \circ \sigma E_1 \circ E_2E_1\pi \circ E_2\sigma \circ \tau_2 E_1 \\
&= E_1\sigma \circ E_1E_2\pi \circ \sigma E_2 \circ E_2\sigma \circ \tau_2^2 E_1 + E_1\sigma \circ \sigma E_1 \circ E_2\tau_1 \circ E_2E_1\pi \circ E_2\sigma \circ \tau_2 E_1 \\
&= E_1\sigma \circ E_1E_2\pi \circ E_1\tau_2 \circ \sigma E_2 \circ E_2\sigma \circ \tau_2 E_1 + E_1\sigma \circ \sigma E_1 \circ E_2\tau_1 \circ E_2E_1\pi \circ E_2\sigma \circ \tau_2 E_1 = b_{12},
\end{aligned}$$

$$a_{21} = 0 = b_{21} \text{ and}$$

$$\begin{aligned}
a_{22} &= \tau_1 E_1 \circ E_1\tau_1 \circ E_1^2\pi \circ E_1\sigma \circ \sigma E_1 \circ E_2\tau_1 \circ E_2E_1\pi \circ E_2\sigma \\
&= \tau_1 E_1 \circ E_1\tau_1 \circ E_1^2\pi \circ E_1\sigma \circ \sigma E_1 \circ E_2\tau_1 \circ E_2E_1\pi \circ E_2\sigma \\
&= \tau_1 E_1 \circ E_1\tau_1 \circ \tau_1 E_1 \circ E_1^2\pi \circ E_1\sigma \circ E_1E_2\pi \circ \sigma E_2 \circ E_2\sigma \\
&= E_1\tau_1 \circ \tau_1 E_1 \circ E_1\tau_1 \circ E_1^2\pi \circ E_1\sigma \circ E_1E_2\pi \circ \sigma E_2 \circ E_2\sigma \\
&= E_1\tau_1 \circ \tau_1 E_1 \circ E_1^2\pi \circ E_1\sigma \circ E_1E_2\pi \circ E_1\tau_2 \circ \sigma E_2 \circ E_2\sigma \\
&= E_1\tau_1 \circ E_1^2\pi \circ \tau_1 E_2 \circ E_1\sigma \circ E_1E_2\pi \circ \sigma E_2 \circ E_2\sigma \circ \tau_2 E_1 \\
&= E_1\tau_1 \circ E_1^2\pi \circ \tau_1 E_2 \circ E_1\sigma \circ \sigma E_1 \circ E_2E_1\pi \circ E_2\sigma \circ \tau_2 E_1 \\
&= E_1\tau_1 \circ E_1^2\pi \circ E_1\sigma \circ \sigma E_1 \circ E_2\tau_1 \circ E_2E_1\pi \circ E_2\sigma \circ \tau_2 E_1 = b_{22}.
\end{aligned}$$

The lemma follows. ☐

Remark 4.3.6. The equalities established in the proof of the lemma above have the following graphical description:



We put $E(m, \pi) = (m', \pi')$.

- Given $f \in \mathrm{Hom}_{\mathcal{V}}((m,\pi),(\tilde{m},\tilde{\pi}))$, we put $E(f) = \begin{pmatrix} E_2 f & 0 \\ 0 & E_1 f \end{pmatrix}$:

$$\begin{tikzcd}
E_2(m) \arrow[rr, bend left, "\pi"] \arrow[d, "E_2 f"'] & \oplus & E_1(m) \arrow[d, "E_1 f"] \\
E_2(\tilde{m}) \arrow[rr, bend right, "\tilde{\pi}"'] & \oplus & E_1(\tilde{m})
\end{tikzcd}$$

Lemma 4.3.7. *We have $E(f) \in \mathrm{Hom}_{\mathcal{V}}(E(m,\pi), E(\tilde{m},\tilde{\pi}))$. The construction makes E into a differential endofunctor of $\mathcal{V}$.*

Proof. The lemma follows from the commutativity of the following diagram:

$$\begin{tikzcd}
E_2^2(m) \arrow[rrrr, bend left, "\sigma\circ E_2\pi\circ\tau_2"] \arrow[d, "E_2^2 f"'] & \oplus & E_2E_1(m) \arrow[rr, bend left, "\sigma"] \arrow[rrrr, bend left, "\tau_1\circ E_1\pi\circ\sigma"] \arrow[d, "E_2E_1 f"] & & E_1E_2(m) \arrow[d, "E_1E_2 f"'] & \oplus & E_1^2(m) \arrow[d, "E_1^2 f"] \\
E_2^2(\tilde{m}) \arrow[rrrr, bend right, "\sigma\circ E_2\tilde{\pi}\circ\tau_2"'] & \oplus & E_2E_1(\tilde{m}) \arrow[rr, bend right, "\sigma"'] \arrow[rrrr, bend right, "\tau_1\circ E_1\tilde{\pi}\circ\sigma"'] & & E_1E_2(\tilde{m}) & \oplus & E_1^2(\tilde{m})
\end{tikzcd}$$

□

4.3.3. 2-*arrows.* We assume in §4.3.3 that σ is invertible.

We define an endomorphism τ of ωE^2. Let $(m,\pi) \in \mathcal{V}$. We have $E^2(m,\pi) = (m'',\pi'')$ where $m'' = [E_2^2(m) \oplus E_2E_1(m) \oplus E_1E_2(m) \oplus E_1^2(m), \partial]$ and

$$\partial = \begin{pmatrix} 0 & & & \\ E_2\pi & 0 & & \\ \sigma\circ E_2\pi\circ\tau_2 & \sigma & 0 & \\ 0 & \tau_1\circ E_1\pi\circ\sigma & E_1\pi & 0 \end{pmatrix}.$$

We define an endomorphism τ of m'' by

$$\tau = \begin{pmatrix} \tau_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_1 \end{pmatrix}. \tag{4.3.2}$$

Theorem 4.3.8. *The endomorphism τ of m'' defines an endomorphism of E^2. The data $(\Delta_\sigma\mathcal{W}, E, \tau)$ is an idempotent-complete strongly pretriangulated 2-representation.*

Proof. The non-zero coefficients of π'' are

$$\begin{aligned}
\pi''_{11} &= \sigma E_2 \circ E_2\sigma \circ E_2^2\pi \circ E_2\tau_2 \circ \tau_2 E_2 \\
\pi''_{22} &= \sigma E_1 \circ E_2\tau_1 \circ E_2E_1\pi \circ E_2\sigma \circ \tau_2 E_1 \\
\pi''_{33} &= \tau_1 E_2 \circ E_1\sigma \circ E_1E_2\pi \circ E_1\tau_2 \circ \sigma E_2 \\
\pi''_{44} &= \tau_1 E_1 \circ E_1\tau_1 \circ E_1^2\pi \circ E_1\sigma \circ \sigma E_1
\end{aligned}$$

$$\pi''_{12} = \sigma E_2 \circ E_2\sigma \circ \tau_2 E_1,\ \pi''_{13} = \sigma E_2,\ \pi''_{24} = \sigma E_1,\ \pi''_{34} = \tau_1 E_2 \circ E_1\sigma \circ \sigma E_1.$$

Let $a = E_1\tau \circ \pi''$ and $b = \pi'' \circ E_2\tau$. We have

$$\begin{aligned}
a_{11} &= \sigma E_2 \circ E_2\sigma \circ E_1\tau_2 \circ E_2^2\pi \circ E_2\tau_2 \circ \tau_2 E_2 = \sigma E_2 \circ E_2\sigma \circ E_2^2\pi \circ \tau_2 E_2 \circ E_2\tau_2 \circ \tau_2 E_2 \\
&= \sigma E_2 \circ E_2\sigma \circ E_2^2\pi \circ E_2\tau_2 \circ \tau_2 E_2 \circ E_2\tau_2 = b_{11}
\end{aligned}$$

$$a_{12} = E_1\tau_2 \circ \sigma E_2 \circ E_2\sigma \circ \tau_2 E_1 = 0 = b_{12}$$

$$a_{13} = E_1\tau_2 \circ \sigma E_2 = b_{13}$$

$$\begin{aligned}
a_{23} &= E_1\sigma^{-1} \circ \tau_1 E_2 \circ E_1\sigma \circ E_1E_2\pi \circ E_1\tau_2 \circ \sigma E_2 \\
&= E_1\sigma^{-1} \circ \tau_1 E_2 \circ E_1\sigma \circ E_1E_2\pi \circ \sigma E_2 \circ E_2\sigma \circ \tau_2 E_1 \circ E_2\sigma^{-1} \\
&= E_1\sigma^{-1} \circ \tau_1 E_2 \circ E_1\sigma \circ \sigma E_2 \circ E_2E_1\pi \circ \circ E_2\sigma \circ \tau_2 E_1 \circ E_2\sigma^{-1} \\
&= \sigma E_1 \circ E_2\tau_1 \circ E_2E_1\pi \circ E_2\sigma \circ \tau_2 E_1 \circ E_2\sigma^{-1} = b_{23}
\end{aligned}$$

$$a_{24} = E_1\sigma^{-1} \circ \tau_1 E_2 \circ E_1\sigma \circ \sigma E_1 = \sigma E_1 \circ E_2\tau_1 = b_{24}$$

$$\begin{aligned}
a_{44} &= E_1\tau_1 \circ \tau_1 E_1 \circ E_1\tau_1 \circ E_1^2\pi \circ E_1\sigma \circ \sigma E_1 = \tau_1 E_1 \circ E_1\tau_1 \circ \tau_1 E_1 \circ E_1^2\pi \circ E_1\sigma \circ \sigma E_1 \\
&= \tau_1 E_1 \circ E_1\tau_1 \circ E_1^2\pi \circ E_1\sigma \circ \sigma E_1 \circ E_2\tau_1 = b_{44}
\end{aligned}$$

All the other coefficients of a and b vanish. We deduce that $a = b$, hence τ is an endomorphism of $E^2(m,\pi)$. It follows easily that τ defines an endomorphism of E^2.

We have $\tau^2 = 0$ and

$$\begin{aligned}
d(\tau) &= \begin{pmatrix} d(\tau_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d(\tau_1) \end{pmatrix} + \tau \circ \partial + \partial \circ \tau \\
&= \begin{pmatrix} \mathrm{id} & & & \\ 2E_2\pi \circ \tau_2 & \mathrm{id} & & \\ \sigma \circ E_2\pi \circ \tau_2^2 & & \mathrm{id} & \\ & \tau_1^2 \circ E_1\pi & 2\tau_1 \circ E_1\pi \circ \sigma & \mathrm{id} \end{pmatrix} \\
&= \mathrm{id}\,.
\end{aligned}$$

We have $E^3(m,\pi) = ([m''', \delta'], \pi''')$, where

$$m''' = E_2^3(m) \oplus E_2^2E_1(m) \oplus E_2E_1E_2(m) \oplus E_2E_1^2(m) \oplus E_1E_2^2(m) \oplus E_1E_2E_1(m) \oplus E_1^2E_2(m) \oplus E_1^3(m).$$

We have

$$\tau E = \begin{pmatrix} \tau_2 E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_2 E_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^{-1}E_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^{-1}E_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau_1 E_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_1 E_1 \end{pmatrix}$$

and

$$E\tau = \begin{pmatrix} E_2\tau_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_2\sigma^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_2\tau_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_1\tau_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E_1\sigma^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_1\tau_1 \end{pmatrix}$$

Let $a = (E\tau)\circ(\tau E)\circ(E\tau)$ and $b = (\tau E)\circ(E\tau)\circ(\tau E)$. We have

$$a_{11} = E_2\tau_2 \circ \tau_2 E_2 \circ E_2\tau_2 = \tau_2 E_2 \circ E_2\tau_2 \circ \tau_2 E_2 = b_{11}$$

$$a_{44} = E_1\tau_1 \circ \tau_1 E_1 \circ E_1\tau_1 = \tau_1 E_1 \circ E_1\tau_1 \circ \tau_1 E_1 = b_{44}$$

$$a_{25} = E_2\sigma^{-1} \circ \sigma^{-1}E_2 \circ E_1\tau_2 = \tau_2 E_1 \circ E_2\sigma^{-1} \circ \sigma^{-1}E_2 = b_{25}$$

$$a_{47} = E_2\tau_1 \circ \sigma^{-1}E_1 \circ E_1\sigma^{-1} = \sigma^{-1}E_1 \circ E_1\sigma^{-1} \circ \tau_1 E_2 = b_{47}$$

and all the other coefficients of a and b vanish. It follows that $a = b$. This completes the proof of the theorem. ☐

4.3.4. *Functoriality.* We consider two differential categories $\mathcal{W}$ and $\mathcal{W}'$ endowed with actions (E_i, τ_i) and (E'_i, τ'_i) of $\mathcal{U}$ for $i \in \{1,2\}$ and closed morphism of functors $\sigma : E_2E_1 \to E_1E_2$ and $\sigma' : E'_2E'_1 \to E'_1E'_2$ making (4.3.1) and the similar diagram for σ' commute.

Let $\Phi : \mathcal{W} \to \mathcal{W}'$ be a differential functor and $\varphi_i : \Phi E_i \xrightarrow{\sim} E'_i\Phi$ be closed isomorphisms of functors making (Φ, φ_i) into morphisms of 2-representations for $i \in \{1,2\}$. Assume

$$(E'_1\varphi_2)\circ(\varphi_1 E_2)\circ(\Phi\sigma) = (\sigma'\Phi)\circ(E'_2\varphi_1)\circ(\varphi_2 E_1) : \Phi E_2 E_1 \to E'_1E'_2\Phi. \tag{4.3.3}$$

Proposition 4.3.9. *There is a differential functor $\Delta\Phi : \Delta_\sigma\mathcal{W} \to \Delta_{\sigma'}\mathcal{W}'$ given by $(m,\pi) \mapsto (\Phi(m), \varphi_1(m)\circ\Phi(\pi)\circ\varphi_2(m)^{-1})$.*

There is a closed isomorphism of functors

$$\varphi = \begin{pmatrix} \varphi_2 & \\ & \varphi_1 \end{pmatrix} : \Delta\Phi E \xrightarrow{\sim} E'\Delta\Phi.$$

If σ and σ' are invertible, then $(\Delta\Phi, \varphi)$ defines a morphism of 2-representations $\Delta_\sigma\mathcal{W} \to \Delta_{\sigma'}\mathcal{W}'$.

Proof. Let (m, π) be an object of $\Delta_\sigma \mathcal{W}$. Let $\pi' = \varphi_1(m) \circ \Phi(\pi) \circ \varphi_2^{-1}(m)$, an element of $Z \operatorname{Hom}_{\overline{\mathcal{W}'}^i}(E_2'\Phi(m), E_1'\Phi(m))$.

We have

$$(E_1'\pi') \circ (\sigma'\Phi(m)) \circ (E_2'\pi') =$$

$$\begin{aligned}
&= (E_1'\varphi_1(m)) \circ (E_1'\Phi\pi) \circ (E_1'\varphi_2^{-1}(m)) \circ (\sigma'\Phi(m)) \circ (E_2'\varphi_1(m)) \circ (E_2'\Phi\pi) \circ (E_2'\varphi_2^{-1}(m)) \\
&= (E_1'\varphi_1(m)) \circ (E_1'\Phi\pi) \circ (\varphi_1(E_2(m))) \circ (\Phi\sigma(m)) \circ (\varphi_2^{-1}(E_1(m))) \circ (E_2'\Phi\pi) \circ (E_2'\varphi_2^{-1}(m)) \\
&= (E_1'\varphi_1(m)) \circ (\varphi_1(E_1(m))) \circ \Phi\big((E_1\pi) \circ \sigma(m) \circ (E_2\pi)\big) \circ (\varphi_2^{-1}(E_2(m))) \circ (E_2'\varphi_2^{-1}(m))
\end{aligned}$$

It follows that

$$(E_1'\pi') \circ (\sigma'\Phi(m)) \circ (E_2'\pi') \circ (\tau_2'\Phi(m)) = (\tau_1'\Phi(m)) \circ (E_1'\pi') \circ (\sigma'\Phi(m)) \circ (E_2'\pi'),$$

hence $(\Phi(m), \pi')$ is an object of $\Delta_{\sigma'}\mathcal{W}'$. We put $\Delta\Phi(m, \pi) = (\Phi(m), \pi')$.

Let $f \in \operatorname{Hom}_{\Delta_\sigma \mathcal{W}}((m, \pi), (\tilde{m}, \tilde{\pi}))$. We have a commutative diagram

$$\begin{tikzcd}
E_2'\Phi(m) \arrow[r, "\varphi_2^{-1}(m)"] \arrow[d, "E_2'\Phi(f)"'] & \Phi E_2(m) \arrow[r, "\Phi\pi"] \arrow[d, "\Phi E_2(f)"] & \Phi E_1(m) \arrow[r, "\varphi_1(m)"] \arrow[d, "\Phi E_1(f)"] & E_1'\Phi(m) \arrow[d, "E_1'\Phi(f)"] \\
E_2'\Phi(\tilde{m}) \arrow[r, "\varphi_2^{-1}(\tilde{m})"'] & \Phi E_2(\tilde{m}) \arrow[r, "\Phi\tilde{\pi}"'] & \Phi E_1(\tilde{m}) \arrow[r, "\varphi_1(\tilde{m})"'] & E_1'\Phi(\tilde{m})
\end{tikzcd}$$

and it follows that $\Phi(f) \in \operatorname{Hom}_{\Delta_{\sigma'}\mathcal{W}'}(\Delta\Phi(m, \pi), \Delta\Phi(\tilde{m}, \tilde{\pi}))$. We put $(\Delta\Phi)(f) = \Phi(f)$. This makes $\Delta\Phi$ into a differential functor $\Delta_\sigma \mathcal{W} \to \Delta_{\sigma'}\mathcal{W}'$.

We have

$$(\Delta\Phi)(E(m, \pi)) = (\Phi(E_2(m)) \oplus \Phi(E_1(m)), \beta),$$

$$\beta = \begin{pmatrix} \varphi_1 E_2 \circ \Phi\sigma \circ \Phi E_2\pi \circ \Phi\tau_2 \circ \varphi_2^{-1} E_2 & \varphi_1 E_2 \circ \Phi\sigma \circ \varphi_2^{-1} E_1 \\ 0 & \varphi_1 E_1 \circ \Phi\tau_1 \circ \Phi E_1\pi \circ \Phi\sigma \circ \varphi_2^{-1} E_1 \end{pmatrix}$$

and

$$E'((\Delta\Phi)(m, \pi)) = (E_2'(\Phi(m)) \oplus E_1'(\Phi(m)), \beta'),$$

$$\beta' = \begin{pmatrix} \sigma'\Phi \circ E_2'(\varphi_1 \circ \Phi\pi \circ \varphi_2^{-1}) \circ \tau_2'\Phi & \sigma'\Phi \\ 0 & \tau_1'\Phi \circ E_1'(\varphi_1 \circ \Phi\pi \circ \varphi_2^{-1}) \circ \sigma'\Phi \end{pmatrix})$$

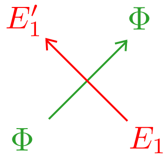
We have

$$\beta' \begin{pmatrix} E_2'\varphi_2 & 0 \\ 0 & E_2'\varphi_1 \end{pmatrix} = \begin{pmatrix} E_1'\varphi_2 & 0 \\ 0 & E_1'\varphi_1 \end{pmatrix} \beta,$$

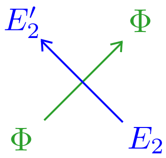
hence $\begin{pmatrix} \varphi_2(m) & \\ & \varphi_1(m) \end{pmatrix}$ defines a closed isomorphism $\Delta\Phi(E(m, \pi)) \xrightarrow{\sim} E'(\Delta\Phi(m, \pi))$. The naturality of φ_1 and φ_2 implies immediately that of φ.

We have $\tau_i'\Phi \circ E_i'\varphi_i \circ \varphi_i E_i = E_i'\varphi_i \circ \varphi_i E_i \circ \Phi\tau_i$ for $i \in \{1, 2\}$. Together with (4.3.3), it follows that $\tau'(\Delta\Phi) \circ E'\varphi \circ \varphi E = E'\varphi \circ \varphi E \circ (\Delta\Phi)\tau$, hence $(\Delta\Phi, \varphi)$ defines a morphism of 2-representations. $\square$

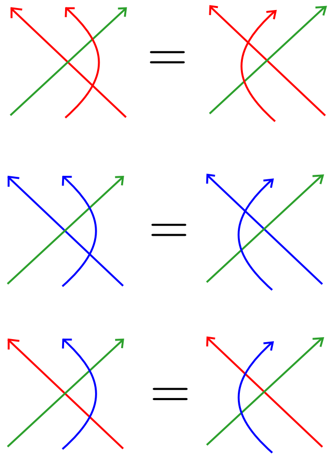
Remark 4.3.10. The data of φ_1 and φ_2, the relations they are required to satisfy, and the map π' in the proof of the proposition are described graphically as:

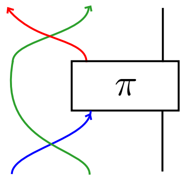


φ_1



φ_2





π'

The following proposition is immediate.

Proposition 4.3.11. *If Φ is faithful, then $\Delta\Phi$ is faithful.*

4.4. **Dual diagonal action.**

4.4.1. *Category.* Consider two actions of $\mathcal{U}$ given by (F_1, τ_1) and (E_2, τ_2) on $\mathcal{W}$ and a closed morphism of functors $\lambda : F_1E_2 \to E_2F_1$ such that diagrams (4.2.1) commute. As in §4.2.1, we have maps $\mu_{i,j} = \mu_{(i,i),(j,j)} : E_2^iF_1^iE_2^jF_1^j \to E_2^{i+j}F_1^{i+j}$.

We define a differential category $\Delta_\lambda\mathcal{W}$. Its objects are pairs (m, ς) where $m \in \overline{\mathcal{W}}^i$ and $\varsigma = (\varsigma_i)_{i\geqslant 1}$, $\varsigma_i \in Z\operatorname{Hom}_{\overline{\mathcal{W}}^i}(E_2^iF_1^i(m), m)$, satisfies that

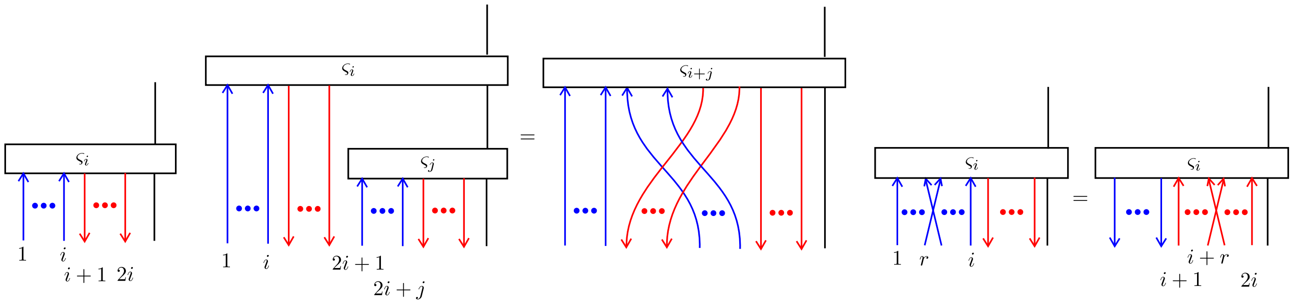
- for all $i, j \geqslant 1$, we have $\varsigma_i \circ E_2^iF_1^i\varsigma_j = \varsigma_{i+j} \circ \mu_{i,j}$
- $\varsigma_i \circ T_rF_1^i = \varsigma_i \circ E_2^iT_r$ for all $1 \leqslant r < i$.

We define $\operatorname{Hom}_{\Delta_\lambda\mathcal{W}}((m,\varsigma),(m',\varsigma'))$ to be the differential submodule of $\operatorname{Hom}_{\overline{\mathcal{W}}^i}(m, m')$ of elements f such that for all $i \geqslant 1$, the following diagram commutes

$$\begin{tikzcd}
E_2^iF_1^i(m) \arrow[r, "\varsigma_i"] \arrow[d, "E_2^iF_1^if"'] & m \arrow[d, "f"] \\
E_2^iF_1^i(m') \arrow[r, "\varsigma_i'"'] & m'
\end{tikzcd}$$

The composition of maps is defined to be that of $\overline{\mathcal{W}}^i$.

Remark 4.4.1. The structure of objects in $\Delta_\lambda\mathcal{W}$ can be described graphically as follows:



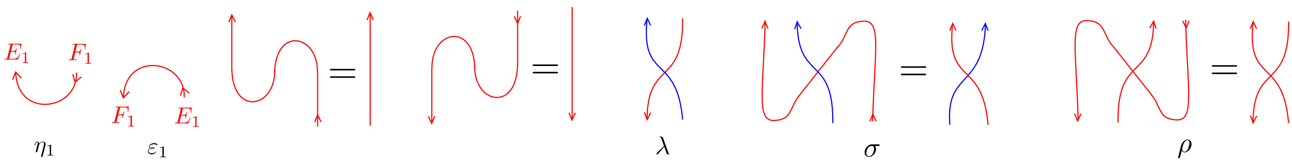
Remark 4.4.2. The maps $\mu_{i,j}$ make $A = \bigoplus_{i\geqslant 0} E_2^i F_1^i$ into a monoid in the monoidal category of endofunctors of $\overline{\mathcal{W}}^i$, when $\overline{\mathcal{W}}^i$ has enough direct sums. If $\overline{\mathcal{W}}^i$ has enough colimits, we have an induced monoid $\bar{A} = \bigoplus_{i\geqslant 0}(E_2^i F_1^i) \otimes_{H_i\otimes H_i} H_i$. Now, the category $\Delta_\lambda\mathcal{W}$ is the category of $\bar{A}$-modules in $\overline{\mathcal{W}}^i$.

Remark 4.4.3. Let us define a lax bi-2-representation $E_{i,j} = E_2^j F_1^i$ on $\mathcal{W}$ as deduced from the one defined in §4.2.1 by applying the swap automorphism of $\mathcal{U}\times\mathcal{U}$ (cf Remark 4.2.3).

There is a faithful differential functor $\Delta_\lambda\mathcal{W} \to \Delta_E\mathcal{W}$, $(m,\varsigma)\mapsto(m,\varsigma_1)$.

4.4.2. *Adjoint.* We assume F_1 has a right adjoint E_1 and denote by ε_1 and η_1 the counit and unit of the adjunction. We denote by τ_1 the endomorphism of E_1^2 corresponding by adjunction to the endomorphism τ_1 of F_1^2. The pair (E_1,τ_1) provides an action of $\mathcal{U}$ on $\mathcal{W}$.

Remark 4.4.4. The maps η_1, ε_1, the relations they satisfy, and λ, σ and ρ are described graphically as:



We denote by σ the composition

$$\sigma : E_2E_1 \xrightarrow{\eta_1 E_2 E_1} E_1F_1E_2E_1 \xrightarrow{E_1\lambda E_2} E_1E_2F_1E_1 \xrightarrow{E_1E_2\varepsilon_1} E_1E_2 \tag{4.4.1}$$

and by ρ the composition

$$\rho : F_1E_1 \xrightarrow{F_1E_1\eta_1} F_1E_1^2F_1 \xrightarrow{F_1\tau_1F_1} F_1E_1^2F_1 \xrightarrow{\varepsilon_1E_1F_1} E_1F_1. \tag{4.4.2}$$

The diagram (4.3.1) is commutative.

Lemma 4.4.5. *We have*

$$E_1\lambda\circ\rho E_2\circ F_1\sigma = \sigma F_1\circ E_2\rho\circ\lambda E_1 \textit{ and } \rho F_1\circ F_1\rho\circ\tau_1E_1 = E_1\tau_1\circ\rho F_1\circ F_1\rho.$$

Proof. We have

$$E_1\lambda\circ\rho E_2\circ F_1\sigma =$$

$= E_1E_2F_1\varepsilon_1 \circ E_1\lambda F_1E_1 \circ F_1E_1^2F_1\lambda E_1 \circ F_1\tau_1F_1^2E_2E_1 \circ F_1E_1\eta_1F_1E_2E_1 \circ F_1\eta_1E_2E_2$
$= E_1E_2F_1\varepsilon_1 \circ E_1\lambda F_1E_1 \circ F_1E_1^2F_1\lambda E_1 \circ F_1E_1^2\tau_1E_2E_1 \circ F_1E_1\eta_1F_1E_2E_1 \circ F_1\eta_1E_2E_2$
$= E_1E_2F_1\varepsilon_1 \circ E_1\lambda F_1E_1 \circ E_1F_1\lambda E_1 \circ E_1\tau_1E_2E_1 \circ \eta_1F_1E_2E_1 \circ \varepsilon_1F_1E_2E_1 \circ F_1\eta_1E_2E_2$
$= E_1E_2F_1\varepsilon_1 \circ E_1\lambda F_1E_1 \circ E_1F_1\lambda E_1 \circ E_1\tau_1E_2E_1 \circ \eta_1F_1E_2E_1$
$= E_1E_2F_1\varepsilon_1 \circ E_1E_2\tau_1E_1 \circ E_1\lambda F_1E_1 \circ E_1F_1\lambda E_1 \circ \eta_1F_1E_2E_1$
$= E_1E_2F_1\varepsilon_1 \circ E_1E_2F_1\varepsilon_1E_1F_1 \circ E_1E_2F_1^2E_1\eta_1 \circ E_1E_2\tau_1E_1 \circ E_1\lambda F_1E_1 \circ E_1F_1\lambda E_1 \circ \eta_1F_1E_2E_1$
$= E_1E_2F_1\varepsilon_1 \circ E_1E_2F_1\varepsilon_1E_1F_1 \circ E_1E_2\tau_1E_1^2F_1 \circ E_1E_2F_1^2E_1\eta_1 \circ E_1\lambda F_1E_1 \circ E_1F_1\lambda E_1 \circ \eta_1F_1E_2E_1$
$= E_1E_2F_1\varepsilon_1 \circ E_1E_2F_1\varepsilon_1E_1F_1 \circ E_1E_2F_1^2\tau_1F_1 \circ E_1E_2F_1^2E_1\eta_1 \circ E_1\lambda F_1E_1 \circ E_1F_1\lambda E_1 \circ \eta_1F_1E_2E_1$
$= \sigma F_1 \circ E_2\rho \circ \lambda E_1.$

We have

$$\begin{aligned}
\rho F_1 \circ F_1\rho \circ \tau_1E_1 &= \varepsilon_1E_1F_1^2 \circ F_1\varepsilon_1E_1^2F_1^2 \circ \tau_1E_1^3F_1^2 \circ F_1^2E_1\tau_1F_1^2 \circ F_1^2E_1^2\eta_1F_1 \circ F_1^2\tau_1F_1 \circ F_1^2E_1\eta_1\\
&= \varepsilon_1E_1F_1^2 \circ F_1\varepsilon_1E_1^2F_1^2 \circ F_1^2\tau_1E_1F_1^2 \circ F_1^2E_1\tau_1F_1^2 \circ F_1^2E_1^2\eta_1F_1 \circ F_1^2\tau_1F_1 \circ F_1^2E_1\eta_1\\
&= \varepsilon_1E_1F_1^2 \circ F_1\varepsilon_1E_1^2F_1^2 \circ F_1^2(\tau_1E_1 \circ E_1\tau_1 \circ \tau_1E_1)F_1^2 \circ F_1^2E_1^2\eta_1F_1 \circ F_1^2E_1\eta_1\\
&= \varepsilon_1E_1F_1^2 \circ F_1\varepsilon_1E_1^2F_1^2 \circ F_1^2(E_1\tau_1 \circ \tau_1E_1 \circ E_1\tau_1)F_1^2 \circ F_1^2E_1^2\eta_1F_1 \circ F_1^2E_1\eta_1\\
&= \varepsilon_1E_1F_1^2 \circ F_1\varepsilon_1E_1^2F_1^2 \circ F_1^2(E_1\tau_1 \circ \tau_1E_1)F_1^2 \circ F_1^2E_1^3\tau_1 \circ F_1^2E_1^2\eta_1F_1 \circ F_1^2E_1\eta_1\\
&= E_1\tau_1 \circ \rho F_1 \circ F_1\rho.
\end{aligned}$$

□

4.4.3. *Relations.* Let $\mathcal{M}$ be the strict monoidal pointed category generated by objects a_l for $1 \leqslant l \leqslant 3$ and maps $\lambda_{lm} : a_la_m \to a_ma_l$ for $l \leqslant m$ with relations $\lambda_{ll}^2 = 0$ and

$$\lambda_{mn}l \circ m\lambda_{ln} \circ \lambda_{lm}n = n\lambda_{lm} \circ \lambda_{ln}m \circ l\lambda_{mn} \text{ for } l \leqslant m \leqslant n.$$

Lemma 4.4.6. *We have a pointed faithful strict monoidal functor*

$$H : \mathcal{M} \to \mathcal{U}^\bullet,\ a_l \mapsto e,\ \lambda_{lm} \mapsto \tau.$$

Given $l_1, \ldots, l_r, m_1, \ldots, m_r \in \{1, 2, 3\}$, the non-zero elements of $H(\mathrm{Hom}_{\mathcal{M}}(a_{l_1} \cdots a_{l_r}, a_{m_1} \cdots a_{m_r})) \subset H_r^\bullet$ are those T_w with $w \in \mathfrak{S}_r$ such that for all $i, j \in \{1, \ldots, r\}$ with $i < j$ and $w(i) > w(j)$, we have $l_i \leqslant l_j$.

Proof. Given the defining relations for $\mathcal{U}^\bullet$, the construction of the lemma does define (uniquely) a monoidal functor H.

Fix $l_1, \ldots, l_n \in \{1, \ldots, 3\}$. Given $i \in \{1, \ldots, n-1\}$ such that $l_i \leqslant l_{i+1}$, we put $\tilde{T}_i = a_{l_1} \cdots a_{l_{i-1}}\lambda_{l_i,l_{i+1}}a_{l_{i+2}} \cdots a_{l_n}$. Note that $\tilde{T}_i\tilde{T}_{i+1}\tilde{T}_i$ is well-defined if and only if $l_i \leqslant l_{i+1} \leqslant l_{i+2}$, hence if and only if $\tilde{T}_{i+1}\tilde{T}_i\tilde{T}_{i+1}$ is well-defined. As a consequence, given $i_1, \ldots, i_r, j_1, \ldots, j_s \in \{1, \ldots, n-1\}$ such that $\tilde{T}_{i_1} \cdots \tilde{T}_{i_r}$ and $\tilde{T}_{j_1} \cdots \tilde{T}_{j_s}$ are well-defined and $T_{i_1} \cdots T_{i_r} = T_{j_1} \cdots T_{j_s}$, then we have $\tilde{T}_{i_1} \cdots \tilde{T}_{i_r} = \tilde{T}_{j_1} \cdots \tilde{T}_{j_s}$. This shows the faithfulness of H.

Consider $i_1, \ldots, i_r$ such that $\tilde{T}_{i_1} \cdots \tilde{T}_{i_r}$ is well-defined and non-zero. Let $w = s_{i_1} \cdots s_{i_r} \in \mathfrak{S}_n$. We show by induction on r that given $(i, j) \in \tilde{L}(w)$, we have $l_i \leqslant l_j$.

Let $w' = s_{i_1} \cdots s_{i_{r-1}}$. Put $d = i_r$ and $w' = ws_d$. Since $T_{i_1} \cdots T_{i_r} \neq 0$, we have $r = \ell(w)$. We have $\tilde{L}(w) = \{(d, d+1)\} \coprod s_d(\tilde{L}(w'))$ by Lemma 3.2.3. We have a well-defined map $\tilde{T}_{i_1} \cdots \tilde{T}_{i_{r-1}}$

from $a_{l_1} \cdots a_{l_{d-1}} a_{l_{d+1}} a_{l_d} a_{l_{d+2}} \cdots a_{l_n}$. It follows by induction that given $(i,j) \in \tilde{L}(w')$, we have $l_{s_d(i)} \leqslant l_{s_d(j)}$. Since $\tilde{L}(w) = \{(d,d+1)\} \coprod s_d(\tilde{L}(w'))$ (Lemma 3.2.3), we deduce that $l_i \leqslant l_j$ for all $(i,j) \in \tilde{L}(w)$.

Consider now $w \in \mathfrak{S}_n$ such that given $(i,j) \in \tilde{L}(w)$, we have $l_i \leqslant l_j$. Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced decomposition of w. We show by induction on r that $\tilde{T}_{i_1} \cdots \tilde{T}_{i_r}$ is well-defined. As before, we define d and w'. By induction on r, the element $\tilde{T}_{i_1} \cdots \tilde{T}_{i_{r-1}}$ gives a well-defined map from $a_{l_1} \cdots a_{l_{d-1}} a_{l_{d+1}} a_{l_d} a_{l_{d+2}} \cdots a_{l_n}$. Since $(d,d+1) \in \tilde{L}(w)$, it follows that $l_d \leqslant l_{d+1}$, hence $\tilde{T}_d$ is a well-defined map from $a_{l_1} \cdots a_{l_n}$. We deduce that $\tilde{T}_{i_1} \cdots \tilde{T}_{i_r}$. This shows that T_w is in the image of H. □

Given $l_1, \ldots, l_r, m_1, \ldots, m_r \in \{1,2,3\}$ and $w \in \mathfrak{S}_r$ satisfying the assumptions of Lemma 4.4.6, we put $\lambda_w = H^{-1}(T_w)$.

We denote by $\mathcal{M}'$ the strict monoidal k-linear category obtained from $k[\mathcal{M}]$ by adding maps $\varepsilon : a_1 a_3 \to 1$ and $\eta : 1 \to a_3 a_1$ and relations

$$a_3\varepsilon \circ \eta a_3 = \mathrm{id},\ \ \varepsilon a_1 \circ a_1\eta = \mathrm{id}$$

$$\lambda_{23} = a_3 a_2 \varepsilon \circ a_3 \lambda_{12} a_3 \circ \eta a_2 a_3,\ \ \lambda_{13} = \varepsilon a_3 a_1 \circ a_1 \lambda_{33} a_1 \circ a_1 a_3 \eta$$

$$\lambda_{11} = \varepsilon a_1^2 \circ a_1 \varepsilon a_3 a_1^2 \circ a_1^2 \lambda_{33} a_1^2 \circ a_1^2 a_3 \eta a_1 \circ a_1^2 \eta.$$

There is a monoidal duality, i.e. a monoidal equivalence $\mathcal{M}'^{\mathrm{opp}} \xrightarrow{\sim} \mathcal{M}'$ given by

$$a_1 \mapsto a_3,\ a_2 \mapsto a_2,\ a_3 \mapsto a_1,\ \lambda_{12} \mapsto \lambda_{23},\ \lambda_{23} \mapsto \lambda_{12},\ \lambda_{13} \mapsto \lambda_{13}$$

$$\lambda_{11} \mapsto \lambda_{33},\ \lambda_{22} \mapsto \lambda_{22},\ \lambda_{33} \mapsto \lambda_{11},\ \varepsilon \mapsto \eta,\ \eta \mapsto \varepsilon.$$

Lemma 4.4.7. *Let $G_1, \ldots, G_n \in \{a_1, a_2, a_3\}$. We have*

$$\lambda_{(1\cdots n+1)} \circ G_1 \cdots G_n \eta = \lambda_{(n+2\cdots 2)} \circ \eta G_1 \cdots G_n : G_1 \cdots G_n \to a_3 G_1 \cdots G_n a_1$$

and

$$\varepsilon G_1 \cdots G_n \circ \lambda_{(2\cdots n+2)} = G_1 \cdots G_n \varepsilon \circ \lambda_{(n+1\cdots 1)} : a_1 G_1 \cdots G_n a_3 \to G_1 \cdots G_n.$$

Proof. We have

$$\begin{aligned}
a_3\lambda_{13} \circ \eta a_3 &= a_3 \varepsilon a_3 a_1 \circ a_3 a_1 \lambda_{33} a_1 \circ a_3 a_1 a_3 \eta \circ \eta a_3 \\
&= a_3 \varepsilon a_3 a_1 \circ \eta a_3^2 a_1 \circ \lambda_{33} a_1 \circ a_3 \eta \\
&= \lambda_{33} a_1 \circ a_3 \eta
\end{aligned}$$

$$\begin{aligned}
\lambda_{13} a_1 \circ a_1 \eta &= \varepsilon a_3 a_1^2 \circ a_1 \lambda_{33} a_1^2 \circ a_1 a_3 \eta a_1 \circ a_1 \eta \\
&= \varepsilon a_3 a_1^2 \circ a_1 a_3^2 \lambda_{11} \circ a_1 a_3 \eta a_1 \circ a_1 \eta \\
&= a_3 \lambda_{11} \circ \varepsilon a_3 a_1^2 \circ a_1 a_3 \eta a_1 \circ a_1 \eta \\
&= a_3 \lambda_{11} \circ \eta a_1 \circ \varepsilon a_1 \circ a_1 \eta \\
&= a_3 \lambda_{11} \circ \eta a_1
\end{aligned}$$

$$\begin{aligned}\lambda_{23}a_1 \circ a_2\eta &= a_3a_2\varepsilon a_1 \circ a_3\lambda_{12}a_3a_1 \circ \eta a_2a_3a_1 \circ a_2\eta \\ &= a_3a_2\varepsilon a_1 \circ a_3a_2a_1\eta \circ a_3\lambda_{12} \circ \eta a_2 \\ &= a_3\lambda_{12} \circ \eta a_2\end{aligned}$$

It follows that the first statement of the lemma holds when $n = 1$. Consider now $n \geqslant 2$. We prove the first statement of the lemma by induction on n. We have

$$\begin{aligned}\lambda_{(n+2\cdots 2)} \circ \eta G_1 \cdots G_n &= \lambda_{(n+2\cdots 3)} \circ (\lambda_{(23)} \circ \eta G_1)G_2 \cdots G_n \\ &= \lambda_{(n+2\cdots 3)} \circ (\lambda_{(12)} \circ G_1\eta)G_2 \cdots G_n \\ &= \lambda_{(12)} \circ G_1(\lambda_{(n+1\cdots 2)} \circ \eta G_2 \cdots G_n) \\ &= \lambda_{(12)} \circ G_1(\lambda_{(1\cdots n)} \circ G_2 \cdots G_n\eta) \\ &= \lambda_{(1\cdots n+1)} \circ G_1 \cdots G_n\eta\end{aligned}$$

The second statement of the lemma follows by applying the duality of $\mathcal{M}'$. □

Lemmas 4.4.5 and 4.4.6 show that there is a k-linear monoidal functor $R : \mathcal{M}' \to \mathcal{W}$

$$a_1 \mapsto F_1,\ a_2 \mapsto E_2,\ a_3 \mapsto E_1,\ \lambda_{12} \mapsto \lambda, \lambda_{23} \mapsto \sigma,\ \lambda_{13} \mapsto \rho,\ \lambda_{11} \mapsto \tau_1,\ \lambda_{22} \mapsto \tau_2,\ \lambda_{33} \mapsto \tau_1$$

$$\eta \mapsto \eta_1,\ \varepsilon \mapsto \varepsilon_1.$$

Given $l_1, \ldots, l_r, m_1, \ldots, m_r \in \{1, 2, 3\}$ and $w \in \mathfrak{S}_r$ satisfying the assumptions of Lemma 4.4.6, we still denote by λ_w the element $R(\lambda_w)$.

Lemma 4.4.7 has the following consequence.

Lemma 4.4.8. *Let $G_1, \ldots, G_n \in \{E_1, E_2, F_1\}$. We have*

$$\lambda_{(1\cdots n+1)} \circ G_1 \cdots G_n\eta_1 = \lambda_{(n+2\cdots 2)} \circ \eta_1 G_1 \cdots G_n : G_1 \cdots G_n \to E_1G_1 \cdots G_nF_1$$

and

$$\varepsilon_1 G_1 \cdots G_n \circ \lambda_{(2\cdots n+2)} = G_1 \cdots G_n\varepsilon_1 \circ \lambda_{(n+1\cdots 1)} : F_1G_1 \cdots G_nE_1 \to G_1 \cdots G_n.$$

4.4.4. 1-*arrows.* Let $(m, \varsigma) \in \Delta_\lambda\mathcal{W}$. Let $\pi = \pi(\varsigma)$ be the composition

$$\pi : E_2(m) \xrightarrow{E_2\eta_1} E_2E_1F_1(m) \xrightarrow{\sigma F_1} E_1E_2F_1(m) \xrightarrow{E_1\varsigma} E_1(m).$$

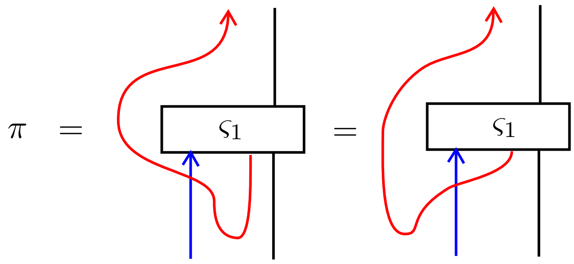
Note that π is also equal to the composition

$$\pi : E_2(m) \xrightarrow{\eta_1 E_2} E_1F_1E_2(m) \xrightarrow{E_1\lambda} E_1E_2F_1(m) \xrightarrow{E_1\varsigma} E_1(m)$$

since $E_1\lambda E_1F_1 \circ \eta_1E_2E_1F_1 \circ E_2\eta_1 = E_1E_2F_1\eta_1 \circ E_1\lambda \circ \eta_1E_2$ and $E_1E_2F_1\eta_1 \circ E_1E_2F_1\varepsilon_1 = \mathrm{id}_{E_1E_2F_1}$.

The pair (m, π) defines an object of $\Delta_\sigma\mathcal{W}$. We obtain a faithful differential functor $\Gamma : \Delta_\lambda\mathcal{W} \to \Delta_\sigma\mathcal{W},\ (m, \varsigma) \mapsto (m, \pi)$.

Remark 4.4.9. The construction of π from ς_1 is illustrated below.



We define now a differential functor $E : \Delta_\lambda \mathcal{W} \to \Delta_\lambda \mathcal{W}$.

Let $(m, \varsigma) \in \Delta_\lambda \mathcal{W}$. Let $m' = E_2(m) \oplus E_1(m)$ where $\pi = \pi(\varsigma_1)$. Given $i \geqslant 1$, we define

$$\varsigma'_i = \begin{pmatrix} E_2\varsigma_i \circ \lambda_{(1\cdots 2i+1)} & \sum_{r=1}^{i} E_2\varsigma_{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ 0 & E_1\varsigma_i \circ \lambda_{(1\cdots 2i+1)} \end{pmatrix} : E_2^i F_1^i(m') \to m'$$

Lemma 4.4.10. *(m', ς') is an object of $\Delta_\lambda \mathcal{W}$.*

Proof. We have

$$\begin{aligned} d((\varsigma'_i)_{11}) &= E_2\varsigma_i \circ d(\tau_2 E_2^{i-1} \circ \cdots \circ E_2^{i-1}\tau_2)F_1^i \circ \lambda_{(i+1\cdots 2i+1)} \\ &= \sum_{r=1}^{i} E_2\varsigma_i \circ \lambda_{(1\cdots r)(r+1\cdots i+1)} \circ \lambda_{(i+1\cdots 2i+1)} \\ &= \sum_{r=1}^{i} E_2\varsigma_i \circ \lambda_{(1\cdots r)(r+1\cdots i+1)} \circ \lambda_{(i+1\cdots 2i+1)} \\ &= \sum_{r=1}^{i} E_2\varsigma_i \circ \lambda_{(1\cdots r)(2i+1\cdots i+r+1)} \circ \lambda_{(i+1\cdots 2i+1)} \\ &= E_2\varsigma_i \circ \lambda_{(1\cdots i)} \end{aligned}$$

$$(\varsigma'_i)_{12} \circ E_2^i F_1^i \pi =$$

$$\begin{aligned} &= \sum_{r=1}^{i} E_2\varsigma_{i-1} \circ E_2^i F_1^{i-1}\varsigma_1 \circ \lambda_{(2i,2i+1)} \circ E_2^i F_1^{i-1}\varepsilon_1 F_1 E_2 \circ E_2^i F_1^i \eta_1 E_2 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= \sum_{r=1}^{i} E_2\varsigma_i \circ \lambda_{(i+1\cdots 2i)} \circ \lambda_{(2i,2i+1)} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= E_2\varsigma_i \circ \lambda_{(1\cdots i)} \\ &= d((\varsigma'_i)_{11}). \end{aligned}$$

$$
\begin{aligned}
d((\varsigma_i')_{22}) &= E_1\varsigma_i \circ \lambda_{(1\cdots i+1)} \circ E_2^i d(\rho F_1^{i-1} \circ F_1\rho F_1^{i-2} \circ \cdots \circ F_1^{i-1}\rho) \\
&= \sum_{r=1}^{i} E_1\varsigma_i \circ \lambda_{(1\cdots i+r)} \circ E_2^i F_1^{r-1}\eta_1 F_1^{i-r} \circ E_2^i F_1^{r-1}\varepsilon_1 F_1^{i-r} \circ \lambda_{(i+r+1\cdots 2i+1)} \\
&= \sum_{r=1}^{i} E_1\varsigma_i \circ \lambda_{(i+r+1\cdots 2)} \circ \eta_1 E_2^i F_1^{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(2i\cdots i+r)} \\
&= \sum_{r=1}^{i} E_1\varsigma_i \circ \lambda_{(i+r+1\cdots i+2)} \circ \lambda_{(i+2\cdots 2)} \circ \eta_1 E_2^i F_1^{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(2i\cdots i+r)} \\
&= \sum_{r=1}^{i} E_1\varsigma_i \circ \lambda_{(2\cdots r+1)} \circ \lambda_{(i+2\cdots 2)} \circ \eta_1 E_2^i F_1^{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(2i\cdots i+r)} \\
&= E_1\varsigma_i \circ \lambda_{(i+2\cdots 2)} \circ \eta_1 E_2^i F_1^{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(2i\cdots i+1)}
\end{aligned}
$$

$$
\begin{aligned}
\pi \circ (\varsigma_i')_{12} &= \sum_{r=1}^{i} E_1\varsigma_1 \circ \lambda_{(12)} \circ E_2\eta_1 \circ E_2\varsigma_{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\
&= \sum_{r=1}^{i} E_1\varsigma_1 \circ \lambda_{(23)} \circ \eta_1 E_2 \circ E_2\varsigma_{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\
&= \sum_{r=1}^{i} E_1\varsigma_1 \circ E_1E_2F_1\varsigma_{i-1} \circ \lambda_{23} \circ \eta_1 E_2^i F_1^{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\
&= \sum_{r=1}^{i} E_1\varsigma_i \circ \lambda_{(i+2\cdots 3)} \circ \lambda_{23} \circ \lambda_{(3\cdots r+2)} \circ \eta_1 E_2^i F_1^{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(2i\cdots i+r)} \\
&= E_1\varsigma_i \circ \lambda_{(i+2\cdots 2)} \circ \eta_1 E_2^i F_1^{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(2i\cdots i+r)} = d((\varsigma_i')_{22}).
\end{aligned}
$$

We have

$$d((\varsigma_i')_{12}) = A + B$$

where

$$
\begin{aligned}
A &= \sum_{1\leqslant s<r\leqslant i} E_2\varsigma_{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots s)(s+1\cdots r)(2i\cdots i+r)} \\
&= \sum_{1\leqslant s<r\leqslant i} E_2\varsigma_{i-1} \circ \lambda_{(s+1\cdots r)} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots s)(2i\cdots i+r)} \\
&= \sum_{1\leqslant s<r\leqslant i} E_2\varsigma_{i-1} \circ \lambda_{(i+r-1\cdots s+i)} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots s)(2i\cdots i+r)} \\
&= \sum_{1\leqslant s<r\leqslant i} E_2\varsigma_{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots s)(2i\cdots i+r)(i+r-1\cdots i+s)}
\end{aligned}
$$

and

$$B = \sum_{\substack{1\leqslant r'\leqslant i \\ 1\leqslant s'\leqslant i-r'}} E_2\varsigma_{i-1} \circ E_2^i F_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots r')(2i\cdots i+r'+s')(i+r'+s'-1\cdots i+r')}$$

So $A = B$ and $d((\varsigma_i')_{12}) = 0$.

We have shown that $d(\varsigma_i') = 0$,

Fix $r \in \{1, \ldots, i\}$. We put $b_r = E_2\varsigma_{i-1} \circ E_2^i F_{i-1}\varepsilon_1 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} : E_2^i F_1^i E_1(m) \to E_2(m)$. Consider $s \in \{1, \ldots, i-1\}$.

If $s > r$, we have

$$\begin{aligned} b_r(T_s \otimes 1) &= E_2\varsigma_{i-1} \circ \lambda_{(s,s+1)} \circ E_2^i F_{i-1} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= E_2\varsigma_{i-1} \circ \lambda_{(i+s-1,i+s)} \circ E_2^i F_{i-1} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= E_2\varsigma_{i-1} \circ E_2^i F_{i-1} \circ \lambda_{(i+s-1,i+s)}\lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= E_2\varsigma_{i-1} \circ E_2^i F_{i-1} \circ \lambda_{(1\cdots r)(2i\cdots i+r)}\lambda_{(i+s,i+s+1)} \\ &= b_r(1 \otimes T_s). \end{aligned}$$

If $s < r-1$, we have

$$\begin{aligned} b_r(1 \otimes T_s) &= E_2\varsigma_{i-1} \circ \lambda_{(i+s,i+s+1)} \circ E_2^i F_{i-1} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= E_2\varsigma_{i-1} \circ \lambda_{(s+1,s+2)} \circ E_2^i F_{i-1} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= E_2\varsigma_{i-1} \circ E_2^i F_{i-1} \circ \lambda_{(s+1,s+2)} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= E_2\varsigma_{i-1} \circ E_2^i F_{i-1} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \circ \lambda_{(s,s+1)} \\ &= b_r(T_s \otimes 1). \end{aligned}$$

We have

$$b_r(T_{r-1} \otimes 1) = E_2\varsigma_{i-1} \circ E_2^i F_{i-1} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \circ \lambda_{(r-1,r)} = 0$$

$$b_r(1 \otimes T_r) = E_2\varsigma_{i-1} \circ E_2^i F_{i-1} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \circ \lambda_{(i+r,i+r+1)} = 0$$

$$b_r(1 \otimes T_{r-1}) = E_2\varsigma_{i-1} \circ E_2^i F_{i-1} \circ \lambda_{(1\cdots r)(2i\cdots i+r-1)} = b_{r-1}(T_{r-1} \otimes 1).$$

We have shown that $(\varsigma_i)_{12}(1 \otimes T_s) = (\varsigma_i)_{12}(T_s \otimes 1)$.

We have

$$\begin{aligned} (\varsigma_i')_{11}(T_s \otimes 1) &= E_2\varsigma_i \circ \lambda_{(s+1,s+2)} \circ \lambda_{(1\cdots 2i+1)} \\ &= E_2\varsigma_i \circ \lambda_{(i+s+1,i+s+2)} \circ \lambda_{(1\cdots 2i+1)} \\ &= E_2\varsigma_i \circ \lambda_{(1\cdots 2i+1)}\lambda_{(i+s,i+s+1)} \\ &= (\varsigma_i')_{11}(1 \otimes T_s). \end{aligned}$$

Similarly,

$$(\varsigma_i')_{22}(T_s \otimes 1) = (\varsigma_i')_{11}(1 \otimes T_s).$$

So $\varsigma_i(1 \otimes T_s) = \varsigma_i(T_s \otimes 1)$.

Let $l \in \{1, 2\}$. We have

$$(\varsigma_{i+j}')_{ll} \circ \mu_{ij} = E_l\varsigma_{i+j} \circ \lambda_w$$

where $w(r) = r$ and $w(i+r) = i+r+j+1$ for $1 \leqslant r \leqslant i$, $w(2i+r) = i+r$ and $w(2i+j+r) = 2i+j+r+1$ for $1 \leqslant r \leqslant j$ and $w(2i+2j+1) = i+j+1$.

We have

$$\begin{aligned}(\varsigma'_i)_{ll} \circ (\varsigma'_j)_{ll} &= E_l\varsigma_i \circ E_l E_2^i F_1^i \varsigma_j \circ \lambda_{(1\cdot 2i+1)} \circ \lambda_{(2i+1\cdots 2i+2j+1)} \\ &= E_l \varsigma_{i+j} \circ \lambda_{w'} \circ \lambda_{(1\cdot 2i+1)} \circ \lambda_{(2i+1\cdots 2i+2j+1)}\end{aligned}$$

where $w'(r) = r$ for $1 \leqslant r \leqslant i+1$, $w'(i+1+r) = i+j+1+r$ for $1 \leqslant r \leqslant i$, $w'(1+2i+r) = 1+i+r$ and $w'(1+2i+j+r) = 1+2i+j+r$ for $1 \leqslant r \leqslant j$.

It follows that $(\varsigma'_{i+j})_{ll} \circ \mu_{ij} = (\varsigma'_i)_{ll} \circ (\varsigma'_j)_{ll}$.

Given $l \leqslant l' \leqslant 1$, we put $b_{l',l} = E_2\varsigma_{l'-1} \circ E_2^{l'} F_{l'-1}\varepsilon_1 \circ \lambda_{(1\cdots l)(2l'\cdots l'+l)} : E_2^{l'} F_1^{l'} E_1(m) \to E_2(m)$. We denote by w_{l_1,l_2} the permutation of $\mathfrak{S}_{l_1+l_2}$ given by $s \mapsto s+l_2$ for $1 \leqslant s \leqslant l_1$ and $s \mapsto s-l_1$ for $l_1+1 \leqslant s \leqslant l_1+l_2$.

Consider $r \in \{1, \ldots, i\}$. We have

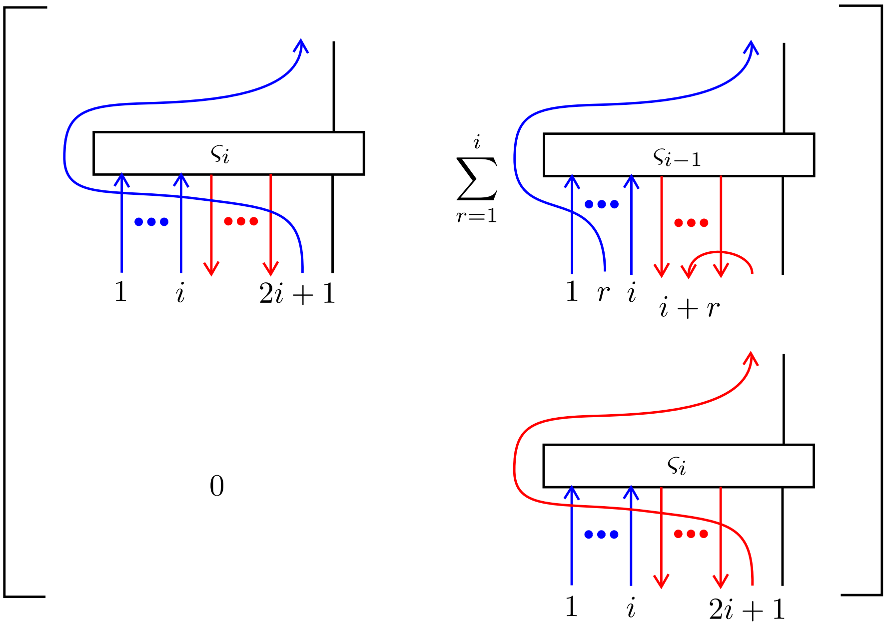
$$\begin{aligned}b_{i,r} \circ (\varsigma'_j)_{22} &= E_2\varsigma_{i-1} \circ E_2^i F_1^{i-1}\varsigma_j \circ E_2^i F_1^{i-1}\varepsilon_1 E_2^j F_1^j \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \circ \lambda_{(2i+1\cdots 2i+2j+1)} \\ &= E_2\varsigma_{i+j-1} \circ \lambda_{w_{i-1,j}} \circ E_2^i F_1^{i-1}\varepsilon_1 E_2^j F_1^j \circ \lambda_{(2i+1\cdots 2i+2j+1)} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= E_2\varsigma_{i+j-1} \circ E_2^i\lambda_{w_{i-1,j}} F_1^j \circ E_2^i F_1^{i-1} E_2^j F_1^{j-1}\varepsilon_1 \circ \lambda_{(2i+2j\cdots 2i)} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= E_2\varsigma_{i+j-1} \circ E_2^i\lambda_{w_{i-1,j}} F_1^j \circ E_2^i F_1^{i-1} E_2^j F_1^{j-1}\varepsilon_1 \circ \lambda_{(2i+2j\cdots i+r)} \\ &= E_2\varsigma_{i+j-1} \circ E_2^{i+j} F_1^{i+j-1}\varepsilon_1 \circ E_2^i \lambda_{w_{i-1,j}} F_1^{j+1} E_1 \circ \lambda_{(2i+2j\cdots i+r)} \\ &= E_2\varsigma_{i+j-1} \circ E_2^{i+j} F_1^{i+j-1}\varepsilon_1 \circ \lambda_{(2i+2j\cdots i+j+r)} \circ E_2^i\lambda_{w_{i,j}} F_1^j E_1 \\ &= b_{i+j,r} \circ \mu_{i,j}.\end{aligned}$$

Consider $r \in \{1, \ldots, j\}$. We have

$$\begin{aligned}(\varsigma'_i)_{11} \circ b_{j,r} &= E_2\varsigma_i \circ E_2^{i+1} F_1^i \varsigma_{j-1} \circ \lambda_{(1\cdots 2i+1)} \circ E_2^i F_1^i E_2^j F_1^{j-1}\varepsilon_1 \circ \lambda_{(2i+1\cdots 2i+r)(2i+2j\cdots 2i+j+r)} \\ &= E_2\varsigma_{i+j-1} \circ E_2^{i+1}\lambda_{w_{i,j-1}} F_1^{j-1} \circ \lambda_{(1\cdots 2i+1)} \circ E_2^i F_1^i E_2^j F_1^{j-1}\varepsilon_1 \circ \lambda_{(2i+1\cdots 2i+r)(2i+2j\cdots 2i+j+r)} \\ &= E_2\varsigma_{i+j-1} \circ E_2^{i+j} F_1^{i+j-1}\varepsilon_1 \circ E_2^{i+1}\lambda_{w_{i,j-1}} F_1^j E_1 \circ \lambda_{(1\cdots 2i+1)} \circ \lambda_{(2i+1\cdots 2i+r)(2i+2j\cdots 2i+j+r)} \\ &= E_2\varsigma_{i+j-1} \circ E_2^{i+j} F_1^{i+j-1}\varepsilon_1 \circ \lambda_{(1\cdots i+r)(2i+2j\cdots 2i+j+r)} \circ E_2^i \lambda_{w_{i,j}} F_1^j E_1 \\ &= b_{i+j,j+r} \circ \mu_{i,j}.\end{aligned}$$

It follows that for all $i, j \geqslant 1$, we have $\varsigma_i \circ E_2^i F_1^i \varsigma_j = \varsigma_{i+j} \circ \mu_{i,j}$. □

Remark 4.4.11. The graphical description of ς' is the following:



Given $f \in \mathrm{Hom}_{\Delta_\lambda\mathcal{W}}((m,\varsigma),(\tilde{m},\tilde{\varsigma}))$, we put $E(f) = \begin{pmatrix} E_2(f) & 0 \\ 0 & E_1(f) \end{pmatrix}$.

Lemma 4.4.12. *We have $E(f) \in \mathrm{Hom}_{\Delta_\lambda\mathcal{W}}(E(m,\varsigma), E(\tilde{m},\tilde{\varsigma}))$. The construction makes E into a differential endofunctor of $\Delta_\lambda\mathcal{W}$.*

Proof. The lemma follows from the commutativity of the following diagram:

$$\begin{tikzcd}
E_2^iF_1^iE_2(m) \arrow[d, "E_2^iF_1^iE_2f"'] \arrow[rrrr, bend left=20, "E_2\varsigma_i\circ\lambda_{(1\cdots 2i+1)}"] & \oplus & E_2^iF_1^iE_1(m) \arrow[d, "E_2^iF_1^iE_1f"] \arrow[rr, "\sum_{r=1}^{i} E_2\varsigma_{i-1}\circ E_2^iF_1^{i-1}\varepsilon_1\circ\lambda_{(1\cdots r)(2i\cdots i+r)}"] \arrow[rrrr, bend left=20, "E_1\varsigma_i\circ\lambda_{(1\cdots 2i+1)}"] & & E_2(m) \arrow[d, "E_2f"'] & \oplus & E_1(m) \arrow[d, "E_1f"] \\
E_2^iF_1^iE_2(\tilde{m}) \arrow[rrrr, bend right=20, "E_2\tilde{\varsigma}_i\circ\lambda_{(1\cdots 2i+1)}"'] & \oplus & E_2^iF_1^iE_1(\tilde{m}) \arrow[rr, "\sum_{r=1}^{i} E_2\tilde{\varsigma}_{i-1}\circ E_2^iF_1^{i-1}\varepsilon_1\circ\lambda_{(1\cdots r)(2i\cdots i+r)}"'] \arrow[rrrr, bend right=20, "E_1\tilde{\varsigma}_i\circ\lambda_{(1\cdots 2i+1)}"'] & & E_2(\tilde{m}) & \oplus & E_1(\tilde{m})
\end{tikzcd}$$

∎

Lemma 4.4.13. *We have $E \circ \Gamma = \Gamma \circ E$.*

Proof. Let $(m,\varsigma) \in \Delta_\lambda\mathcal{W}$. We have $E(m,\pi) = (m',\pi')$ where $m' = \mathrm{cone}(\pi)$ and π' is given in §4.3.2. We have $\Gamma \circ E(m,\varsigma) = (m',\pi'')$ where

$$\pi''_{12} = E_1E_2\varepsilon_1 \circ E_1\lambda E_1 \circ \eta_1E_2E_1 = \sigma,\ \pi''_{21} = 0$$

$$\begin{aligned}\pi''_{11} &= E_1E_2\varsigma_1 \circ E_1E_2\lambda \circ E_1\lambda E_2 \circ E_1F_1\tau_2 \circ \eta_1E_2^2\\ &= E_1E_2\varsigma_1 \circ E_1E_2\lambda \circ E_1\lambda E_2 \circ \eta_1E_2^2 \circ \tau_2\\ &= E_1E_2\varsigma_1 \circ E_1E_2\lambda \circ \sigma F_1E_2 \circ E_2\eta_1E_2 \circ \tau_2\\ &= \sigma \circ E_2E_1\varsigma_1 \circ E_2E_1\lambda \circ E_2\eta_1E_2 \circ \tau_2\\ &= \pi'_{11}\end{aligned}$$

$$\begin{aligned}\pi''_{22} &= E_1^2\varsigma_1 \circ E_1^2\lambda \circ E_1\rho E_2 \circ E_1F_1\sigma \circ \eta_1E_2E_1\\ &= E_1^2\varsigma_1 \circ E_1^2\lambda \circ E_1\rho E_2 \circ \eta_1E_1E_2 \circ \sigma\\ &= E_1^2\varsigma_1 \circ E_1^2\lambda \circ \tau_1F_1E_2 \circ E_1\eta_1E_2 \circ \sigma\\ &= \tau_1 \circ E_1^2\varsigma_1 \circ E_1^2\lambda \circ E_1\eta_1E_2 \circ \sigma\\ &= \pi'_{22}\end{aligned}$$

It follows that $\pi'' = \pi'$. □

4.4.5. *2-arrows.* We assume in §4.4.5 that σ is invertible.

Given $(m, \varsigma) \in \Delta_\lambda\mathcal{W}$, write $E^2(m, \varsigma) = (m'', \varsigma'')$. The formula (4.3.2) defines an endomorphism τ of m''.

Lemma 4.4.14. *Given $i \geqslant 1$, we have $\tau \circ \varsigma''_i = \varsigma''_i \circ E_2^iF_1^i\tau$.*

Proof. Let $A = \tau \circ \varsigma''_i$ and $B = \varsigma''_i \circ E_2^iF_1^i\tau$.

We have

$$a_{21} = a_{22} = a_{31} = a_{32} = a_{33} = a_{34} = a_{41} = a_{42} = a_{43} = 0$$

$$\begin{aligned}a_{11} &= \lambda_{(12)} \circ E_2^2\varsigma_i \circ \lambda_{(2\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)}\\ &= E_2^2\varsigma_i \circ \lambda_{(1\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)}\end{aligned}$$

$$\begin{aligned}a_{12} &= \sum_{r=1}^{i} \lambda_{(12)} \circ E_2^2\varsigma_{i-1} \circ E_2^{i+1}F_1^{i-1}\varepsilon_1 \circ \lambda_{(2\cdots r+1)} \circ \lambda_{(2i+1\cdots i+r+1)} \circ \lambda_{(1\cdots 2i+1)}\\ &= \sum_{r=1}^{i} E_2^2\varsigma_{i-1} \circ E_2^{i+1}F_1^{i-1}\varepsilon_1 \circ \lambda_{(12)} \circ \lambda_{(1\cdots i+1)} \circ \lambda_{(1\cdots r)} \circ \lambda_{(2i+1\cdots i+r+1)} \circ \lambda_{(i+1\cdots 2i+1)}\\ &= 0\end{aligned}$$

$$\begin{aligned}
a_{13} &= \sum_{r=1}^{i} \lambda_{(12)} \circ E_2^2 \varsigma_{i-1} \circ \lambda_{(2\cdots 2i)} \circ E_2^i F_1^{i-1} \varepsilon_1 E_2 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\
&= \sum_{r=1}^{i} E_2^2 \varsigma_{i-1} \circ \lambda_{(1\cdots 2i)} \circ E_2^i F_1^{i-1} \varepsilon_1 E_2 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \circ \lambda_{(2i+1,2i+2)} \circ E_2^i F_1^{i-1} \sigma^{-1} \\
&= \sum_{r=1}^{i} E_2^2 \varsigma_{i-1} \circ \lambda_{(1\cdots 2i)} \circ E_2^i F_1^{i-1} E_2 \varepsilon_1 \circ \lambda_{(2i,2i+1)} \circ \lambda_{(2i\cdots i+r)} \circ \lambda_{(1\cdots r)} \circ E_2^i F_1^{i-1} \sigma^{-1} \\
&= \sum_{r=1}^{i} E_2^2 \varsigma_{i-1} \circ E_2^{i+1} F_1^{i-1} \varepsilon_1 \circ \lambda_{(1\cdots 2i+1)} \circ \lambda_{(2i\cdots i+r)} \circ \lambda_{(1\cdots r)} \circ E_2^i F_1^{i-1} \sigma^{-1}
\end{aligned}$$

$$\begin{aligned}
a_{14} &= \sum_{\substack{1\leqslant r\leqslant i\\ 1\leqslant s<i}} \lambda_{(12)} \circ E_2^2 \varsigma_{i-2} \circ E_2^i F_1^{i-2} \varepsilon_1 \circ \lambda_{(2\cdots s+1)(2i-1\cdots i+s)} \circ E_2^i F_1^{i-1} \varepsilon_1 E_1 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\
&= \sum_{\substack{1\leqslant r\leqslant i\\ 1\leqslant s<i}} E_2^2 \varsigma_{i-2} \circ E_2^i F_1^{i-2} \varepsilon_1 \circ E_2^i F_1^{i-1} \varepsilon_1 E_1 \circ \lambda_{(1\cdots s+1)} \circ \lambda_{(1\cdots r)} \circ \lambda_{(2i-1\cdots i+s)} \circ \lambda_{(2i\cdots i+r)} \\
&= \sum_{1\leqslant r\leqslant s<i} E_2^2 \varsigma_{i-2} \circ E_2^i F_1^{i-2} \varepsilon_1 \circ E_2^i F_1^{i-1} \varepsilon_1 E_1 \circ \lambda_{(2\cdots r+1)} \circ \lambda_{(1\cdots s+1)} \circ \lambda_{(2i\cdots i+r)} \circ \lambda_{(2i\cdots i+s+1)}
\end{aligned}$$

$$\begin{aligned}
a_{23} &= \sigma^{-1} \circ E_1 E_2 \varsigma_i \circ \lambda_{(2\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)} \\
&= \sigma^{-1} \circ E_1 E_2 \varsigma_i \circ \lambda_{(2\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)} \circ \lambda_{(2i+1,2i+2)} \circ E_2^i F_1^i \sigma^{-1} \\
&= \sigma^{-1} \circ E_1 E_2 \varsigma_i \circ \lambda_{(12)} \circ \lambda_{(2\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)} \circ E_2^i F_1^i \sigma^{-1} \\
&= E_1 E_2 \varsigma_i \circ \lambda_{(2\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)} \circ E_2^i F_1^i \sigma^{-1}
\end{aligned}$$

$$\begin{aligned}
a_{24} &= \sum_{r=1}^{i} \sigma^{-1} \circ E_1 E_2 \varsigma_{i-1} \circ E_1 E_2^i F_1^{i-1} \varepsilon_1 \circ \lambda_{(2\cdots r+1)} \circ \lambda_{(2i+1\cdots i+r+1)} \circ \lambda_{(1\cdots 2i+1)} \\
&= \sum_{r=1}^{i} E_2 E_1 \varsigma_{i-1} \circ E_2 E_1 E_2^{i-1} F_1^{i-1} \varepsilon_1 \circ \sigma^{-1} E_2^{i-1} F_1^i E_1 \circ \lambda_{(1\cdots 2i+1)} \circ \lambda_{(1\cdots r)} \circ \lambda_{(2i\cdots i+r)} \\
&= \sum_{r=1}^{i} E_2 E_1 \varsigma_{i-1} \circ E_2 E_1 E_2^{i-1} F_1^{i-1} \varepsilon_1 \circ \lambda_{(2\cdots 2i+1)} \circ \lambda_{(1\cdots r)} \circ \lambda_{(2i\cdots i+r)}
\end{aligned}$$

$$\begin{aligned}
a_{44} &= \lambda_{(12)} \circ E_1^2 \varsigma_i \circ \lambda_{(2\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)} \\
&= E_1^2 \varsigma_i \circ \lambda_{(1\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)}
\end{aligned}$$

We have

$$b_{21} = b_{12} = b_{22} = b_{31} = b_{32} = b_{33} = b_{41} = b_{42} = b_{43} = 0$$

$$\begin{aligned} b_{11} &= E_2^2\varsigma_i \circ \lambda_{(2\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)} \circ \lambda_{(2i+1,2i+2)} \\ &= E_2^2\varsigma_i \circ \lambda_{(1\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)} \end{aligned}$$

$$\begin{aligned} b_{13} &= \sum_{r=1}^{i} E_2^2\varsigma_{i-1} \circ E_2^{i+1}F_1^{i-1}\varepsilon_1 \circ \lambda_{(2\cdots r+1)} \circ \lambda_{(2i+1\cdots i+r+1)} \circ \lambda_{(1\cdots 2i+1)} \circ E_2^iF_1^i\sigma^{-1} \\ &= \sum_{r=1}^{i} E_2^2\varsigma_{i-1} \circ E_2^{i+1}F_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots 2i+1)} \circ \lambda_{(2i\cdots i+r)} \circ \lambda_{(1\cdots r)} \circ E_2^iF_1^i\sigma^{-1} \end{aligned}$$

$$\begin{aligned} b_{14} &= \sum_{\substack{1\leqslant r\leqslant i \\ 1\leqslant s<i}} E_2^2\varsigma_{i-2} \circ E_2^iF_1^{i-2}\varepsilon_1 \circ \lambda_{(2\cdots s+1)(2i-1\cdots i+s)} \circ E_2^iF_1^{i-1}\varepsilon_1E_1 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \circ \lambda_{(2i+1,2i+2)} \\ &= \sum_{\substack{1\leqslant r\leqslant i \\ 1\leqslant s<i}} E_2^2\varsigma_{i-2} \circ E_2^iF_1^{i-2}\varepsilon_1 \circ E_2^iF_1^{i-1}\varepsilon_1E_1 \circ \lambda_{(2i+1,2i+2)} \circ \lambda_{(2\cdots s+1)(2i-1\cdots i+s)} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= \sum_{\substack{1\leqslant r\leqslant i \\ 1\leqslant s<i}} E_2^2\varsigma_{i-2} \circ E_2^iF_1^{i-2}\varepsilon_1 \circ E_2^iF_1^{i-1}\varepsilon_1E_1 \circ \lambda_{(2i-1,2i)} \circ \lambda_{(2\cdots s+1)(2i-1\cdots i+s)} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= \sum_{1\leqslant s<r\leqslant i} E_2^2\varsigma_{i-2} \circ E_2^iF_1^{i-2}\varepsilon_1 \circ E_2^iF_1^{i-1}\varepsilon_1E_1 \circ \lambda_{(2\cdots s+1)} \circ \lambda_{(1\cdots r)} \circ \lambda_{(2i\cdots i+s)} \circ \lambda_{(2i\cdots i+r)} \\ &= \sum_{1\leqslant r'\leqslant s'\leqslant i} E_2^2\varsigma_{i-2} \circ E_2^iF_1^{i-2}\varepsilon_1 \circ E_2^iF_1^{i-1}\varepsilon_1E_1 \circ \lambda_{(2\cdots r'+1)} \circ \lambda_{(1\cdots s'+1)} \circ \lambda_{(2i\cdots i+r')} \circ \lambda_{(2i\cdots i+s'+1)} \end{aligned}$$

$$b_{23} = E_2E_1\varsigma_i \circ \lambda_{(2\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)} \circ E_2^iF_1^i\sigma^{-1}$$

$$\begin{aligned} b_{24} &= \sum_{r=1}^{i} E_2E_1\varsigma_{i-1} \circ \lambda_{(2\cdots 2i)} \circ E_2^iF_1^{i-1}\varepsilon_1E_1 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \circ \lambda_{(2i+1,2i+2)} \\ &= \sum_{r=1}^{i} E_2E_1\varsigma_{i-1} \circ \lambda_{(2\cdots 2i)} \circ E_2^iF_1^{i-1}E_1\varepsilon_1 \circ \lambda_{(2i,2i+1)} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ &= \sum_{r=1}^{i} E_2E_1\varsigma_{i-1} \circ E_2^iF_1^{i-1}E_1\varepsilon_1 \circ \lambda_{(2\cdots 2i+1)} \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \end{aligned}$$

$$\begin{aligned}
b_{34} &= \sum_{r=1}^{i} E_1E_2\varsigma_{i-1} \circ E_1E_2^iF_1^{i-1}\varepsilon_1 \circ \lambda_{(2\cdots r+1)} \circ \lambda_{(2i+1\cdots i+r+1)} \circ \lambda_{(1\cdots 2i+1)} \circ \lambda_{(2i+1,2i+2)} \\
&= \sum_{r=1}^{i} E_1E_2\varsigma_{i-1} \circ E_1E_2^iF_1^{i-1}\varepsilon_1 \circ \lambda_{(1\cdots 2i+2)} \circ \lambda_{(1\cdots r)} \circ \lambda_{(2i\cdots i+r)} \\
&= \sum_{r=1}^{i} E_1E_2\varsigma_{i-1} \circ \lambda_{(1\cdots 2i)} \circ E_2^iF_1^{i-1}E_1\varepsilon_1 \circ \lambda_{(2i,2i+1)} \circ \lambda_{(2i+1,2i+2)} \circ \lambda_{(1\cdots r)} \circ \lambda_{(2i\cdots i+r)} \\
&= \sum_{r=1}^{i} E_1E_2\varsigma_{i-1} \circ \lambda_{(1\cdots 2i)} \circ E_2^iF_1^{i-1}\varepsilon_1E_1 \circ \lambda_{(2i+1,2i+2)} \circ \lambda_{(2i+1,2i+2)} \circ \lambda_{(1\cdots r)} \circ \lambda_{(2i\cdots i+r)} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
b_{44} &= E_1^2\varsigma_i \circ \lambda_{(2\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)} \circ \lambda_{(2i+1,2i+2)} \\
&= E_1^2\varsigma_i \circ \lambda_{(1\cdots 2i+2)} \circ \lambda_{(1\cdots 2i+1)}
\end{aligned}$$

We deduce that $A = B$ and the lemma follows. ☐

Lemma 4.4.14 shows that τ defines an endomorphism of $E^2(m,\varsigma)$ for all $(m,\varsigma) \in \Delta_\lambda\mathcal{W}$. The functor Γ is faithful, $\Gamma E^2 = E^2\Gamma$ (Lemma 4.4.13) and τ commutes with Γ. It follows that τ is functorial.

Theorem 4.3.8 has the following consequence.

Theorem 4.4.15. *The data $(\Delta_\lambda\mathcal{W}, E, \tau)$ is an idempotent-complete strongly pretriangulated 2-representation.*

The following proposition is a consequence of Lemma 4.4.13 and the construction of τ.

Proposition 4.4.16. *The functor $\Gamma : \Delta_\lambda\mathcal{W} \to \Delta_\sigma\mathcal{W}$ induces a morphism of 2-representations.*

4.5. **Tensor product and internal Hom.** Let us give two applications of the construction of §4.3. Let $(\mathcal{V}_1, E_1, \tau_1)$ and $(\mathcal{V}_2, E_2, \tau_2)$ be idempotent-complete strongly pretriangulated 2-representations.

We view $\mathcal{V}_1 \otimes \mathcal{V}_2$ as endowed with two strictly commuting actions of $\mathcal{U}$ given by $(E_1 \otimes 1, \tau_1 \otimes 1)$ and $(1 \otimes E_2, 1 \otimes \tau_2)$: the isomorphism $\sigma : (1 \otimes E_2) \circ (E_1 \otimes 1) \xrightarrow{\sim} (E_1 \otimes 1) \circ (1 \otimes E_2)$ is the identity.

We define the *tensor product 2-representation*

$$\mathcal{V}_1 \bigotimes \mathcal{V}_2 = \Delta_\sigma(\mathcal{V}_1 \otimes \mathcal{V}_2).$$

Given $(\Phi_i, \varphi_i) : \mathcal{V}_i \to \mathcal{V}_i'$ a morphism of 2-representations for $i \in \{1,2\}$, Proposition 4.3.9 provides a morphism of 2-representations $\mathcal{V}_1 \bigotimes \mathcal{V}_2 \to \mathcal{V}_1' \bigotimes \mathcal{V}_2'$.

We have constructed a monoidal structure on the differential 2-category of idempotent-complete strongly pretriangulated 2-representations.

Consider now $\mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_2)$. It is endowed with two strictly commuting structures of 2-representations: the first one is given by $((\Phi \mapsto \Phi \circ E_1), \Phi\tau_1)$ and the second one by $((\Phi \mapsto E_2 \circ \Phi), \tau_2\Phi)$. The isomorphism σ is the identity.

We define the *internal* Hom *2-representation*

$$\mathcal{H}om(\mathcal{V}_1,\mathcal{V}_2)=\Delta\operatorname{Hom}(\mathcal{V}_1,\mathcal{V}_2).$$

The category $\mathcal{H}om(\mathcal{V}_1,\mathcal{V}_2)$ has objects pairs (Φ,π) where $\Phi:\mathcal{V}_1\to\overline{\mathcal{V}_2}^i$ is a differential functor and $\pi:E_2\Phi\to\Phi E_1$ is a closed natural transformation of functors such that

$$\tau_1\Phi\circ\pi E_1\circ E_2\pi=\pi E_1\circ E_2\pi\circ\tau_2\Phi:E_2^2\Phi\to\Phi E_1^2.$$

Note that $\operatorname{Hom}_{\mathcal{U}}(\mathcal{V}_1,\mathcal{V}_2)$ is the full subcategory of $\mathcal{H}om(\mathcal{V}_1,\mathcal{V}_2)$ with objects pairs (Φ,π) where Φ takes values in $\mathcal{V}_2$ and π is invertible.

Given $(\Phi_1,\varphi_1):\mathcal{V}_1'\to\mathcal{V}_1$ and $(\Phi_2,\varphi_2):\mathcal{V}_2\to\mathcal{V}_2'$ two morphisms of 2-representations, Proposition 4.3.9 provides a morphism of 2-representations $\mathcal{H}om(\mathcal{V}_1,\mathcal{V}_2)\to\mathcal{H}om(\mathcal{V}_1',\mathcal{V}_2')$.

5. Bimodule 2-representations

5.1. **Differential algebras.**

5.1.1. *2-representations.* Let A be a differential algebra.

Definition 5.1.1. *A 2-representation on A is the data of a differential (A,A)-bimodule E and of an endomorphism τ of the (A,A)-bimodule $E\otimes_A E$ such that*

$$\tau^2=0,\ d(\tau)=\mathrm{id}\ \textit{and}\ (E\otimes\tau)\circ(\tau\otimes E)\circ(E\otimes\tau)=(\tau\otimes E)\circ(E\otimes\tau)\circ(\tau\otimes E).$$

We say that the 2-representation is right finite *if E is finitely generated and projective as a (non-differential) A^{opp}-module.*

Consider a 2-representation on A. Note that $E\otimes_A-$ is a differential endofunctor of A-diff, and τ defines an endomorphism of $(E\otimes_A-)^2$. This gives a structure of 2-representation on A-diff. It restricts to a 2-representation on $(\bar{A})^i$ if E is strictly perfect as a differential A-module.

Note that there is a morphism of differential algebras

$$H_n\to\operatorname{End}_{A\otimes A^{\mathrm{opp}}}(E^n),\ T_i\mapsto E^{n-i-1}\otimes\tau\otimes E^{i-1}.$$

Let A' be another differential algebra with a 2-representation (E',τ'). We define a *morphism of 2-representations* from (A,E,τ) to (A',E',τ') to be an (A',A)-bimodule P together with a closed isomorphism of (A',A)-bimodules $\varphi:P\otimes_A E\xrightarrow{\sim}E'\otimes_{A'}P$ such that

$$\tau'P\circ E'\varphi\circ\varphi E=E'\varphi\circ\varphi E\circ P\tau:PE^2\to E'^2P. \tag{5.1.1}$$

Note that such a pair (P,φ) gives rise to a morphism of 2-representations $(P\otimes_A-,\varphi):(A\text{-diff},E\otimes_A-,\tau)\to(A'\text{-diff},E'\otimes_{A'}-,\tau')$.

We obtain a differential 2-category of 2-representations on differential algebras.

The *opposite* *2-representation* is the data (A',E',τ') where $A'=A^{\mathrm{opp}}$, $E'=E$ and $\tau'=\tau$. Note that (A,E,τ) coincides with its double dual.

Assume now the 2-representation is right finite. We have two morphisms of (A,A)-bimodules $\eta:A\to E\otimes_A E^\vee$ and $\varepsilon:E^\vee\otimes_A E\to A$ (unit and counit of adjunction). We have a morphism of (A,A)-bimodules $\rho:E^\vee E\to EE^\vee$ defined as the composition

$$\rho:E^\vee E\xrightarrow{\bullet\eta}E^\vee EEE^\vee\xrightarrow{E^\vee\tau E^\vee}E^\vee EEE^\vee\xrightarrow{\varepsilon\bullet}EE^\vee.$$

There is a canonical isomorphism of differential algebras $\operatorname{End}(E^2)^{\mathrm{opp}}\xrightarrow{\sim}\operatorname{End}((E^\vee)^2)$ and we still denote by τ the endomorphism of $(E^\vee)^2$ corresponding to τ.

We define the *left dual 2-representation* on A with the bimodule $E^\vee$ and the endomorphism τ.

5.2. **Lax cocenter.** Let B be a differential algebra. A *lax bi-2-representation* on B is the data of

- differential (B,B)-bimodules $E_{i,j}$ for $i,j \geqslant 0$
- morphisms of differential algebras $H_i \otimes H_j \to \mathrm{End}(E_{i,j})$
- morphisms $\mu_{(i,j),(i',j')} : E_{i,j}E_{i',j'} \to E_{i+i',j+j'}$ satisfying properties (1) and (2) of §4.2.1.

Consider a lax bi-2-representation E. Note that the functors $(E_{i,j} \otimes_B -)$ provide a structure of lax bi-2-representation on B-diff.

We define the differential algebra $A = \Delta_E(B)$ as the quotient of the tensor algebra $T_B(E_{0,1}E_{1,0})$ by the two-sided ideal generated by $\bigoplus_{i\geqslant 0} K_i$, where K_i is the kernel of the composition

$$(E_{0,1}E_{1,0})^i \xrightarrow{\text{can}} E_{i,i} \xrightarrow{\text{can}} E_{i,i}/((T_r \otimes 1)x - (1 \otimes T_r)x)_{x\in E_{i,i},\ 1\leqslant r<i}.$$

We have $A^0 = B$ and A is generated by A^0 and $A^1 = (E_{0,1}E_{1,0})/K_1$ as an algebra.

Let (M,ς) an object of $\Delta_{E\otimes_B -}(B\text{-diff})$. The action of $T_B(E_{0,1}E_{1,0})$ on M vanishes on K_i for all i, hence defines an action of A on M. This gives a fully faithful differential functor $\Delta_{E\otimes_B -}(B\text{-diff}) \to (\Delta_E(B))\text{-diff}$. If the canonical injective morphism of differential (B,B)-bimodules

$$(E_{0,1}E_{1,0})^i/K_i \to E_{i,i}/((T_r \otimes 1)x - (1 \otimes T_r)x)_{x\in E_{i,i},\ 1\leqslant r<i} \tag{5.2.1}$$

is a split injection for all $i \geqslant 1$, then the functor above is an isomorphism

$$\Delta_{E\otimes_B -}(B\text{-diff}) \xrightarrow{\sim} (\Delta_E(B))\text{-diff}\,.$$

5.3. **Diagonal action.**

5.3.1. *Algebra.* Let B be a differential algebra endowed with two 2-representations (F_1,τ_1) and (E_2,τ_2) together with a closed morphism $\lambda : F_1E_2 \to E_2F_1$ such that the diagrams (4.2.1) commute.

We define the algebra $A = \Delta'_\lambda(B)$ as the quotient of the tensor algebra $T_B(F_1E_2)$ by the by the image of the composition

$$F_1^2E_2^2 \xrightarrow{\tau_1E_2^2 - F_1^2\tau_2} F_1^2E_2^2 \xrightarrow{F_1\lambda E_2} (F_1E_2)^2.$$

We have $A^0 = B$ and $A^1 = F_1E_2$.

Let B' be a differential algebra endowed with two 2-representations (F'_1,τ'_1) and (E'_2,τ'_2) together with a closed morphism $\lambda' : F'_1E'_2 \to E'_2F'_1$ such that the analogs of the diagrams (4.2.1) commute. Let $A' = \Delta'_{\lambda'}(B')$. Let P be a (B',B)-bimodule and $\varphi_1 : PF_1 \xrightarrow{\sim} F'_1P$ and $\varphi_2 : PE_2 \xrightarrow{\sim} E'_2P$ be two closed isomorphisms of bimodules such that (P,φ_1) and (P,φ_2) are morphisms of 2-representations and such that

$$\lambda'P \circ F'_1\varphi_2 \circ \varphi_1E_2 = E'_2\varphi_1 \circ \varphi_2F_1 \circ P\lambda : PF_1E_2 \to E'_2F'_1P.$$

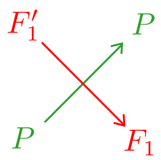
The isomorphism $F'_1\varphi_2 \circ \varphi_1E_2 : PF_1E_2 \xrightarrow{\sim} F'_1E'_2P$ induces an isomorphism of (B',B)-bimodules $f : P \otimes_B T_B(F_1E_2) \xrightarrow{\sim} T_{B'}(F'_1E'_2) \otimes_{B'} P$. This isomorphism f endows the right

$T_B(F_1E_2)$-module $P \otimes_B T_B(F_1E_2)$ with a commuting left action of $T_{B'}(F_1'E_2')$. The isomorphism f induces an isomorphism

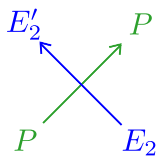
$$P \otimes_B T_B(F_1E_2) \otimes_{T_B(F_1E_2)} A \xrightarrow{\sim} A' \otimes_{T_{B'}(F_1'E_2')} T_{B'}(F_1'E_2') \otimes_{B'} P.$$

So, we obtain a structure of (A', A)-bimodule on $P \otimes_B A$.

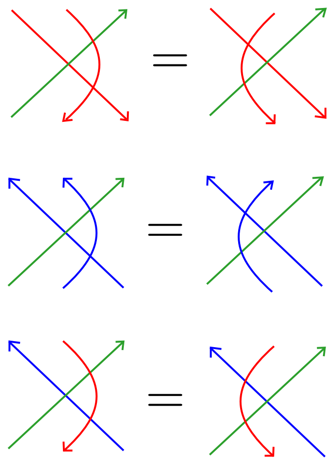
Remark 5.3.1. The data of φ_1 and φ_2 and the relations they are required to satisfy are described graphically as:

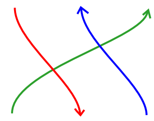


φ_1



φ_2





$F_1'\varphi_2 \circ \varphi_1 E_2$

5.3.2. *Left dual.* Let B be a differential algebra endowed with two 2-representations (E_1, τ_1) and (E_2, τ_2), the first of which is right finite.

We consider the data of $\sigma \in Z\operatorname{Hom}(E_2E_1, E_1E_2)$ such that the diagrams (4.3.1) commute.

We define

$$\lambda : E_1^\vee E_2 \xrightarrow{\bullet\eta_1} E_1^\vee E_2 E_1 E_1^\vee \xrightarrow{E_1^\vee \sigma E_1^\vee} E_1^\vee E_1 E_2 E_1^\vee \xrightarrow{\varepsilon_1 \bullet} E_2 E_1^\vee. \tag{5.3.1}$$

Let $A = \Delta_\sigma(B) = \Delta'_\lambda(B)$. This is the graded quotient of the tensor algebra $T_B(E_1^\vee E_2)$ by the ideal generated by the image of the composition

$$(E_1^\vee)^2 E_2^2 \xrightarrow{\tau_1 E_2^2 - (E_1^\vee)^2 \tau_2} (E_1^\vee)^2 E_2^2 \xrightarrow{E_1^\vee \lambda E_2} (E_1^\vee E_2)^2.$$

The algebra A is generated by $A^0 = B$ and $A^1 = E_1^\vee E_2$.

Let L be a differential B-module. The data of a structure of A-module on L extending the action of B is the same as the data of a morphism of B-modules $\varsigma : E_1^\vee E_2 \otimes_B L \to L$ such that $d(\varsigma) = 0$ and the following diagram commutes

$$\begin{tikzcd}
 & (E_1^\vee)^2 E_2^2 L \arrow[r, "E_1^\vee \lambda E_2"] & (E_1^\vee E_2)^2 L \arrow[r, "E_1^\vee E_2 \varsigma"] & E_1^\vee E_2 L \arrow[dr, "\varsigma"] & \\
(E_1^\vee)^2 E_2^2 L \arrow[ur, "\tau_1 E_2^2"] \arrow[dr, "(E_1^\vee)^2 \tau_2"'] & & & & L \\
 & (E_1^\vee)^2 E_2^2 L \arrow[r, "E_1^\vee \lambda E_2"'] & (E_1^\vee E_2)^2 L \arrow[r, "E_1^\vee E_2 \varsigma"'] & E_1^\vee E_2 L \arrow[ur, "\varsigma"'] &
\end{tikzcd} \tag{5.3.2}$$

This gives us an identification (isomorphism of categories) between differential A-modules and pairs consisting of a differential B-module L and a map ς as above.

Consider the adjunction isomorphism

$$\phi : \mathrm{Hom}_B(E_2L, E_1L) \xrightarrow{\sim} \mathrm{Hom}_B(E_1^\vee E_2L, L)$$

Let $\pi \in Z\,\mathrm{Hom}_B(E_2L, E_1L)$ and let $\varsigma = \phi(\pi) \in Z\,\mathrm{Hom}_B(E_1^\vee E_2L, L)$. The commutativity of the diagram (5.3.2) is equivalent to the commutativity of the diagram

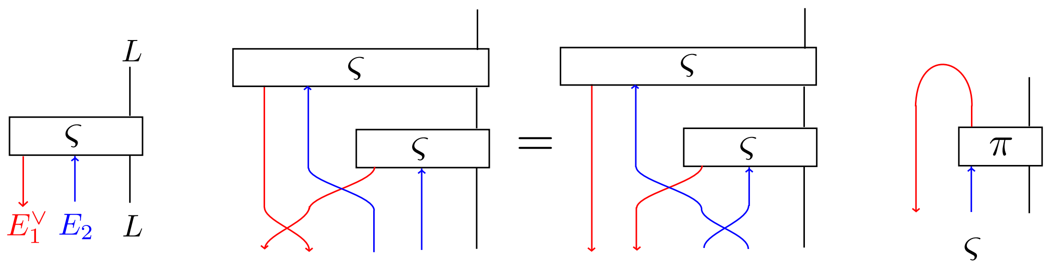
$$\begin{tikzcd} E_2^2L \arrow[r, "E_2\pi"] \arrow[d, "\tau_2"'] & E_2E_1L \arrow[r, "\sigma"] & E_1E_2L \arrow[r, "E_1\pi"] & E_1^2L \arrow[d, "\tau_1"] \\ E_2^2L \arrow[r, "E_2\pi"'] & E_2E_1L \arrow[r, "\sigma"'] & E_1E_2L \arrow[r, "E_1\pi"'] & E_1^2L \end{tikzcd} \tag{5.3.3}$$

This gives us an identification (isomorphism of categories) between differential A-modules and pairs $[L,\pi]$ where L is a differential B-module, $\pi \in Z\,\mathrm{Hom}_B(E_2L, E_1L)$ and the diagram (5.3.3) commutes. We have obtained the following lemma.

Lemma 5.3.2. *The construction* $(m,\pi) \mapsto [m,\pi]$ *defines an isomorphism of differential categories* $\Phi : \Delta_\sigma(B\text{-diff}) \to (\Delta_\sigma B)\text{-diff}$.

We will show that the structure of 2-representation on $\Delta_\sigma(B\text{-diff})$ comes from a structure of 2-representation on $\Delta_\sigma B$, when σ is invertible.

Remark 5.3.3. The map ς, the relations it is required to satisfy, and the relation $\varsigma = \phi(\pi)$ are described graphically as:



5.3.3. *Action.* We define the closed morphism of (B,A)-bimodules $u : E_2 \otimes_B A \to E_1 \otimes_B A$ as the adjoint to the multiplication map $E_1^\vee E_2 \otimes_B A \to A$. We define E as the cone of u.

We define a morphism of (B,A)-bimodules $v : E_2 \otimes_B E \to E_1 \otimes_B E$ by

$$v_{11} : E_2^2 \otimes_B A \xrightarrow{\tau_2 \otimes 1} E_2^2 \otimes_B A \xrightarrow{E_2\eta_1\bullet} E_2E_1E_1^\vee E_2 \otimes_B A \xrightarrow{\sigma\bullet} E_1E_2E_1^\vee E_2 \otimes_B A \xrightarrow{E_1E_2\mathrm{mult.}} E_1E_2 \otimes_B A$$

$$v_{12} : E_2E_1 \otimes_B A \xrightarrow{\sigma\otimes 1} E_1E_2 \otimes_B A$$

$$v_{21} = 0$$

$$v_{22} : E_2E_1 \otimes_B A \xrightarrow{\sigma\otimes 1} E_1E_2 \otimes_B A \xrightarrow{E_1\eta_1\bullet} E_1^2E_1^\vee E_2 \otimes_B A \xrightarrow{\tau_1\bullet} E_1^2E_1^\vee E_2 \otimes_B A \xrightarrow{E_1^2\mathrm{mult.}} E_1^2 \otimes_B A$$

Note that the morphism v corresponds, by adjunction, to the morphism $w : E_1^\vee E_2 \otimes_B E \to E$ defined as follows

$$w_{11} : E_1^\vee E_2^2 \otimes_B A \xrightarrow{E_1^\vee \tau_2 \otimes 1} E_1^\vee E_2^2 \otimes_B A \xrightarrow{\lambda E_2 \otimes 1} E_2E_1^\vee E_2 \otimes_B A \xrightarrow{E_2\mathrm{mult.}} E_2 \otimes_B A$$

$$w_{12} : E_1^\vee E_2E_1 \otimes_B A \xrightarrow{E_1^\vee \sigma\bullet} E_1^\vee E_1E_2 \otimes_B A \xrightarrow{\varepsilon_1\bullet} E_2 \otimes_B A,\ \ w_{21} = 0$$

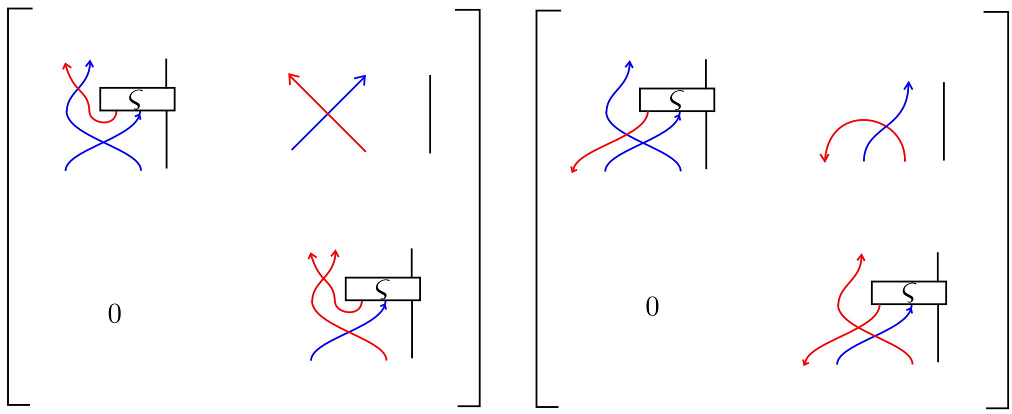
$$w_{22} : E_1^\vee E_2 E_1 \otimes_B A \xrightarrow{E_1^\vee \sigma \otimes 1} E_1^\vee E_1 E_2 \otimes_B A \xrightarrow{\rho_1 \bullet} E_1 E_1^\vee E_2 \otimes_B A \xrightarrow{E_1 \text{mult.}} E_1 \otimes_B A.$$

Lemma 5.3.4. *The pair $[E, v]$ gives E a structure of differential (A, A)-bimodule via Lemma 5.3.2. Furthermore, there is an isomorphism of functors $\Phi\mathcal{E} \xrightarrow{\sim} (E \otimes_A -)\Phi : \Delta_\sigma(B\text{-diff}) \to A\text{-diff}$.*

Proof. The vanishing of $d(v)_{11}$ and $d(v)_{22}$ follows from $d(\tau_1) = \mathrm{id}$ and $d(\tau_2) = \mathrm{id}$. The vanishing of $d(v)_{12}$ is clear. Finally, the vanishing of $d(v)_{21}$ follows from the commutativity of the diagram (5.3.3). Since $d(v) = 0$, we have obtained a structure of differential $(T_B(E_1^\vee E_2), A)$-bimodule on E.

The object of $\Delta_\sigma(B\text{-diff})$ corresponding to A via Lemma 5.3.2 is (A, u). We have $\mathcal{E}(A, u) = (E, v)$, where $\mathcal{E}$ is the endofunctor defining the 2-representation on $\Delta_\sigma(B\text{-diff})$. Since (E, v) is an object of $\Delta_\sigma(B\text{-diff})$, it follows that the action of $T_B(E_1^\vee E_2)$ on E factors through an action of A. So, E has a structure of differential (A, A)-bimodule and we have an isomorphism of functors $\Phi\mathcal{E} \xrightarrow{\sim} (E \otimes_A -)\Phi : \Delta_\sigma(B\text{-diff}) \to A\text{-diff}$. ☐

Remark 5.3.5. The maps v and w are described graphically as:



v w

We assume now that σ is invertible. We define τ an endomorphism of (B, A)-bimodules of $E_2^2 \otimes_B A \oplus E_2 E_1 \otimes_B A \oplus E_1 E_2 \otimes_B A \oplus E_1^2 \otimes_B A$

$$\tau = \begin{pmatrix} \tau_2 \otimes 1 & 0 & 0 & 0 \\ 0 & 0 & \sigma^{-1} \otimes 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_1 \otimes 1 \end{pmatrix}. \tag{5.3.4}$$

Proposition 5.3.6. *The pair (E, τ) defines a 2-representation on A and Φ induces a isomorphism of 2-representations $\Delta_\sigma(B\text{-diff}) \xrightarrow{\sim} (\Delta_\sigma B)\text{-diff}$. If E_2 is right finite, then E is right finite.*

Proof. The fact that τ defines an endomorphism of (A, A)-bimodules of E^2 satsifying the appropriate relations follows from the fact that it agrees with the endomorphism of $\mathcal{E}^2$ defining the 2-representation on $\Delta_\sigma(B\text{-diff})$. We deduce that (E, τ) is a 2-representation on A and Φ is a morphism of 2-representations.

Note that E is finitely generated and projective as a (non-differential) A^{opp}-module if E_1 and E_2 are finitely generated and projective B^{opp}-modules. ☐

5.3.4. *Tensor product case.* Let A_1 and A_2 be two differential algebras equiped with structures of 2-representations (E_i, τ_i), $i = 1, 2$.

Let $B = A_1 \otimes A_2$. It is endowed with commuting 2-representations $(E_1 \otimes A_2, \tau_1 \otimes 1)$ and $(A_1 \otimes E_2, 1 \otimes \tau_2)$: the isomorphism σ is induced by the swap map $E_2 \otimes E_1 \xrightarrow{\sim} E_1 \otimes E_2$, $a_2 \otimes a_1 \mapsto a_1 \otimes a_2$. The tensor product identifies $(A_1\text{-diff}) \otimes (A_2\text{-diff})$ with a full subcategory of B-diff.

Assume E_1 is right finite. The map λ is an isomorphism. We put $A_1 \circledcirc A_2 = \Delta'_\lambda(B)$. It is the quotient of the tensor algebra $T_{A_1 \otimes A_2}(E_1^\vee \otimes E_2)$ by the ideal generated by $p\tau_2(q) - \tau_1(p)q$ for $p \in (E_1^\vee)^{\otimes 2}$ and $q \in (E_2)^{\otimes 2}$. The underlying differential module is

$$A = \bigoplus_{i \geqslant 0} (E_1^\vee)^i \otimes_{H_i} E_2^i.$$

The multiplication is defined by

$$\big((E_1^i)^\vee \otimes_{H_i} E_2^i\big) \otimes \big((E_1^j)^\vee \otimes_{H_j} E_2^j\big) \to (E_1^{i+j})^\vee \otimes_{H_{i+j}} E_2^{i+j},\ (a_1 \otimes a_2) \otimes (b_1 \otimes b_2) \mapsto (a_1 b_1) \otimes (a_2 b_2).$$

This construction provides the differential 2-category of right finite 2-representations on differential algebras with a monoidal structure.

5.4. **Dual diagonal action.**

5.4.1. *Algebra.* Let B be a differential algebra endowed with two 2-representations (F_1, τ_1) and (E_2, τ_2) together with a closed morphism $\lambda : F_1 E_2 \to E_2 F_1$ such that the diagrams (4.2.1) commute.

We define the differential algebra

$$A = \Delta_\lambda(B) = \bigoplus_{i \geqslant 0} (E_2^i F_1^i)/((T_r \otimes 1)x - (1 \otimes T_r)x)_{x \in E_2^i F_1^i,\ 1 \leqslant r < i}.$$

Its multiplication is given by the maps $\mu_{i,j} = \mu_{(i,i),(j,j)} E_2^i F_1^i E_2^j F_1^j \to E_2^{i+j} F_1^{i+j}$ defined in §4.2.1.

Given M a differential A-module and given $i \geqslant 1$, we have differential B-module maps $\varsigma_i : E_2^i F_1^i \otimes_B M \to M$. These make $(M, (\varsigma_i)_i)$ into an object of $\Delta_\lambda(B\text{-diff})$ and provides an isomorphism of differential categories $\Delta_\lambda(B)\text{-diff} \xrightarrow{\sim} \Delta_\lambda(B\text{-diff})$.

Remark 5.4.1. As in Remark 4.4.3, we obtain a lax bi-2-representation on B by setting $E_{i,j} = E_2^j F_1^i$. We have an injective morphism of differential algebras $\Delta_E(B) \to \Delta_\lambda(B)$.

Assume the morphisms (5.2.1) are isomorphisms for all i (this holds for example if λ is an isomorphism). Then we have a canonical isomorphism $\Delta_E(B) \xrightarrow{\sim} \Delta_\lambda(B)$. The algebra $\Delta_\lambda(B)$ is generated by B and $E_2 F_1$.

The map λ extends (uniquely) to a morphism of algebras $\Delta'_\lambda(B) \to \Delta_\lambda(B)$ that is the identity on B. If λ is an isomorphism holds, then this map is an isomorphism $\Delta'_\lambda(B) \xrightarrow{\sim} \Delta_\lambda(B)$.

5.4.2. *Left dual.* We assume now that E_1 is right finite. Consider $\sigma \in Z\operatorname{Hom}(E_2 E_1, E_1 E_2)$ defined as in (4.4.1).

Let $\pi : E_2 \otimes_B A \to E_1 \otimes_B A$ be the closed morphism of (B, A)-bimodules given as a composition

$$\pi : E_2 \otimes_B A \xrightarrow{E_2 \eta_1} E_2 E_1 E_1^\vee \otimes_B A \xrightarrow{\sigma E_1^\vee} E_1 E_2 E_1^\vee \otimes_B A \xrightarrow{E_1 \mathrm{mult}} E_1 \otimes_B A.$$

We put $E = \operatorname{cone}(\pi)$. Given $i \geqslant 1$, we define a morphism of (B, B)-bimodules $\varsigma_i : E_2^i F_1^i E \to E$

$$\varsigma_i = \begin{pmatrix} E_2\mathrm{mult} \circ \lambda_{(1\cdots 2i+1)} & \sum_{r=1}^{i} E_2\mathrm{mult} \circ E_2^i F_1^{i-1} \varepsilon_1 \circ \lambda_{(1\cdots r)(2i\cdots i+r)} \\ 0 & E_1\mathrm{mult} \circ \lambda_{(1\cdots 2i+1)} \end{pmatrix}$$

The following lemma is a consequence of Lemmas 4.3.5 and 4.3.7 applied to $m = A$.

Lemma 5.4.2. *The ς_i's define a left action of A on E, giving E a structure of differential (A, A)-bimodule.*

Note that the isomorphism of differential categories $\Delta_\lambda(B)$-diff $\xrightarrow{\sim} \Delta_\lambda(B\text{-diff})$ commutes with E.

Assume now σ is an isomorphism. We define τ a (B, A)-bimodule endomorphism of E^2 as in (5.3.4).

Theorem 4.4.15 has the following consequence.

Theorem 5.4.3. *The data (E, τ) defines a 2-representation on $\Delta_\lambda(B)$.*

Note that we have an isomorphism of 2-representations $\Delta_\lambda(B)$-diff $\xrightarrow{\sim} \Delta_\lambda(B\text{-diff})$.

Consider the $(\Delta_\sigma(B), \Delta_\lambda(B))$-bimodule $\Delta_\lambda(B)$, where the right action is given by multiplication and the left action by multiplication preceded by the morphism of algebras $\Delta_\sigma(B) = \Delta'_\lambda(B) \to \Delta_\lambda(B)$. It follows from Proposition 4.4.16 that this bimodule induces a morphism of 2-representations from $\Delta_\lambda(B)$ to $\Delta_\sigma(B)$.

5.5. **Differential categories.**

5.5.1. *Bimodule 2-representations.* All the definitions and constructions of §5.1–5.4 extend from the setting of differential algebras to that of differential categories. We will describe this explicitly.

We view the monoidal category $\mathcal{U}$ as a 2-category with one object $*$.

Definition 5.5.1. *A* bimodule 2-representation *is the data of a 2-functor $\Upsilon : \mathcal{U} \to \mathrm{Bimod}$.*
It is right finite *if $\Upsilon(e)$ is right finite.*

We say that Υ is a bimodule 2-representation on $\Upsilon(*)$.

Bimodule 2-representations form a differential 2-category.

Let $\mathcal{C}$ be a differential category. There are equivalences of differential 2-categories between

- the 2-category of bimodule 2-representations Υ on $\mathcal{C}$
- the 2-category with objects differential functors $M : \mathcal{C} \times \mathcal{C}^{\mathrm{opp}} \times \mathcal{U} \to k$-diff together with
  - isomorphisms $\mu_{m,n} : M(c, -, e^m) \otimes_{\mathcal{C}} M(-, c', e^n) \xrightarrow{\sim} M(c, c', e^{n+m})$ functorial in c and c', compatible with the canonical morphism $\mathrm{End}(e^m) \otimes \mathrm{End}(e^n) \to \mathrm{End}(e^{n+m})$ and satisfying $\mu_{l,n+m} \circ (\mathrm{id} \otimes \mu_{m,n}) = \mu_{m+l,n} \circ (\mu_{l,m} \otimes \mathrm{id})$
  - an isomorphism $\mu_0 : M(-, -, e^0) \xrightarrow{\sim} \mathrm{Id}$ such that $\mu_{m,0} = \mathrm{mult} \circ (M(c, -, e^m) \otimes \mu_0)$ and $\mu_{0,m} = \mathrm{mult} \circ (\mu_0 \otimes M(-, c, e^m))$
- the 2-category of pairs (E, τ) where E is a $(\mathcal{C}, \mathcal{C})$-bimodule and $\tau \in \mathrm{End}(E^2)$ satisfies (4.1.1).

The category $\mathcal{H}om((\mathcal{C},E,\tau),(\mathcal{C}',E',\tau'))$ of 1-arrows in the third 2-category above has objects pairs (P,φ) where P is a $(\mathcal{C}',\mathcal{C})$-bimodule and $\varphi : P\otimes_{\mathcal{C}} E \xrightarrow{\sim} E'\otimes_{\mathcal{C}'} P'$ is a closed isomorphism of $(\mathcal{C}',\mathcal{C})$-bimodules satisfying (5.1.1). We leave it to the reader to describe 1-arrows in the second 2-category above. In these 2-categories, the 2-arrows are morphisms of (non-differential) bimodules or functors compatible with the additional structure.

The equivalences are given by

$$\Upsilon \mapsto (M : (c_1,c_2,e^n)\mapsto \Upsilon(e^n)(c_1,c_2)),\ \ M\mapsto (E=M(-,-,e),\tau=M(-,-,\tau))$$
$$E\mapsto (\Upsilon : e^n\mapsto E^n).$$

We will use the terminology "bimodule 2-representation" for either one of those three equivalent structures.

Note that a 2-representation $\Upsilon : \mathcal{U}\to \mathrm{End}(\mathcal{C})$ gives rise to a bimodule 2-representation M on $\mathcal{C}$ given by $M(c_1,c_2,e^n) = \mathrm{Hom}_{\mathcal{C}}(c_2,\Upsilon\circ \mathrm{rev}(e^n)(c_1))$ (cf §2.2.3). Note also that a bimodule 2-representation M on a differential category $\mathcal{C}$ gives rise to a 2-representation $\Upsilon : \mathcal{U}\to \mathrm{End}(\mathcal{C}\text{-diff})$ given by $\Upsilon(e^n)=M(-,-,e^n)\otimes_{\mathcal{C}} -$.

5.5.2. A *bimodule lax bi-2-representation* is a lax differential 2-functor $\Upsilon : \mathcal{U}\otimes\mathcal{U}\to \mathrm{Bimod}$. We say it is a bimodule lax bi-2-representation on $\Upsilon(*\otimes *)$.

A bimodule lax bi-2-representation on $\mathcal{C}$ is the same as the data of

- $(\mathcal{C},\mathcal{C})$-bimodules $E_{i,j}$ for $i,j\geqslant 0$
- morphisms of differential algebras $H_i\otimes H_j\to \mathrm{End}(E_{i,j})$
- morphisms $\mu_{(i,j),(i',j')} : E_{i,j}E_{i',j'}\to E_{i+i',j+j'}$ satisfying properties (1) and (2) of §4.2.1.

We define the differential category $\Delta_E(\mathcal{C})$ as the additive category quotient of $T_{\mathcal{C}}(E_{0,1}E_{1,0})$ by the ideal of maps generated by the kernels of the compositions

$$(E_{0,1}E_{1,0})^i(c_1,c_2)\xrightarrow{\mathrm{can}} E_{i,i}(c_1,c_2)\xrightarrow{\mathrm{can}} E_{i,i}(c_1,c_2)/((T_r\otimes 1)x-(1\otimes T_r)x)_{x\in E_{i,i},\ 1\leqslant r<i}.$$

Assume now $\mathcal{C}$ is a differential category endowed with two structures (F_1,τ_1) and (E_2,τ_2) of bimodule 2-representations together with a closed morphism $\lambda : F_1E_2\to E_2F_1$ such that the diagrams (4.2.1) commute.

We define the differential category $\Delta'_\lambda(\mathcal{C})$ as the additive category quotient of $T_{\mathcal{C}}(F_1E_2)$ by the ideal of maps generated by the image of the composition

$$F_1^2E_2^2(c_1,c_2)\xrightarrow{\tau_1E_2^2-F_1^2\tau_2} F_1^2E_2^2(c_1,c_2)\xrightarrow{F_1\lambda E_2}(F_1E_2)^2(c_1,c_2).$$

We have a differential category $\mathcal{C}'=\bigoplus_{i\geqslant 0}E_2^iF_1^i$. Its objects are those of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}'}(c_1,c_2)=\bigoplus_{i\geqslant 0}E_2^iF_1^i(c_1,c_2)$. The multiplication is induced by the maps $\mu_{i,j}$. We define the differential category $\Delta_\lambda(\mathcal{C})$ as the additive category quotient of $\bigoplus_{i\geqslant 0}E_2^iF_1^i$ by the ideal of maps generated by the images of $T_r\otimes 1-1\otimes T_r : E_2^iF_1^i\to E_2^iF_1^i$ for $1\leqslant r<i$.

Assume now $\mathcal{C}$ is a differential category endowed with two structures (E_1,τ_1) and (E_2,τ_2) of bimodule 2-representations, the first of which is right finite. Consider $\sigma : E_2E_1\to E_1E_2$ closed such that the diagrams (4.3.1) commute. We define $\lambda : E_1^\vee E_2\to E_2E_1^\vee$ as in (5.3.1).

- We put $\Delta_\sigma(\mathcal{C})=\Delta'_\lambda(\mathcal{C})$. As in §5.3.3, we define a $(\Delta_\sigma\mathcal{C},\mathcal{C})$-bimodule E and extend it to a $(\Delta_\sigma\mathcal{C},\Delta_\sigma\mathcal{C})$-bimodule. Assume finally that σ is invertible. We construct in addition an endomorphism τ of E^2. We obtain a bimodule 2-representation on $\Delta_\sigma\mathcal{C}$ and an isomorphism of 2-representations $\Delta_\sigma(\mathcal{C}\text{-diff}) \xrightarrow{\sim} \Delta_\sigma(\mathcal{C})\text{-diff}$. The 2-representation is right finite if E_2 is right finite.

As in §5.3.4, we have a monoidal structure on the differential 2-category of right finite bimodule 2-representations.

• We drop now the assumption that σ is invertible. We define as in §5.4.2 a $(\Delta_\lambda\mathcal{C}, \Delta_\lambda\mathcal{C})$-bimodule E. Assume σ is invertible. We obtain an endomorphism τ of E^2 and a bimodule 2-representation on $\Delta_\lambda\mathcal{C}$.

5.6. **Pointed categories.** Let $\mathcal{V}$ be a differential pointed category. A *bimodule 2-represesentation* on $\mathcal{V}$ is the data of a strict monoidal differential pointed functor from the 2-category with one object given by $\mathcal{U}^\bullet$ to $\mathrm{Bimod}^\bullet$. Note that a bimodule 2-representation on $\mathcal{V}$ gives rise to a bimodule 2-representation on $k[\mathcal{V}]$.

A *bimodule lax bi-2-representation* is a lax differential pointed 2-functor $\Upsilon : \mathcal{U}^\bullet \wedge \mathcal{U}^\bullet \to \mathrm{Bimod}^\bullet$. We say it is a bimodule lax bi-2-representation on $\Upsilon(* \wedge *)$.

A bimodule lax bi-2-representation on $\mathcal{V}$ is the same as the data of

- $(\mathcal{V}, \mathcal{V})$-bimodules $E_{i,j}$ for $i, j \geqslant 0$
- morphisms of differential pointed algebras $H_i \wedge H_j \to \mathrm{End}(E_{i,j})$
- morphisms $\mu_{(i,j),(i',j')} : E_{i,j}E_{i',j'} \to E_{i+i',j+j'}$ satisfying properties (1) and (2) of §4.2.1.

We define the differential pointed category $\Delta_E(\mathcal{V})$ as the quotient of $T_{\mathcal{V}}(E_{0,1}E_{1,0})$ by the equivalence relation generated by $f \sim f'$ if (f, f') is in the equalizer of a composition

$$(E_{0,1}E_{1,0})^i(c_1, c_2) \xrightarrow{\text{can}} E_{i,i}(c_1, c_2) \xrightarrow{\text{can}} E_{i,i}(c_1, c_2)/((T_r \wedge 1)x \sim (1 \wedge T_r)x)_{x \in E_{i,i},\ 1 \leqslant r < i}.$$

Consider a differential pointed category $\mathcal{V}$ endowed with two bimodule 2-representations (F_1, τ_1) and (E_2, τ_2) and a closed morphism $\lambda : F_1E_2 \to E_2F_1$ such that the diagrams (4.2.1) commute.

We define the differential pointed category $\Delta'_\lambda(\mathcal{V})$ as the quotient of $T_{\mathcal{V}}(F_1E_2)$ by the equivalence relation generated by

$$(F_1\lambda E_2) \circ (\tau_1 E_2^2)(f) \sim (F_1\lambda E_2) \circ (F_1^2\tau_2)(f) \text{ for } f \in F_1^2E_2^2(c_1, c_2) \text{ and } c_1, c_2 \in \mathcal{V}.$$

We define the differential pointed category $\Delta_\lambda(\mathcal{V})$. We consider first the differential pointed category with same objects as $\mathcal{V}$ and pointed set of maps $v_1 \to v_2$ given by $\bigvee_{i \geqslant 0} E_2^iF_1^i(v_1, v_2)$. The category $\Delta_\lambda(\mathcal{V})$ is the quotient of that category by the equivalence relation generated by $(T_r \wedge 1)(f) \sim (1 \wedge T_r)(f)$ for $f \in E_2^iF_1^i$ and $1 \leqslant r < i$.

Note that there is a canonical isomorphism of differential categories for $? \in \{\varnothing, \prime\}$

$$k[\Delta_\lambda^?(\mathcal{V})] \xrightarrow{\sim} \Delta_\lambda^?(k[\mathcal{V}])$$

5.7. **Douglas-Manolescu's algebra-modules.** Let us recall some aspects of Douglas-Manolescu's theory [DouMa].

Note that Douglas and Manolescu work in the differential graded setting, and we translate their constructions to the differential setting.

Their nil-Coxeter 2-algebra [DouMa, §2.2] can be viewed as the same data as our monoidal category $\mathcal{U}$ (cf [DouMa, Remark 2.4]). A bottom-algebra module [DouMa, §2.4] for the nil-Coxeter 2-algebra is the same data as a lax bimodule 2-representation on a differential algebra A, where a lax bimodule 2-representation on A is defined to be a lax 2-functor $\Upsilon : \mathcal{U} \to \mathrm{Bimod}$

with $\Upsilon(1)$ the differential category with one object whose endomorphism ring is A. They also consider top-algebra modules, where $\mathcal{U}$ above is replaced by $\mathcal{U}^{\mathrm{opp}}$. Using the isomorphism $\mathcal{U} \xrightarrow{\sim} \mathcal{U}^{\mathrm{opp}}$ (§4.1.1), a top-algebra module can be viewed as a bottom-algebra module, hence as a lax bimodule 2-representation.

Douglas and Manolescu define a tensor product of a top algebra-module and a bottom algebra-module [DouMa, Definition 2.11]. This corresponds to our construction of a differential algebra A as a tensor product $\bigotimes$. Note that they do not endow this tensor product with any algebra-module structure.

6. Hecke 2-representations

6.1. Regular 2-representations.

6.1.1. *Bimodules.* Fix $r, n \geqslant 0$. We define some bimodules $L^{\pm}(r,n)$ and $R^{\pm}(r,n)$ with underlying differential graded module H_{r+n}, following §3.1.3 and Proposition 3.1.6.

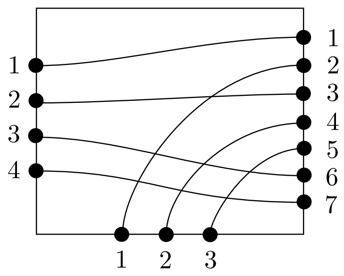
We endow $L^{+}(r,n)$ (resp. $L^{-}(r,n)$) with a structure of differential graded $(H_r \otimes H_n, H_{r+n})$-bimodule where

- H_{r+n} acts by right multiplication
- $h \in H_r$ acts by left multiplication by h (resp. by $f_n(h)$)
- $h \in H_n$ acts by left multiplication by $f_r \circ \iota_n(h)$ (resp. by h).

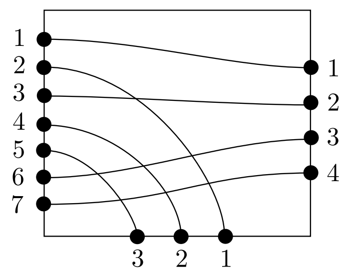
We endow $R^{+}(r,n)$ (resp. $R^{-}(r,n)$) with a structure of differential graded $(H_{r+n}, H_r \otimes H_n)$-bimodule where

- H_{r+n} acts by left multiplication
- $h \in H_r$ acts by right multiplication by h (resp. by $f_n(h)$)
- $h \in H_n$ acts by right multiplication by $f_r \circ \iota_n(h)$ (resp. by h).

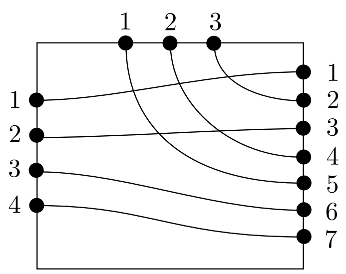
Example 6.1.1. Elements of $L^{\pm}(r,n)$ and $R^{\pm}(r,n)$ can be represented by good strand diagrams in a rectangle, as in the examples below.



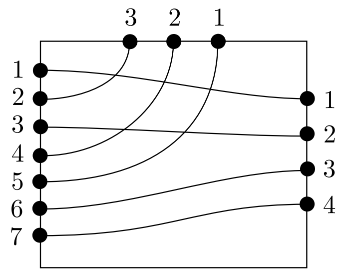
$L^{+}(4,3)$



$R^{+}(4,3)$

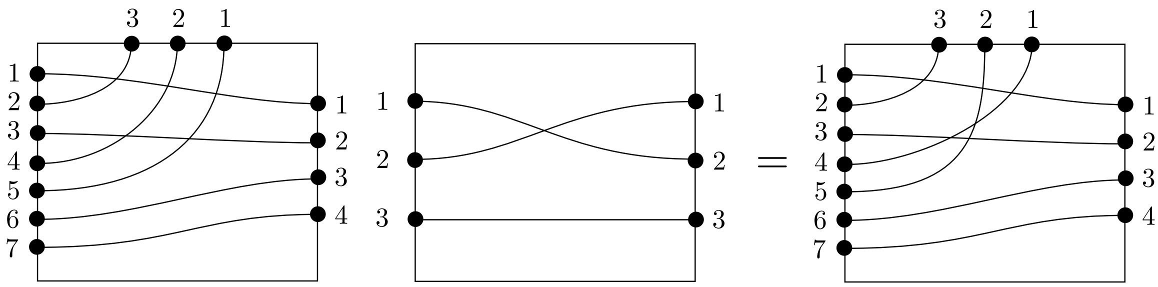


$L^{-}(4,3)$



$R^{-}(4,3)$

The actions are obtained by concatenation of diagrams (note that a diagram that is not good represents 0), as in the example below, where we first apply the reflection of the rectangle swapping the top and the bottom, then rotate 90 degrees anticlockwise the diagram of h':



$h \in R^{-}(4,3)$ $\qquad$ $h' \in H_3$ $\qquad$ $hh' \in R^{-}(4,3)$

These bimodules coincide with (the nil version of) the bimodules introduced in §3.1.3, after restricting the action of $H_r \otimes H_n$ to H_r:

$L^{\pm}(r,n) = L^{\pm}(I,S)$ and $R^{\pm}(r,n) = L^{\pm}(S,I)$ where $S = \{s_1,\ldots,s_{r+n-1}\}$ and $I = \{s_1,\ldots,s_{r-1}\}$.

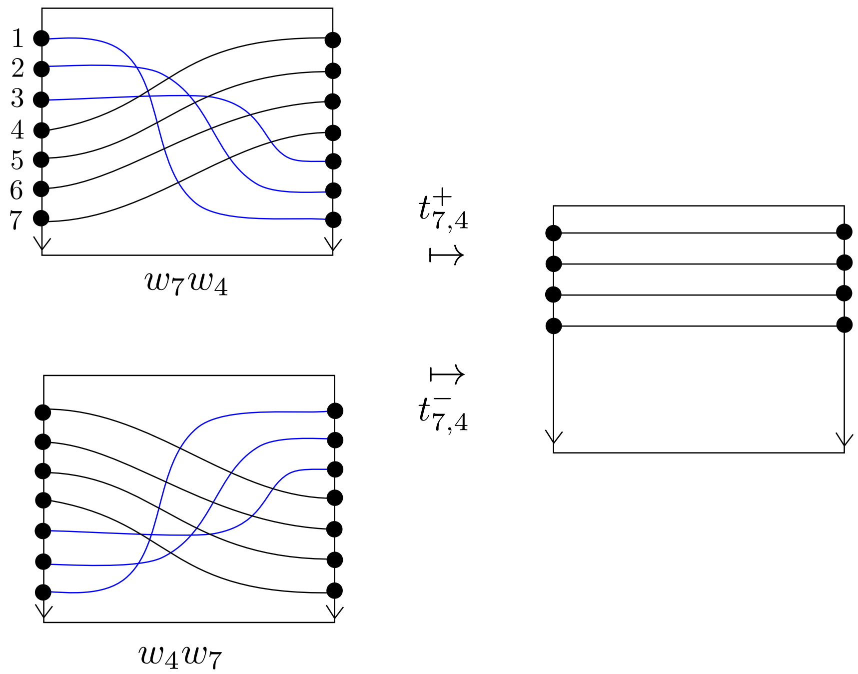
Given $m \geqslant 0$, we denote by $w_m \in \mathfrak{S}_m$ the longest element, *i.e.*, $w_m(i) = m-i+1$. We have two morphisms of differential graded $\mathbf{F}_2$-modules (cf Proposition 3.1.6)

$$t^{\pm}_{r+n,r} = t^{\pm}_{S,I} : H_{r+n} \to H_r\langle \frac{1}{2}n(2r+n-1)\rangle$$

given by

$$t^+_{r+n,r}(T_w) = \begin{cases} T_{w_r w_{r+n} w} & \text{if } w \in w_{r+n}\mathfrak{S}_r \\ 0 & \text{otherwise} \end{cases} \text{ and } t^-_{r+n,r}(T_w) = \begin{cases} T_{w w_{r+n} w_r} & \text{if } w \in \mathfrak{S}_r w_{r+n} \\ 0 & \text{otherwise} \end{cases}$$

Example 6.1.2. Let us describe some examples of $t^{\pm}_{7,4}(T_w)$:



It is immediate that there is an isomorphism of differential graded $(H_{r+n}, H_r \otimes H_n)$-modules

$$\mathrm{Hom}_{H^{\mathrm{opp}}_{r+n}}(L^{\pm}(r,n), H_{r+n}) \xrightarrow{\sim} R^{\pm}(r,n),\, f \mapsto f(1)$$

and it follows from Proposition 3.1.6 that there is an isomorphism of differential graded $(H_r \otimes H_n, H_{r+n})$-modules

$$L^{\mp}(r,n) \xrightarrow{\sim} \mathrm{Hom}_{H^{\mathrm{opp}}_r}(R^{\pm}(r,n), H_r)\langle \frac{1}{2}n(2r+n-1)\rangle,\ h \mapsto (h' \mapsto t^{\pm}_{n+r,r}(hh')).$$

6.1.2. *Twisted description.* We describe now $L^+(r,n)$ as a twisted free $(H_r \otimes H_n)$-module.

Consider $E \subset \{1,\ldots,r+n\}$ with $|E| = r$. Let $w_E \in \mathfrak{S}_{r+n}$ be the permutation such that $w_E(E) = \{1,\ldots,r\}$ and the restrictions of w_E to E and to $\{1,\ldots,r+n\}\backslash E$ are increasing. If $E = \{i_1 < \cdots < i_r\}$, then we have a reduced decomposition

$$w_E = (s_r \cdots s_{i_r-1})(s_{r-1} \cdots s_{i_{r-1}-1}) \cdots (s_2 \cdots s_{i_2-1})(s_1 \cdots s_{i_1-1})$$

and

$$\tilde{L}(w_E) = \coprod_{b=1}^{r} \big((\{1,\ldots,i_b-1\}\backslash\{i_1,\ldots,i_{b-1}\}) \times \{i_b\}\big).$$

There is a bijection

$$\beta : \mathfrak{S}_r \times \mathfrak{S}_n \times \{E \subset \{1,\ldots,r+n\} \mid |E| = r\} \xrightarrow{\sim} \mathfrak{S}_{r+n},\, (v,v',E) \mapsto v f_r(v') w_E$$

where $f_r(v') \in \mathfrak{S}_{n+r}$ is given by $f_r(v')(i) = i$ for $i \leqslant r$ and $f_r(v')(r+i) = r + v'(i)$ for $1 \leqslant i \leqslant n$. We have $\ell(\beta(v,v',E)) = \ell(v) + \ell(v') + \ell(w_E)$.

Given $(a, i_b) \in \tilde{L}(w_E)$, we define $v(E, a, b) \in \mathfrak{S}_r$ and $v'(E, a, b) \in \mathfrak{S}_n$ as follows. Let $b' \in \{1, \ldots, r\}$ be minimal such that $a < i_{b'}$. We define $v(E, a, b)$ to be the cycle $(b, b-1, \ldots, b')$ and $v'(E, a, b)$ to be the cycle $(a - b' + 1, a - b' + 2, \ldots, i_b - b)$. We have

$$w_E s_{a,i_b} = v(E, a, b) f_r(v'(E, a, b)) w_{(E \cup \{a\}) \setminus \{i_b\}}$$

and $\ell(w_E) - \ell(w_{(E \cup \{a\}) \setminus \{i_b\}}) = i_b - a$.

Given $m \geqslant 1$, we define a free differential $(H_r \otimes H_n)$-module

$$V_m = \bigoplus_{E \subset \{1, \ldots, r+n\},\ |E| = r,\ \ell(w_E) = m-1} (H_r \otimes H_n) b_E.$$

Given $m' < m$, we define $f_{m',m} : V_m \to V_{m'}$ as the morphism of $(H_r \otimes H_n)$-modules given by

$$b_E \mapsto \sum_{\substack{i \in E,\ j \in \{1, \ldots, r+n\} \setminus E \\ i - j = m - m'}} (T_{v(E,j,i)} \otimes T_{v'(E,j,i)}) b_{(E \cup \{j\}) \setminus \{i\}}.$$

We will show below (Lemma 6.1.3) that $d(f_{m',m}) = \sum_{m > m'' > m'} f_{m'm''} \circ f_{m''m}$. We denote by V the differential $(H_r \otimes H_n)$-module obtained as the corresponding twisted object $[\bigoplus V_m, (f_{m'm})]$ (cf §2.1.3). We have $V = \bigoplus_m V_m$ as a $(H_r \otimes H_n)$-module and $d_V = \sum_m d_{V_m} + \sum_{m,m'} f_{m',m}$.

Lemma 6.1.3. *The maps $(f_{m'm})$ define a twisted object $V = [\bigoplus V_m, (f_{m'm})]$. There is an isomorphism of differential $(H_r \otimes H_n)$-modules*

$$V \xrightarrow{\sim} L^+(r, n),\ (h \otimes h') b_E \mapsto h f_r(\iota_n(h')) T_{w_E} \text{ for } h \in H_r \text{ and } h' \in H_n.$$

Proof. The length property of the bijection β above shows that the map of the lemma is an isomorphism of $(H_r \otimes H_n)$-modules. Since

$$d(T_{w_E}) = \sum_{i \in E,\ j \in \{1, \ldots, r+n\} \setminus E,\ j < i} T_{w_E s_{i,j}},$$

it follows that the map of the lemma intertwines d_V and the differential of $L^+(r, n)$. The lemma follows. $\square$

There is a dual version of Lemma 6.1.3. In particular, there is a decomposition of right $(H_r \otimes H_n)$-modules

$$R^+(r, n) = \bigoplus_{E \subset \{1, \ldots, r+n\},\ |E| = r} T_{w_E^{-1}} (H_r \otimes f_r(H_n))$$

6.1.3. *Actions.* There is a "left" 2-representation on $\mathcal{U}$

$$\Upsilon^- : \mathcal{U} \to \mathrm{End}(\mathcal{U}),\ e^n \mapsto e^n \otimes -$$

and a "right" 2-representation on $\mathcal{U}$

$$\Upsilon^+ : \mathcal{U} \xrightarrow[\sim]{\mathrm{rev}} \mathcal{U}^{\mathrm{rev}} \xrightarrow{e^n \mapsto - \otimes e^n} \mathrm{End}(\mathcal{U}).$$

The bimodule 2-representation $L^{\pm}$ associated to $\Upsilon^{\pm}$ is given by

$$L^{\pm}(e^r, e^s, e^n) = \delta_{s,r+n} L^{\pm}(r, n)$$

and it is left and right finite. Its right dual is isomorphic to the bimodule 2-representation $R^{\pm}$ given by

$$R^{\pm}(e^s, e^r, e^n) = \delta_{s,r+n} R^{\pm}(r, n)$$

while its left dual is isomorphic to $R^{\mp}\langle -\frac{1}{2}n(2r+n-1)\rangle$ (note that the action of $\mathcal{U}$ on the duals is obtained from the natural action of $\mathcal{U}^{\mathrm{revopp}}$ by applying the isomorphism $\mathrm{rev}\circ\mathrm{opp}$).

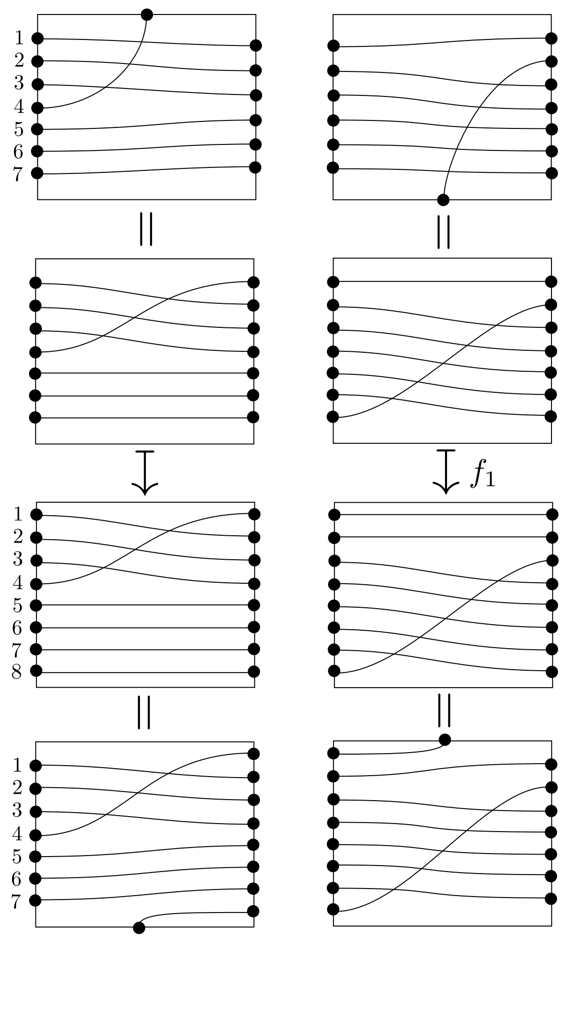
6.1.4. *Gluing.* Consider the morphism of functors

$$\lambda : R^-(-_1,-,e)\otimes L^+(-,-_2,e)\to L^+(-_1,-,e)\otimes R^-(-,-_2,e)$$

where $\lambda(e^s,e^s)$ is given by the following morphism of differential graded (H_s,H_s)-bimodules

$$\begin{array}{ccc} R^-(e^s,-,e)\otimes L^+(-,e^s,e) = R^-(s-1,1)\otimes_{H_{s-1}} L^+(s-1,1) & & a\otimes b \\ \downarrow & & \downarrow \\ L^+(e^s,-,e)\otimes R^-(-,e^s,e) = L^+(s,1)\otimes_{H_{s+1}} R^-(s,1) & & a\otimes f_1(b) \end{array}$$

Remark 6.1.4. An example of diagrammatic description of λ is given below:



The morphism

$$(R^-(-,-,e)\lambda L^+(-,-,e))\circ(R^-(-,-,e)^2\tau-\tau L^+(-,-,e)^2):$$
$$R^-(-,-,e)^2L^+(-,-,e)^2\to(R^-(-,-,e)L^+(-,-,e))^2\langle -1\rangle$$

is on (e^s, e^s) the morphism of differential graded (H_s, H_s)-bimodules

$$R^-(s-1,1)\otimes_{H_{s-1}} R^-(s-2,1)\otimes_{H_{s-2}} L^+(s-2,1)\otimes_{H_{s-1}} L^+(s-1,1)\to \\ R^-(s-1,1)\otimes_{H_{s-1}} L^+(s-1,1)\otimes_{H_s} R^-(s-1,1)\otimes_{H_{s-1}} L^+(s-1,1)\langle -1\rangle$$

given by

$$1\otimes 1\otimes 1\otimes 1\mapsto T_1\otimes 1\otimes 1\otimes 1-1\otimes 1\otimes 1\otimes T_{s-1}.$$

Given $s\geqslant 1$, let $M_s=R^-(s-1,1)\otimes_{H_{s-1}} L^+(s-1,1)$, a differential graded (H_s,H_s)-bimodule. When $s\geqslant 2$, we define $\kappa=1\otimes 1\otimes 1\otimes T_{s-1}-T_1\otimes 1\otimes 1\otimes 1\in M_s\otimes_{H_s} M_s$. We put $\kappa=0$ when $s=1$. We put $M_0=0$.

Lemma 6.1.5. *There is a morphism of differential graded (H_s,H_s)-bimodules $M_s\to \hat{H}_s^+$ given by $a\otimes b\mapsto acb$ for $a,b\in H_s$. It induces an isomorphism of differential graded algebras and of differential graded (H_s,H_s)-bimodules $T_{H_s}(M_s)/(\kappa)\xrightarrow{\sim}\hat{H}_s^+$.*

Proof. We have $acT_ib=aT_{i+1}cb$ for $i\in\{1,\ldots,s-2\}$. This shows the first statement of the lemma.

We have now a morphism of differential graded algebras and of (H_s,H_s)-bimodules f' : $T_{H_s}(M_s)\to\hat{H}_s^+$ induced by the morphism $M_s\to\hat{H}_s^+$. We have

$$f'((1\otimes 1)\otimes(1\otimes T_{s-1}))=c^2T_{s-1}=T_1c^2=f'((T_1\otimes 1)\otimes(1\otimes 1)),$$

hence $f'(\kappa)=0$. So, f induces a morphism of algebras $f:T_{H_s}(M_s)/(\kappa)\to\hat{H}_s^+$.

On the other hand, $\hat{H}_s^+$ is the free algebra generated by H_s and c with the relations $cT_i=T_{i+1}c$ for $i\in\{1,\ldots,s-2\}$ and $c^2T_{s-1}=T_1c^2$ (Proposition 3.2.9). Since $T_{i+1}\otimes 1=1\otimes T_i$ in M_s for $i\in\{1,\ldots,s-2\}$ and $(1\otimes 1)\otimes(1\otimes T_{s-1})=(T_1\otimes 1)\otimes(1\otimes 1)$ in $M_s\otimes M_s$, we deduce that there is a morphism of algebras $g:\hat{H}_s^+\to T_{H_s}(M_s)/(\kappa),\ T_i\mapsto T_i,\ c\mapsto 1\otimes 1$. The morphisms f and g are inverse and we are done. ∎

Let $\mathcal{H}^+$ be the differential graded pointed category with set of objects $\mathbf{Z}_{\geqslant 0}$ and $\mathrm{Hom}_{\mathcal{H}^+}(m,n)=\delta_{mn}\hat{\mathfrak{S}}_n^{+,\mathrm{nil}}$. Lemma 6.1.5 has the following consequence.

Theorem 6.1.6. *The construction of Lemma 6.1.5 induces an isomorphism of differential graded pointed categories $\Theta:\Delta'_\lambda(\mathcal{U}^\bullet)\xrightarrow{\sim}\mathcal{H}^+$.*

6.2. **Nil Hecke category.**

6.2.1. *Definition.* We now define a groupoid of n-periodic bijections.

Given I a subset of $\mathbf{Z}/n$ we denote by $\tilde{I}$ its inverse image in $\mathbf{Z}$.

Let $\mathcal{S}_n$ be the category with objects the subsets of $\mathbf{Z}/n$ and where $\mathrm{Hom}_{\mathcal{S}_n}(I,J)$ is the set of n-periodic bijections $\sigma:\tilde{I}\xrightarrow{\sim}\tilde{J}$. The group $n\mathbf{Z}$ acts by translation on Hom-sets. Note that $\hat{\mathfrak{S}}_n=\mathrm{End}_{\mathcal{S}_n}(\mathbf{Z}/n)$.

Given $i,j\in\tilde{I}$ with $i-j\notin n\mathbf{Z}$, the element $s_{ij}\in\hat{\mathfrak{S}}_n$ restricts to an n-periodic bijection $\tilde{I}\xrightarrow{\sim}\tilde{I}$, which we also denote by s_{ij}.

Let I be a subset of $\mathbf{Z}/n$. There is a unique increasing bijection $\beta_I:\{1,\ldots,|I|\}\xrightarrow{\sim}\tilde{I}\cap\{1,\ldots,n\}$. We extend it to an increasing bijection $\mathbf{Z}\xrightarrow{\sim}\tilde{I}$ by $\beta_I(r+d|I|)=\beta_I(r)+dn$ for $r\in\{1,\ldots,|I|\}$ and $d\in\mathbf{Z}$. There is an isomorphism of groups

$$F_I:\hat{\mathfrak{S}}_{|I|}\xrightarrow{\sim}\mathrm{End}_{\mathcal{S}_n}(I),\ \sigma\mapsto\beta_I\circ\sigma\circ\beta_I^{-1}.$$

6.2.2. *Length.* Consider $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I,J)$. We define
$$L(\sigma) = \{(i,i') \in \tilde{I}^2 \mid i < i',\ \sigma(i) > \sigma(i')\}$$
and $\tilde{L}(\sigma) = \{(i,i') \in L(\sigma) \mid 1 \leqslant i \leqslant n\}$. The canonical map $\tilde{L}(\sigma) \to L(\sigma)/n\mathbf{Z}$ is bijective. We define $\ell(\sigma) = |\tilde{L}(\sigma)|$.

Lemma 6.2.1. *We have $\ell(\sigma' \circ \sigma) \leqslant \ell(\sigma') + \ell(\sigma)$ for all $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I,J)$ and $\sigma' \in \mathrm{Hom}_{\mathcal{S}_n}(J,K)$.*

Proof. We have
$$\begin{aligned} L(\sigma'\circ\sigma) &= \{(i_1,i_2) \in \tilde{I}^2 \mid i_1 < i_2,\ \sigma(i_1) > \sigma(i_2),\ \sigma'\circ\sigma(i_1) > \sigma'\circ\sigma(i_2)\} \sqcup \\ &\qquad \{(i_1,i_2) \in \tilde{I}^2 \mid i_1 < i_2,\ \sigma(i_1) < \sigma(i_2),\ \sigma'\circ\sigma(i_1) > \sigma'\circ\sigma(i_2)\} \\ &= \{(i_1,i_2) \in L(\sigma) \mid \sigma'\circ\sigma(i_1) > \sigma'\circ\sigma(i_2)\} \sqcup (\sigma^{-1}\times\sigma^{-1})\big(\{(j_1,j_2) \in L(\sigma') \mid \sigma^{-1}(j_1) < \sigma^{-1}(j_2)\}\big). \end{aligned}$$
It follows that
$$\ell(\sigma') + \ell(\sigma) - \ell(\sigma'\circ\sigma) = 2|\{(i_1,i_2) \in \tilde{I}^2 \mid i_1 < i_2,\ \sigma(i_1) > \sigma(i_2),\ \sigma'\circ\sigma(i_1) < \sigma'\circ\sigma(i_2)\}/n\mathbf{Z}| \geqslant 0.$$
□

Let $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I,J)$. We have $\ell(\sigma) = 0$ if and only if σ is an increasing bijection.

Given $\tau \in \mathrm{Hom}_{\mathcal{S}_n}(J,I)$ with $\ell(\tau) = 0$, we have $L(\tau\circ\sigma) = L(\sigma) = (\tau\times\tau)\big(L(\sigma\circ\tau)\big)$, hence $\ell(\tau\circ\sigma) = \ell(\sigma\circ\tau) = \ell(\sigma)$.

Since $L(\tau\circ\sigma) = (\beta_I \times \beta_I)(L(F_I^{-1}(\tau\circ\sigma)))$, we have $\ell(\sigma) = \ell(F_I^{-1}(\tau\circ\sigma))$. As a consequence, we deduce the following result from Lemma 3.2.3.

Lemma 6.2.2. *Let $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I,J)$. We have*
$$\ell(\sigma) = \sum_{\substack{0 \leqslant i_1 < i_2 < n \\ i_1, i_2 \in \tilde{I}}} \Big|\Big\lfloor \frac{\sigma(i_2) - \sigma(i_1)}{n} \Big\rfloor\Big|.$$

The next lemma relates length and number of intersections of paths on a cylinder.

Lemma 6.2.3. *Let $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I,J)$ where $I = \{i_1 + n\mathbf{Z}, i_2 + n\mathbf{Z}\}$ and $J = \{j_1 + n\mathbf{Z}, j_2 + n\mathbf{Z}\}$ with $1 \leqslant i_1 \neq i_2 \leqslant n$, $1 \leqslant j_1 \neq j_2 \leqslant n$ and $\sigma(i_r) = j_r \pmod n$ for $r \in \{1,2\}$. Fix $\beta : \{i_1, i_2, j_1, j_2\} \to \mathbf{R}$ increasing with $|\beta(u) - \beta(v)| < 1$ for all u, v.*

Consider $\gamma_r : [0,1] \to \mathbf{R}$ continuous with $\gamma_r(0) = \beta(i_r)$ and $\gamma_r(1) = \beta(j_r) + \frac{\sigma(i_r) - j_r}{n}$ for $r \in \{1,2\}$. We have
$$\ell(\sigma) \leqslant |\{t \in [0,1] \mid e^{2i\pi\gamma_1(t)} = e^{2i\pi\gamma_2(t)}\}|$$
with equality if, for all $r \in \{1,2\}$, the map γ_r is affine.

Proof. Without loss of generality, we can assume $i_1 < i_2$. The lemma follows by applying the intermediate value theorem to $\gamma_2(t) - \gamma_1(t)$ and using Lemma 6.2.2, considering four cases according to the signs of $j_2 - j_1$ and $\sigma(i_2) - \sigma(i_1)$. □

6.2.3. *Filtration.* Given $I, J \subset \mathbf{Z}/n$, we define $\mathrm{Hom}_{\mathcal{S}_n^{\geqslant -r}}(I,J) = \{\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I,J) \mid l(\sigma) \leqslant r\}$ for $r \in \mathbf{Z}_{\geqslant 0}$. It follows from Lemma 6.2.1 that this defines a structure of $\mathbf{Z}_{\leqslant 0}$-filtered category on $\mathcal{S}_n$. We put $\mathcal{H}_n = \mathrm{gr}\mathcal{S}_n^{\bullet}$, a pointed $\mathbf{Z}_{\leqslant 0}$-graded category.

Note that a map σ of length 0 is invertible in $\mathcal{H}_n$. Note also that F_I induces an isomorphism of graded pointed monoids $\hat{\mathfrak{S}}_{|I|}^{\mathrm{nil}} \xrightarrow{\sim} \mathrm{End}_{\mathcal{H}_n}(I)$.

6.2.4. *Non-commutative degree.* Let us consider the free abelian groups $R_n = \bigoplus_{a\in\mathbf{Z}/n}\mathbf{Z}\alpha_a$ and $L_n = \bigoplus_{a\in\mathbf{Z}/n}\mathbf{Z}\varepsilon_a$. We define a linear map $\rho : R_n \to L_n$ by $\rho(\alpha_a) = \varepsilon_{a+1} - \varepsilon_a$ and a representation of the group R_n on L_n given by

$$\alpha_a \cdot \varepsilon_b = (\delta_{a,b} + \delta_{a+1,b})\varepsilon_b.$$

Note that $\delta = \sum_{a\in\mathbf{Z}/n}\alpha_a \in \ker\rho$ and $\delta\cdot\varepsilon_b = 2\varepsilon_b$ for all b.

We define a bilinear map

$$\langle -,-\rangle : R_n \times R_n \to L_n,\ \langle\alpha,\alpha'\rangle = \alpha\cdot\rho(\alpha').$$

Let $\Gamma'_n = L_n \times R_n$. We define a group structure on Γ'_n by

$$(l,\alpha)\cdot(l',\alpha') = (l + l' + \langle\alpha,\alpha'\rangle, \alpha+\alpha').$$

Given $I \subset \mathbf{Z}/n$, we put $\varepsilon_I = \sum_{a\in I}\varepsilon_a \in L_n$. Given $i,j \in \mathbf{Z}$, we put

$$\alpha_{i,j} = \sum_{i\leqslant r<j}\alpha_{r+n\mathbf{Z}} - \sum_{j\leqslant r<i}\alpha_{r+n\mathbf{Z}}.$$

Note that $\alpha_{i,i+1} = \alpha_{i+n\mathbf{Z}}$, $\alpha_{i+n,j+n} = \alpha_{i,j}$ and $\alpha_{i,j} + \alpha_{j,k} = \alpha_{i,k}$ for all $i,j,k \in \mathbf{Z}$. Note also that $\delta = \alpha_{i,i+n}$ for all $i \in \mathbf{Z}$. Note finally that $\rho(\alpha_{i,j}) = \varepsilon_{j+n\mathbf{Z}} - \varepsilon_{i+n\mathbf{Z}}$.

Consider $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I,J)$. We put

$$[\![\sigma]\!] = \sum_{i\in\tilde{I}\cap[1,n]}\alpha_{i,\sigma(i)} \in R_n.$$

Note that $\rho([\![\sigma]\!]) = \varepsilon_J - \varepsilon_I$ and $[\![\sigma'\circ\sigma]\!] = [\![\sigma']\!] + [\![\sigma]\!]$ for any $\sigma' \in \mathrm{Hom}_{\mathcal{S}_n}(J,K)$.

We define

$$m(\sigma) = [\![\sigma]\!]\cdot\varepsilon_I \in L_n \text{ and } \mathrm{dm}(\sigma) = (-m(\sigma), [\![\sigma]\!]) \in \Gamma'_n.$$

Lemma 6.2.4. *Let $w \in W_{|I|}$, $m \in \mathbf{Z}$ and let $\sigma = F_I(wc^m)$ be the element of $\mathrm{End}_{\mathcal{S}_n}(I)$ corresponding to wc^m. We have $\ell(\sigma) = \ell(w)$, $[\![\sigma]\!] = m\cdot\delta$ and $m(\sigma) = 2m\varepsilon_I$.*

Proof. The first statement follows from the fact that F_I preserves lengths (cf the discussion before Lemma 6.2.2).

Note that $[\![s_{i,j}]\!] = 0$ for $i,j \in \tilde{I}$ with $i-j \notin n\mathbf{Z}$, while $[\![F_I(c)]\!] = \delta$. We deduce that $[\![\sigma]\!] = m\cdot\delta$. The last statement of the lemma is immediate. □

Lemma 6.2.5. *Consider $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I,J)$ and $\sigma' \in \mathrm{Hom}_{\mathcal{S}_n}(J,K)$. We have $\mathrm{dm}(\sigma'\circ\sigma) = \mathrm{dm}(\sigma')\cdot\mathrm{dm}(\sigma)$.*

Proof. We have

$$m(\sigma'\circ\sigma) = [\![\sigma']\!]\cdot\varepsilon_I + [\![\sigma]\!]\varepsilon_I = m(\sigma') + m(\sigma) + [\![\sigma']\!]\cdot(\varepsilon_I - \varepsilon_J),$$

hence

$$m(\sigma') + m(\sigma) - m(\sigma'\circ\sigma) = [\![\sigma']\!]\cdot\rho([\![\sigma]\!]).$$

The lemma follows. □

We put $\Gamma_n = \frac{1}{2}\mathbf{Z} \times \Gamma'_n$. We endow Γ_n with a structure of $\mathbf{Z}$-monoid by using the embedding $\mathbf{Z} \hookrightarrow \frac{1}{2}\mathbf{Z} \hookrightarrow \Gamma_n$.

Given $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I, J)$, we put $\deg(\sigma) = (-\ell(\sigma), -\mathrm{dm}(\sigma)) \in \Gamma_n$.

Let D be a subset of $\{1, \ldots, n\} \times \{\pm 1\}$ that embeds in its projection on $\{1, \ldots, n\}$. We denote by Γ_D the quotient of Γ_n by the subgroup generated by $(0, \varepsilon_{i+n\mathbf{Z}}) - (\frac{1}{2}\nu_i, 0)$, where $(i, \nu_i) \in D$. We identify $\frac{1}{2}\mathbf{Z}$ with the image of $\frac{1}{2}\mathbf{Z} \times 0$ in Γ_D. We define a partial order on Γ_D by $h \geqslant g$ if hg^{-1} is in $\frac{1}{2}\mathbf{Z}_{\geqslant 0}$. We denote by $\deg_D(\sigma)$ the image of $\deg(\sigma)$ in Γ_D.

Given E a subset of $\{1, \ldots, n\}$, we put $E^+ = \{(i, 1) \mid i \in E\}$.

By Lemmas 6.2.1 and 6.2.5, we obtain a Γ_D-filtration on $\mathcal{S}_n$ by defining

$$\mathrm{Hom}_{\mathcal{S}_n^{\geqslant g}}(I, J) = \{\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I, J) \mid \deg_D(\sigma) \geqslant g\}.$$

It follows from Lemma 6.2.5 that the pointed category $\mathcal{H}_n$ is isomorphic to the graded pointed category associated to the Γ_D-filtration of $\mathcal{S}_n$ (after forgetting the Γ_D-grading to $\mathbf{Z}$).

Note that if $D = \varnothing$, then $\Gamma_D = \Gamma_n$, $\deg_D = \deg$ and the $\mathbf{Z}_{\leqslant 0}$ grading on $\mathcal{H}_n$ given by the length can be recovered from the Γ_n-grading by using the quotient map $\Gamma_n \to \Gamma_n/\Gamma'_n = \frac{1}{2}\mathbf{Z}$.

This quotient map provides a $\mathbf{Z}$-grading on the Γ_n-graded pointed category associated to the Γ_n-filtration of $\mathcal{S}_n$. This $\mathbf{Z}$-graded pointed category is isomorphic to $\mathcal{H}_n$.

Remark 6.2.6. The bilinear form $R_n \times R_n \to \mathbf{Z}$ obtained from $\langle -, - \rangle$ by composing with the quotient map $L_n \mapsto \mathbf{Z}$, $\varepsilon_i \mapsto 1$ is antisymmetric.

Let $D = \{1, \ldots, n\} \times \{+1\}$. The group Γ_D is a central extension of R_n by $\mathbf{Z}$ using the antisymmetric form above. It restricts to the antisymmetric form L of [LiOzTh1, §3.3.1].

6.2.5. *Differential.* Given $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I, J)$, let $D(\sigma)$ be the set of pairs $(i_1, i_2) \in L(\sigma)$ such that

- $i_2 - i_1 < n$ or $\sigma(i_1) - \sigma(i_2) < n$ and
- given $i \in \tilde{I}$ with $i_1 < i < i_2$, we have $\sigma(i_1) < \sigma(i)$ or $\sigma(i) < \sigma(i_2)$.

We put $\tilde{D}(\sigma) = D(\sigma) \cap \tilde{L}(\sigma)$. The diagonal action of $n\mathbf{Z}$ on $L(\sigma)$ preserves $D(\sigma)$ and we have a canonical bijection $\tilde{D}(\sigma) \xrightarrow{\sim} D(\sigma)/n\mathbf{Z}$.

Given $(i_1, i_2) \in L(\sigma)$, we put $\sigma^{i_1,i_2} := \sigma \circ s_{i_1,i_2}$.

We define a partial order on $\mathrm{Hom}_{\mathcal{S}_n}(I, J)$ as the transitive closure of $\sigma' < \sigma$ if $\sigma' = \sigma^{i_1,i_2}$ for some $(i_1, i_2) \in D(\sigma)$.

When $I = J = \mathbf{Z}/n$, this coincides with the extended Chevalley-Bruhat order on $\hat{\mathfrak{S}}_n$ by Lemma 3.2.4 and given $(i_1, i_2) \in L(\sigma)$, we have $\sigma^{i_1,i_2} < \sigma$ (Lemma 3.2.3). The next lemma shows that this holds for general maps in $\mathcal{S}_n$.

Lemma 6.2.7. *Let $\sigma, \sigma' \in \mathrm{Hom}_{\mathcal{S}_n}(I, J)$. Given $\tau \in \mathrm{Hom}_{\mathcal{S}_n}(J, I)$ with $\ell(\tau) = 0$, we have $\sigma' < \sigma$ if and only if $\tau \circ \sigma' < \tau \circ \sigma$ if and only if $\sigma' \circ \tau < \sigma \circ \tau$.*

Proof. Note that τ is an increasing bijection since $\ell(\tau) = 0$. We have $D(\tau \circ \sigma) = D(\sigma)$ and given $(i_1, i_2) \in D(\sigma)$, we have $(\tau \circ \sigma)^{i_1,i_2} = \tau \circ \sigma^{i_1,i_2}$. This shows the first equivalence. The second equivalence follows from the fact that $D(\sigma \circ \tau) = (\tau^{-1} \times \tau^{-1})(D(\sigma))$ and given $(i_1, i_2) \in D(\sigma)$, we have $(\sigma \circ \tau)^{\tau^{-1}(i_1),\tau^{-1}(i_2)} = \sigma^{i_1,i_2} \circ \tau$. $\square$

Lemma 6.2.8. *Given $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I, J)$, there is a bijection*

$$\tilde{D}(\sigma) \xrightarrow{\sim} \{\sigma' \in \mathrm{Hom}_{\mathcal{S}_n}(I, J) \mid \sigma' < \sigma,\ \ell(\sigma') = \ell(\sigma) - 1\},\ (i_1, i_2) \mapsto \sigma^{i_1,i_2}.$$

Note that

$$\{\sigma' \in \mathrm{Hom}_{\mathcal{S}_n}(I,J) \mid \sigma' < \sigma,\ \ell(\sigma') = \ell(\sigma)-1\} = \{\sigma' \in \mathrm{Hom}_{\mathcal{S}_n}(I,J) \mid \sigma' < \sigma,\ \deg(\sigma') = \deg(\sigma)+1\}.$$

Given $(i_1,i_2) \in L(\sigma)$*, we have* $(i_1,i_2) \in D(\sigma)$ *if and only if* $\deg_D(\sigma) = \deg_D(\sigma^{i_1,i_2}) - 1$ *for some subset (equivalently, for any subset)* D *of* $\{1,\ldots,n\} \times \{\pm 1\}$ *that embeds in its projection on* $\{1,\ldots,n\}$*.*

Proof. Let $\tau \in \mathrm{Hom}_{\mathcal{S}_n}(J,I)$ be an increasing bijection. We have $D(\tau\circ\sigma) = D(\sigma)$ and

$$\{\sigma'' \in \mathrm{End}_{\mathcal{S}_n}(I) \mid \sigma'' < \tau\circ\sigma,\ \ell(\sigma'') = \ell(\tau\circ\sigma)-1\} = \{\tau\circ\sigma' \mid \sigma' \in \mathrm{Hom}_{\mathcal{S}_n}(I,J),\ \sigma' < \sigma,\ \ell(\sigma') = \ell(\sigma)-1\}$$

by Lemma 6.2.7. Since the first statement of the lemma holds for $\tau\circ\sigma$ by Lemma 3.2.4, it holds for σ.

The other statements follow from Lemmas 6.2.4 and 6.2.5. □

Lemma 6.2.9. *Consider* $\sigma'' \in \mathrm{Hom}_{\mathcal{S}_n}(I,J)$ *and* $\sigma' \in \mathrm{Hom}_{\mathcal{S}_n}(J,K)$ *and let* $\sigma = \sigma'\sigma''$*. Assume* $\ell(\sigma) = \ell(\sigma') + \ell(\sigma'')$*.*

Let $(i_1,i_2) \in D(\sigma)\backslash(D(\sigma)\cap D(\sigma''))$*. Let* $\alpha'' = \sigma'' s_{i_1,i_2}$ *and* $\alpha'' = (\sigma')^{\sigma''(i_1),\sigma''(i_2)}$*. We have* $\sigma = \alpha'\alpha''$ *and* $\ell(\sigma) = \ell(\alpha') + \ell(\alpha'')$*.*

Proof. Assume first $I = J = K$. The lemma follows in that case from Lemmas 3.2.4 and 3.2.2.

Consider now the general case. There are increasing bijections $\tau : J \to I$ and $\tau' : K \to J$. We have $D(\sigma) = \tau^{-1}(D(\tau'\sigma\tau))$ and $D(\sigma'') = \tau^{-1}(D(\sigma''\tau))$ (proof of Lemma 5.4.7). The lemma follows now from the previous case applied to the decomposition $\tau'\sigma\tau = (\tau'\sigma')(\sigma''\tau)$. □

Consider $\sigma \in \mathrm{Hom}_{\mathcal{H}_n}(I,J)$ non-zero. We put

$$d(\sigma) = \sum_{(i_1,i_2)\in\tilde{D}(\sigma)} \sigma^{i_1,i_2} \in \mathrm{Hom}_{\mathbf{F}_2[\mathcal{H}_n]}(I,J).$$

Proposition 6.2.10. *The maps* d *equip the* $\mathbf{F}_2$*-linear* Γ_n*-graded category* $\mathbf{F}_2[\mathcal{H}_n]$ *with a differential* Γ_n*-graded structure, hence equip* $\mathcal{H}_n$ *with a differential* Γ_n*-graded pointed structure.*

Given $I \subset \mathbf{Z}/n$*, the morphism* F_I *induces an isomorphism of differential* $\mathbf{Z}$*-graded pointed monoids*

$$\hat{\mathfrak{S}}^{\mathrm{nil}}_{|I|} \xrightarrow{\sim} \mathrm{End}_{\mathcal{H}_n}(I).$$

Proof. Note that Lemma 6.2.8 shows that d is homogeneous of degree 1. The compatibility of d with F_I follows from Lemma 3.2.4.

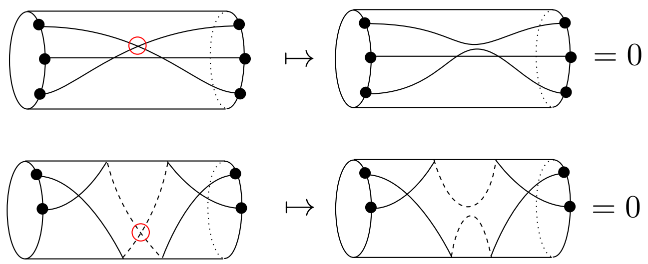
Consider now $\sigma \in \mathrm{Hom}_{\mathcal{H}_n}(I,J)$ non-zero. There exists $\tau \in \mathrm{Hom}_{\mathcal{H}_n}(J,I)$ with $\ell(\tau) = 0$. We have $d(\tau\circ\sigma) = \tau\circ d(\sigma)$, hence $d^2(\tau\circ\sigma) = \tau\circ d^2(\sigma)$. The compatibility of F_I with d shows that $d^2(\tau\circ\sigma) = 0$. Since τ is invertible, we deduce that $d^2(\sigma) = 0$.

Consider finally $\sigma' \in \mathrm{Hom}_{\mathcal{H}_n}(J,K)$ and fix $\tau' \in \mathrm{Hom}_{\mathcal{H}_n}(K,J)$ with $\ell(\tau') = 0$. We have $d(\tau'\circ\sigma'\circ\sigma\circ\tau) = \tau'\circ d(\sigma'\circ\sigma)\circ\tau$ and it follows from the compatibility of F_J with d that

$$\begin{aligned}
d(\tau'\circ\sigma'\circ\sigma\circ\tau) &= F_J\big(d(F_J^{-1}(\tau'\circ\sigma'\circ\sigma\circ\tau))\big) = F_J\big(d(F_J^{-1}(\tau'\circ\sigma')\circ F_J^{-1}(\sigma\circ\tau))\big) \\
&= F_J\big(d(F_J^{-1}(\tau'\circ\sigma'))\circ F_J^{-1}(\sigma\circ\tau)\big) + F_J\big(F_J^{-1}(\tau'\circ\sigma'))\circ d(F_J^{-1}(\sigma\circ\tau)\big) \\
&= d(\tau'\circ\sigma')\circ\sigma\circ\tau + \tau'\circ\sigma'\circ d(\sigma\circ\tau).
\end{aligned}$$

□

Example 6.2.11. Elements of $\tilde{L}(\sigma)$ correspond to intersections in a representing diagram. Given $(i_1, i_2) \in \tilde{L}(\sigma)$, the element σ^{i_1,i_2} correspond to the diagram obtained by smoothing the intersection point corresponding to (i_1, i_2). If $(i_1, i_2) \notin \tilde{D}(\sigma)$, the element associated to the diagram will vanish in $\mathcal{H}_n$.



6.2.6. *Change of* n. Fix a positive integer $n' \leqslant n$ and an increasing injection $\alpha : \{1, \ldots, n'\} \hookrightarrow \{1, \ldots, n\}$. We extend α to an increasing injection $\mathbf{Z} \to \mathbf{Z}$ by $\alpha(r + dn') = \alpha(r) + dn$ for $r \in \{1, \ldots, n'\}$ and $d \in \mathbf{Z}$.

Consider $\alpha : \{1, \ldots, n'\} \hookrightarrow \{1, \ldots, n\}$ an increasing injection as in §6.2.1. We define two injective morphisms of groups

$$R_\alpha : R_n \to R_{n'},\ \alpha_{i+n'\mathbf{Z}} \mapsto \alpha_{\alpha(i),\alpha(i+1)} \text{ and } L_\alpha : L_n \to L_{n'},\ \varepsilon_{i+n'\mathbf{Z}} \mapsto \varepsilon_{i+n\mathbf{Z}}$$

for $1 \leqslant i \leqslant n'$. We have commutative diagrams

$$\begin{tikzcd} R_{n'} \arrow[r, "\rho"] \arrow[d, "R_\alpha"'] & L_{n'} \arrow[d, "L_\alpha"] \\ R_n \arrow[r, "\rho"'] & L_n \end{tikzcd} \quad \begin{tikzcd} R_{n'} \times L_{n'} \arrow[r, "\cdot"] \arrow[d, "R_\alpha \times L_\alpha"'] & L_{n'} \arrow[d, "L_\alpha"] \\ R_n \times L_n \arrow[r, "\cdot"'] & L_n \end{tikzcd} \quad \begin{tikzcd} R_{n'} \times R_{n'} \arrow[r, "\langle -,- \rangle"] \arrow[d, "R_\alpha \times R_\alpha"'] & L_{n'} \arrow[d, "L_\alpha"] \\ R_n \times R_n \arrow[r, "\langle -,- \rangle"'] & L_n \end{tikzcd}$$

As a consequence, we have two injective morphisms of groups

$$\Gamma'_\alpha = L_\alpha \times R_\alpha : \Gamma'_{n'} \to \Gamma'_n \text{ and } \Gamma_\alpha = \mathrm{id} \times L_\alpha \times R_\alpha : \Gamma_{n'} \to \Gamma_n,$$

the last of which induces an injective morphism of groups $\Gamma_D \to \Gamma_{(\alpha\times\mathrm{id})(D)}$, for D be a subset of $\{1, \ldots, n'\} \times \{\pm 1\}$ that embeds in its projection on $\{1, \ldots, n'\}$.

We define now a fully faithful functor $F = F_\alpha : \mathcal{S}_{n'} \to \mathcal{S}_n$. Given I a subset of $\mathbf{Z}/n'$, we define $F(I)$ to be the image of $\alpha(\tilde{I} \cap [1, n'])$ in $\mathbf{Z}/n$. Given $\sigma \in \mathrm{Hom}_{\mathcal{S}_{n'}}(I, J)$, we put $F(\sigma) = \alpha \circ \sigma \circ \alpha^{-1}$.

Note that the isomorphism of group $\hat{\mathfrak{S}}_{n'} = \mathrm{End}_{\mathcal{S}_{n'}}(\mathbf{Z}/n') \xrightarrow{\sim} \mathrm{End}_{\mathcal{S}_n}(F(\mathbf{Z}/n'))$ induced by F coincides with $F_{F(\mathbf{Z}/n')}$ defined in §6.2.1.

As a consequence, F_α induces a fully faithful graded functor $\mathcal{H}_{n'} \to \mathcal{H}_n$.

Lemma 6.2.12. *Given $n' \leqslant n$ and $\alpha : \{1, \ldots, n'\} \hookrightarrow \{1, \ldots, n\}$ an increasing injection, the functor F_α induces a differential Γ_n-graded pointed functor $\mathcal{H}_{n'} \to \mathcal{H}_n$.*

Proof. Let $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I, J)$. We have $L(F_\alpha(\sigma)) = (\alpha \times \alpha)(L(\sigma))$, hence $\ell(F_\alpha(\sigma)) = \ell(\sigma)$. We have $R_\alpha([\![\sigma]\!]) = [\![F_\alpha(\sigma)]\!]$, hence $L_\alpha(m(\sigma)) = m(F_\alpha(\sigma))$. We deduce that $\Gamma_\sigma(\deg(\sigma)) = \deg(F_\alpha(\sigma))$.

We have $D(F_\alpha(\sigma)) = (\alpha \times \alpha)(D(\sigma))$ and $F_\alpha(s_{i_1,i_2}) = s_{\alpha(i_1),\alpha(i_2)}$ for $i_1, i_2 \in \tilde{I}$ with $i_1 - i_2 \notin n\mathbf{Z}$, hence F_α is compatible with d. $\square$

6.3. **Positive and finite variants.**

6.3.1. *Constructions.* We define now positive and finite variants of the categories.

We define $\hat{\mathfrak{S}}_n^{++}$ to be the submonoid of $\hat{\mathfrak{S}}_n$ of elements σ such that $\sigma(r) \geqslant r$ for all $r \in \mathbf{Z}$.

Let $? \in \{+, ++\}$. We define $\mathcal{S}_n^?$ to be the Γ_n-filtered subcategory of $\mathcal{S}_n$ with same objects as $\mathcal{S}_n$ and with maps those $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I, J)$ such that $\sigma(r) > 0$ if $? = +$ (resp. $\sigma(r) \geqslant r$ if $? = ++$) for all $r \in \tilde{I} \cap \mathbf{Z}_{>0}$. We define $\mathcal{H}_n^?$ as the Γ_n-graded pointed subcategory of $\mathcal{H}_n$ with same objects as $\mathcal{H}_n$ and non-zero maps those of $\mathcal{S}_n^?$. Note that there is a canonical isomorphism of Γ_n-graded pointed categories $\mathrm{gr}\mathcal{S}_n^? \xrightarrow{\sim} \mathcal{H}_n^?$.

Note that the usual symmetric group $\mathfrak{S}_n$ identifies with the subgroup of $\hat{\mathfrak{S}}_n$ of elements σ such that $\sigma(\{1, \ldots, n\}) = \{1, \ldots, n\}$. The subalgebra of $\hat{H}_n$ generated by $T_1, \ldots, T_{n-1}$ is isomorphic to H_n.

We denote by $\mathcal{S}_n^f$ the Γ_n-filtered subcategory of $\mathcal{S}_n$ with same objects as $\mathcal{S}_n$ and with maps those $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I, J)$ such that $\sigma(r) \in \{1, \ldots, n\}$ for all $r \in \tilde{I} \cap \{1, \ldots, n\}$. We denote by $\mathcal{H}_n^f$ the corresponding Γ_n-graded pointed subcategory of $\mathcal{H}_n$. There is a canonical isomorphism of Γ_n-graded pointed categories $\mathrm{gr}\mathcal{S}_n^f \xrightarrow{\sim} \mathcal{H}_n^f$.

We have also subcategories $\mathcal{S}_n^{f++} = \mathcal{S}_n^f \cap \mathcal{S}_n^{++}$ of $\mathcal{S}_n$ and $\mathcal{H}_n^{f++} = \mathcal{H}_n^f \cap \mathcal{H}_n^{++}$ of $\mathcal{H}_n$.

Lemma 6.3.1. *$\mathcal{H}_n^f$, $\mathcal{H}_n^+$, $\mathcal{H}_n^{++}$ and $\mathcal{H}_n^{f++}$ are differential Γ_n-graded pointed subcategories of $\mathcal{H}_n$.*

Proof. Let $\sigma \in \mathrm{Hom}_{\mathcal{H}_n^f}(I, J)$. There is $\tau \in \mathrm{Hom}_{\mathcal{H}_n^f}(J, I)$ with $\ell(\tau) = 0$. We have $d(\tau \circ \sigma) = \tau \circ d(\sigma)$. The isomorphism $\hat{H}_n \xrightarrow{\sim} \mathrm{End}_{\mathbf{F}_2[\mathcal{H}_n]}(\mathbf{Z}/n)$ given by Proposition 6.2.10 restricts to an isomorphism of differential graded algebras $H_n \xrightarrow{\sim} \mathrm{End}_{\mathbf{F}_2[\mathcal{H}_n^f]}(\mathbf{Z}/n)$. It follows that $d(\tau \circ \sigma) \in \mathbf{F}_2[\mathcal{H}_n^f]$, hence $d(\sigma) \in \mathbf{F}_2[\mathcal{H}_n^f]$. So, $\mathbf{F}_2[\mathcal{H}_n^f]$ is a differential subcategory of $\mathbf{F}_2[\mathcal{H}_n]$.

One shows similarly that $\mathbf{F}_2[\mathcal{H}_n^+]$ is a differential subcategory of $\mathbf{F}_2[\mathcal{H}_n]$.

Let $\sigma \in \mathrm{Hom}_{\mathcal{H}_n^{++}}(I, J)$. Let $(i_1, i_2) \in D(\sigma)$ and let $\sigma' = \sigma^{i_1, i_2}$. Given $i \in \tilde{I}$, we have $\sigma'(i) = \sigma(i)$ if $i \notin (i_1 + n\mathbf{Z}) \cup (i_2 + n\mathbf{Z})$, while

$$\sigma'(i_1) = \sigma(i_2) \geqslant i_2 > i_1 \text{ and } \sigma'(i_2) = \sigma(i_1) > \sigma(i_2) \geqslant i_2.$$

It follows that $\sigma' \in \mathrm{Hom}_{\mathcal{H}_n^{++}}(I, J)$, hence $d(\sigma) \in \mathbf{F}_2[\mathcal{H}_n^{++}]$. □

We extend all previous constructions to the case $n = 0$ by setting $\hat{\mathfrak{S}}_0 = \hat{\mathfrak{S}}_0^{++} = \mathfrak{S}_0 = 1$, $H_0 = \hat{H}_0 = \mathbf{F}_2$, $\mathcal{S}_0 = \mathcal{S}_0^{++} = \mathcal{S}_0^f$ is the category with one object $\varnothing$ and one map and $\mathcal{H}_0 = \mathcal{H}_0^f = \mathcal{H}_0^{++}$ is its associated pointed category.

6.3.2. *Lipshitz-Ozsváth-Thurston's strands algebras.* Fix $n \geqslant 1$. The strands algebra with n places [LiOzTh1, Definition 3.2] is the differential algebra

$$\mathcal{A}(n) = \mathrm{End}_{\mathrm{add}(\mathbf{F}_2[\mathcal{H}_n^{f++}])}\Big(\bigoplus_{I \subset \mathbf{Z}/n} I\Big).$$

The group $G'(n)$ of [LiOzTh1, Definition 3.33] is an index 2 subgroup of $\Gamma_{[1,n]^+}$ and the extension to $\Gamma_{[1,n]^+}$ of the $G'(n)$-grading on $\mathcal{A}(n)$ [LiOzTh1, Definition 3.38] coincides with our $\Gamma_{[1,n]^+}$-grading.

7. Strand algebras

7.1. 1-dimensional spaces.

7.1.1. *Definitions.* A manifold is defined to be a topological manifold with boundary with finitely many connected components, all of which have the same dimension. A 1-dimensional manifold is a finite disjoint union of copies of S^1, $\mathbf{R}$, $\mathbf{R}_{\geqslant 0}$ and $[0,1]$.

Given a point x of a topological space X, we put $C(x) = C_X(x) = \lim_U \pi_0(U - \{x\})$, where U runs over the set of open neighbourhoods of x. If X' is a subspace of X containing an open neighbourhood of x, then we have a canonical bijection $C_{X'}(x) \xrightarrow{\sim} C_X(x)$ and we identify those two sets.

We put $T(X) = \coprod_{x \in X} C(x)$ and we denote by $\mathrm{pt} : T(X) \to X$ the canonical map.

Definition 7.1.1. *We define a* 1-dimensional space *to be a topological space that is homeomorphic to the complement of a finite set of points in a* 1*-dimensional finite CW-complex, and that has no connected component that is a point.*

Given E a finite subset of $S^1 = \{z \in \mathbf{C} \mid ||z|| = 1\}$, we put $\mathrm{St}(E) = \bigcup_{e \in E} \mathbf{R}_{\geqslant 0} e$ and $\mathrm{St}^\circ(E) = \mathrm{St}(E) - \{0\}$. These are 1-dimensional spaces. Given $n \geqslant 1$, we put $\mathrm{St}(n) = \mathrm{St}(\{e^{2i\pi r/n}\}_{0 \leqslant r < n})$.

Let X be a 1-dimensional space. There is a finite subset E of X such that $X - E$ is homeomorphic to a finite disjoint union of copies of $\mathbf{R}$.

Let $x \in X$. If U is a small enough connected open neighbourhood of x, then there is a homeorphisms $U \xrightarrow{\sim} \mathrm{St}(n_x)$, $x \mapsto 0$ for some $n_x = n_{x,X} \geqslant 1$. In addition, we have a canonical bijection $C(x) \xrightarrow{\sim} \pi_0(U - \{x\})$ and we identify those two sets of cardinality n_x.

We define the boundary $\partial X = \{x \in X \mid n_x = 1\}$. We put $X_{exc} = \{x \in X | n_x \geqslant 3\}$.

Definition 7.1.2. *We say X is* non-singular *if $X_{exc} = \varnothing$. Note that $X - X_{exc}$ is a non-singular* 1*-dimensional space.*

A 1-dimensional space is non-singular if and only if it is a 1-dimensional manifold.

Definition 7.1.3. *We say that an open neighbourhood U of $x \in X$ is* small *if it is homeomorphic to* $\mathrm{St}(n_x)$*, if $|\overline{U} - U| = n_x$ and if $n_{x'} = 2$ for all $x' \in \overline{U} - \{x\}$.*

Note that every point of a 1-dimensional space admits a small open neighbourhood.

7.1.2. *Morphisms.* Let X' be a 1-dimensional space and let $f : X \to X'$ be a continuous map. Let X'_f be the set of points $x' \in X'$ such that there is no open neighbourhood U of x' with the property that $f_{|f^{-1}(U)} : f^{-1}(U) \to U$ is a homeomorphism. Let $X_f = f^{-1}(X'_f)$.

Lemma 7.1.4. *The following conditions are equivalent:*

1. *there is a finite subset E_1 of X such that $f(X - E_1)$ is open in X' and $f_{|X-E_1} : X - E_1 \to f(X - E_1)$ is a homeomorphism*
2. *X_f is finite*
3. *there is a finite subset E_2 of X such that $f_{|X-E_2} : X - E_2 \to f(X - E_2)$ is a homeomorphism*
4. *given $x \in X$, there is a finite subset E_x of $X - \{x\}$ such that $f_{|X-E_x}$ is injective*
5. *there is a finite subset E_3 of X such that $f_{|X-E_3}$ is injective.*

Proof. The implication (1) $\Rightarrow$ (2) follows from the fact that $X_f \subset f^{-1}(f(E_1))$. For the implication (2) $\Rightarrow$ (3), take $E_2 = X_f$. For (3) $\Rightarrow$ (4), take $E_x = (X - \{x\}) \cap (f^{-1}(f(x)) \cup E_2)$. The implication (4) $\Rightarrow$ (5) is immediate.

Let us show that (5) $\Rightarrow$ (1). Note first that an injective continuous map $\mathbf{R} \to \mathbf{R}$ is open and a homeomorphism onto its image. It follows that the implication holds when X and X' are homeomorphic to $\mathbf{R}$ and $E_3 = \varnothing$.

Consider now the general case. There is a finite subset E_1 of X containing E_3 such that $X - E_1$ and $X' - f(E_1)$ are homeomorphic to a finite disjoint union of copies of $\mathbf{R}$. By the discussion above, the restriction of f to a connected component of $X - E_1$ is open and a homeomorphism onto its image, so the same holds for $f_{|X-E_1}$.

□

Definition 7.1.5. *We say that f is a* morphism of 1-dimensional spaces *if it satisfies any of the equivalent conditions of Lemma 7.1.4.*

Note that

- a composition of morphisms of 1-dimensional spaces is a morphism of 1-dimensional spaces
- a morphism of 1-dimensional spaces is invertible if and only if it is a homeomorphism.

Definition 7.1.6. *We define a* 1-dimensional subspace *of X to be a subspace Y with only finitely many connected components, none of which are points.*

Let us record some basic facts on subspaces.

Lemma 7.1.7. (1) *The image of a morphism of 1-dimensional spaces is a 1-dimensional subspace.*
(2) *If Y is a 1-dimensional subspace of X, then Y is a 1-dimensional space and the inclusion map $Y \hookrightarrow X$ is a morphism of 1-dimensional spaces.*
(3) *Let $f : X \to X'$ be a morphism of 1-dimensional spaces and Y' be a 1-dimensional subspace of X'. Let F be the set of connected components of $f^{-1}(Y')$ that are points. Then F is finite, $Y = f^{-1}(Y') - F$ is a 1-dimensional subspace of X and $f_{|Y} : Y \to Y'$ is a morphism of 1-dimensional spaces.*

We now provide a description of the local structure of morphisms of 1-dimensional spaces.

Lemma 7.1.8. *Let $f : X \to X'$ be a morphism of 1-dimensional spaces and let $x' \in X'$. Let $r = |f^{-1}(x')|$. There exists*

- *a small open neighbourhood U of x' and a homeomorphism $a : \mathrm{St}(n_{x'}) \xrightarrow{\sim} U$ with $a(0) = x'$,*
- *a family of disjoint subsets $I_0, I_1, \ldots, I_r$ of $\{e^{2i\pi d/n_{x'}}\}_{0 \leqslant d < n_{x'}}$ with $I_l \neq \varnothing$ for $1 \leqslant l \leqslant r$ and a homeomorphism $b : \mathrm{St}^\circ(I_0) \sqcup \mathrm{St}(I_1) \sqcup \cdots \sqcup \mathrm{St}(I_r) \xrightarrow{\sim} f^{-1}(U)$*

such that $f_{|f^{-1}(U)} = a \circ g \circ b^{-1}$ where $g : \mathrm{St}^\circ(I_0) \sqcup \mathrm{St}(I_1) \sqcup \cdots \sqcup \mathrm{St}(I_r) \to \mathrm{St}(n_{x'})$ is the map whose restriction to $\mathrm{St}^\circ(I_0)$ and $\mathrm{St}(I_l)$ is the inclusion map.

In particular, the canonical map, still denoted by $f : T(X) \to T(X')$ is injective and $f(X_{exc}) \subset X'_{exc}$.

Proof. Let E be a finite subset of X such that $f^{-1}(f(E)) = E$, $f(X - E)$ is open in X' and $f_{|X-E} : X - E \to f(X - E)$ is a homeomorphism. Let U be a small open neighbourhood of x' such that $U - \{x'\} \subset X' - f(E)$. Note that $f(X) \cap (U - \{x'\})$ is open in X' and $f_{|f^{-1}(U-\{x'\})} : f^{-1}(U - \{x'\}) \to f(X) \cap (U - \{x'\})$ is a homeomorphism.

Let L be a connected component of $U - \{x'\}$. Note that $f(f^{-1}(L))$ is an open 1-dimensional subspace of L and L is homeomorphic to $\mathbf{R}$. By shrinking U, we can assume that $f^{-1}(L) = \varnothing$ or $f(f^{-1}(L)) = L$. So, we can assume that given L a connected component of $U - \{x'\}$ with $f^{-1}(L) \neq \varnothing$, the map $f_{|f^{-1}(L)} : f^{-1}(L) \to L$ is a homeomorphism.

Since U is small, there is a homeomorphism $a : \mathrm{St}(n_{x'}) \xrightarrow{\sim} U,\ 0 \mapsto x'$. Let $\{x_1, \ldots, x_r\} = f^{-1}(x')$ and define

$$I_l = \{e^{2i\pi d/n_{x'}} | 0 \leqslant d < n_{x'},\ x_l \in \overline{f^{-1}(a(\mathbf{R}_{>0}e^{2i\pi d/n_{x'}}))}\}$$

for $l \in \{1, \ldots, r\}$. Define

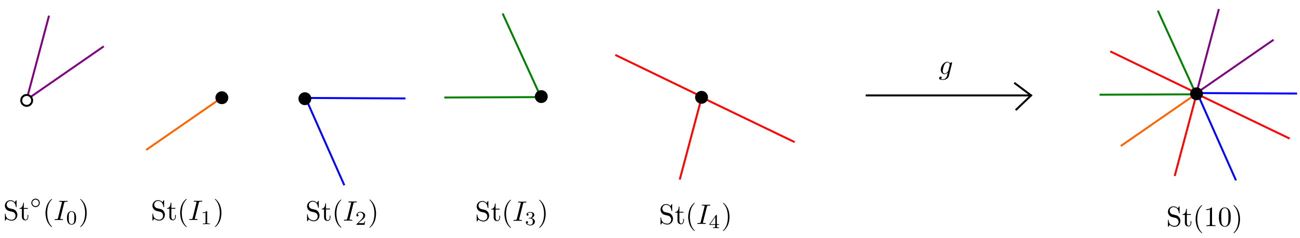
$$I_0 = \{e^{2i\pi d/n_{x'}} | 0 \leqslant d < n_{x'},\ f^{-1}(a(\mathbf{R}_{>0}e^{2i\pi d/n_{x'}})) \neq \varnothing,\ f^{-1}(x') \cap \overline{f^{-1}(a(\mathbf{R}_{>0}e^{2i\pi d/n_{x'}}))} = \varnothing\}.$$

Note that a restricts to a homeomorphism $\mathrm{St}(\bigcup_{0 \leqslant l \leqslant r} I_r) \xrightarrow{\sim} f(f^{-1}(U))$.

The composition $a \circ g$ takes values in $f(f^{-1}(U))$. Its restriction to $\mathrm{St}^\circ(I_0)$ defines a homeomorphism $\mathrm{St}^\circ(I_0) \xrightarrow{\sim} a(\mathrm{St}^\circ(I_0))$. Since $f_{|f^{-1}(a(\mathrm{St}^\circ(I_0)))} : f^{-1}(a(\mathrm{St}^\circ(I_0))) \to a(\mathrm{St}^\circ(I_0))$ is a homeomorphism, we have a homeomorphism $b_0 = (f_{|f^{-1}(a(\mathrm{St}^\circ(I_0)))})^{-1} \circ (a \circ g)_{|\mathrm{St}^\circ(I_0)} : \mathrm{St}^\circ(I_0) \xrightarrow{\sim} f^{-1}(a(\mathrm{St}^\circ(I_0)))$.

Consider now $l \in \{1, \ldots, r\}$. We construct as above a homeomorphism $b'_l : \mathrm{St}^\circ(I_l) \xrightarrow{\sim} f^{-1}(a(\mathrm{St}^\circ(I_l)))$ such that $(a \circ g)_{|\mathrm{St}^\circ(I_l)} = f \circ b'_l$. The homeomorphism b'_l extends uniquely to a homeomorphism $b_l : \mathrm{St}(I_l) \to f^{-1}(a(\mathrm{St}(I_l)))$. We define $b = b_0 \sqcup b_1 \sqcup \cdots \sqcup b_r$. We have $f_{|f^{-1}(U)} = a \circ g \circ b^{-1}$. $\square$

Example 7.1.9. Here is an example of map g as in Lemma 7.1.8:



The next two results follow immediately from Lemma 7.1.8.

Lemma 7.1.10. *Let Y be a 1-dimensional subspace of X and let $y \in Y$. Let $I = \{e^{2i\pi d/n_{y,X}}\}_{0 \leqslant d < n_{y,Y}}$. There is an open neighbourhood U of y in X and a homeomorphism $\mathrm{St}(n_{y,X}) \xrightarrow{\sim} U,\ 0 \mapsto y$ whose restriction to $\mathrm{St}(I)$ is a homeomorphism $\mathrm{St}(I) \xrightarrow{\sim} U \cap Y$. We have a commutative diagram*

$$\begin{tikzcd}
\mathrm{St}(n_{y,X}) \arrow[r, "\sim"] & U \\
\mathrm{St}(I) \arrow[u, hook] \arrow[r, "\sim"'] & U \cap Y \arrow[u, hook]
\end{tikzcd}$$

Lemma 7.1.11. *Let $f : X \to X'$ be a surjective morphism of 1-dimensional spaces. It induces a bijection $T(X) \xrightarrow{\sim} T(X')$.*

7.1.3. *Quotients.* Let $\tilde{X}$ be a 1-dimensional space and $\sim$ be an equivalence relation on $\tilde{X}$.

Definition 7.1.12. *We say that $\sim$ is a* finite relation *if the set of points that are not alone in their equivalence class is finite.*

Assume $\sim$ is a finite relation. Let $q : \tilde{X} \to X = \tilde{X}/\sim$ be the quotient map. Note that X is a 1-dimensional space with

$$X_{exc} = q(\tilde{X}_{exc}) \cup \{x \in X | \ |q^{-1}(x)| > 2\} \cup \{x \in X \mid |q^{-1}(x)| = 2, \ q^{-1}(x) \not\subset \partial\tilde{X}\}$$

and q is a morphism of 1-dimensional spaces.

Given $x \in X$, the quotient map induces a bijection $q : \coprod_{\tilde{x}\in q^{-1}(x)} C(\tilde{x}) \xrightarrow{\sim} C(x)$.

Quotients have a universal property. In particular, we have the following result.

Lemma 7.1.13. *Let $f : X \to X'$ be a morphism of 1-dimensional spaces. Define an equivalence relation on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. This defines a finite relation on X and f factors uniquely as a composition $f = \bar{f} \circ q$ where $\bar{f} : X/\sim \to X'$ is a morphism of 1-dimensional spaces and $q : X \to X/\sim$ is the quotient map.*

The next lemma shows that 1-dimensional spaces X can be viewed (non uniquely) as 1-dimensional manifolds with a finite relation.

Lemma 7.1.14. *Given X a 1-dimensional space, there is a 1-dimensional manifold $\hat{X}$ with a finite relation $\sim$ and an isomorphism $f : \hat{X}/\sim \xrightarrow{\sim} X$ such that $f(\hat{X}_f) = X_{exc}$.*

Proof. Fix, for every $x \in X_{exc}$, a small open neighbourhood U_x of x and a homeomorphism $f_x : U_x \xrightarrow{\sim} \mathrm{St}(E_x)$, where E_x is a finite subset of S^1. We choose now an equivalence relation on E_x whose classes have cardinality at most 2. Note that f_x induces a bijection between $C(x)$ and E_x, hence the equivalence relation can be viewed on $C(x)$.

Define $\hat{U}_x = \coprod_{E'\in E_x/\sim} \mathrm{St}(E')$. The map f_x provides an open embedding

$$U_x - \{x\} \xrightarrow{\sim} \mathrm{St}^\circ(E_x) \xrightarrow{\sim} \coprod_{E'\in E_x/\sim} \mathrm{St}^\circ(E') \hookrightarrow \hat{U}_x.$$

We put

$$\hat{X} = (X - X_{exc}) \coprod_{(\coprod_{x\in X_{exc}}(U_x - \{x\}))} \Big(\coprod_{x\in X_{exc}} \hat{U}_x\Big).$$

Note that $\hat{X}$ is a 1-dimensional manifold. Let $q : \hat{X} \to X$ be the canonical map: it identifies X with the quotient of $\hat{X}$ by the equivalence relation given by $\hat{x}_1 \sim \hat{x}_2$ if $q(\hat{x}_1) = q(\hat{x}_2)$. Up to isomorphism, $\hat{X}$ depends only on the choice of an equivalence relation on $C(x)$ for $x \in X_{exc}$. $\square$

7.1.4. *Paths.*

Lemma 7.1.15. *Let E be a finite subset of X and γ be a path in X such that for all connected components I of $[0,1]\backslash\gamma^{-1}(E)$, the restriction of γ to $\bar{I}$ is nullhomotopic. Then γ is nullhomotopic.*

Proof. Given $e \in E$, let U_e be a connected and simply connected open neighborhood of e. Choose U_e small enough so that $U_e \cap U_{e'} = \varnothing$ for $e \neq e'$. Let $U = \bigcup_{e\in E} U_e$. Let V be an open subset of $X\backslash E$ containing $X\backslash U$.

Let C be the set of connected components I of $[0,1]\backslash\gamma^{-1}(E)$ such that $\bar{I}$ is not contained in $\gamma^{-1}(U)$ nor in $\gamma^{-1}(V)$. By Lebesgue's number Lemma, that set is finite. Since the restriction of γ to $\bar{I}$ is nullhomotopic for $I \in C$, it follows that γ is homotopic to a path γ' that is constant on $\bar{I}$ for $I \in C$ and that coincides with γ on $[0,1] - \bigcup_{I\in C} I$. Let I' be a connected component of $[0,1]\backslash\gamma^{-1}(E)$ with $I' \notin C$. We have $\bar{I} \cap \gamma^{-1}(E) \neq \varnothing$, hence $\bar{I} \subset \gamma^{-1}(U)$. We deduce that $\gamma'([0,1]) \subset U$, hence γ' is nullhomotopic. ☐

Lemma 7.1.16. *Let E be a finite subset of X and γ a path in X. Let B be the set of connected components I of $[0,1]\backslash\gamma^{-1}(E)$ such that $\gamma_{|\bar{I}}$ is not nullhomotopic. Then B is finite and there are paths γ' and γ'' homotopic to γ such that*

- *γ and γ' coincide on $\bigcup_{I\in B} \bar{I}$ and $\gamma'([0,1]\backslash\bigcup_{I\in B} \bar{I}) \subset E$*
- *$\gamma''^{-1}(E)$ is finite.*

Proof. Let $\mathcal{U}$ be an open covering of X by connected and simply connected subsets, each of which contain at most one element of E. By Lebesgue's number Lemma, there are only finitely many $I \in \pi_0([0,1]\backslash\gamma^{-1}(E))$ such that $\bar{I}$ is not contained in an element of $\gamma^{-1}(\mathcal{U})$. So, B is finite.

We can write γ as a finite composition of its restrictions to $\bar{I}$ for $I \in B$ interlaced with finitely many paths that satisfy the assumptions of Lemma 7.1.15. Thanks to that lemma, we obtain a path γ' satisfying the requirements of the lemma. By shrinking the intervals on which γ' is constant to points, we obtain a path γ'' as desired. ☐

Definition 7.1.17. *We say that a path γ in a 1-dimensional space X is* minimal *if there is a finite covering of $[0,1]$ by open subsets such that the restriction of γ to any of those open subsets is injective.*

Given a continuous map $f : X \to X'$ and a path $\gamma : [0,1] \to X$, we will usually denote by $f(\gamma)$ the path $f \circ \gamma$.

We denote by $[\gamma]$ the homotopy class of a path γ. Note that we always consider homotopies relative to the endpoints. We denote by $\Pi(X)$ the fundamental groupoid of X.

Given $x_0, x_1 \in X$ such that there is a unique homotopy class of paths from x_0 to x_1 in X, we denote by $[x_0 \to x_1]$ that homotopy class.

The following lemma is classical for 1-dimensional finite CW-complexes.

Lemma 7.1.18. *Let X be a 1-dimensional space. A homotopy class of paths in X contains a minimal path if and only if it is not an identity.*

Given γ, γ' two homotopic minimal paths in X, there is a homeomorphism $\phi : [0,1] \xrightarrow{\sim} [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ such that $\gamma' = \gamma \circ \phi$.

Proof. Let γ_1, γ_2 be two minimal paths in X with $\gamma_1(1) = \gamma_2(0)$. The path $\gamma_2 \circ \gamma_1$ is minimal if and only if there are $t_1, t_2 \in (0,1)$ such that $\gamma_1((t_1,1)) \cap \gamma_2((0,t_2)) = \varnothing$. If $\gamma_2 \circ \gamma_1$ is not minimal, then there are unique elements $t_1 \in [0,1)$ and $t_2 \in (0,1]$ such that $(\gamma_2)_{|[0,t_2]} \circ (\gamma_1)_{|[t_1,1]}$ is homotopic to a constant path and $(\gamma_2)_{|[t_2,1]} \circ (\gamma_1)_{|[0,t_1]}$ is minimal (if $t_2 \neq 1$ or $t_1 \neq 0$).

We deduce by induction that a composition of minimal paths is homotopic to a minimal path or to a constant path.

Let γ be a path in X. If X is homeomorphic to an interval of $\mathbf{R}$, then γ is homotopic to a minimal path or a constant path. In general there is a finite subset E of X such that given U a connected component of $X\backslash E$, the space $\bar{U}$ is homeomorphic to an interval of $\mathbf{R}$. By Lemma

7.1.16 there is a path γ' homotopic to γ and such that $\gamma'^{-1}(E)$ is finite. So, γ' is a composition of paths contained in subspaces of X that are homeomorphic to intervals of $\mathbf{R}$. Consequently, γ' is a composition of minimal paths. It follows that γ', hence γ, is homotopic to a minimal or constant path.

Let γ be a path homotopic to a constant path. The image $\bar{\gamma}$ of γ in $\bar{X} = X/(X_{exc} \cup \{\gamma(0), \gamma(1)\})$ is homotopic to a constant path. Since $\bar{X}$ is homotopy equivalent to a wedge of circles, its fundamental group is free and $\bar{\gamma}$ cannot be a minimal path. It follows that γ is not minimal.

Let γ be a minimal path. Let $\{0 = t_0 < t_1 < \ldots < t_n = 1\} = \{0,1\} \cup \gamma^{-1}(X_{exc})$. Note that $\gamma((t_i, t_{i+1}))$ is contained in a connected component U_i of $X \backslash X_{exc}$ and it is a connected component if $\gamma(t_i),\ \gamma(t_{i+1}) \in X_{exc}$. If $\bar{U}_i$ is homeomorphic to an interval of $\mathbf{R}$, then $U_i \neq U_{i+1}$ and $U_i \neq U_{i-1}$. Otherwise, $\bar{U}_i$ is homeomorphic to S^1 and if $U_i = U_{i+1}$, then the paths $\gamma_{|U_i}$ and $\gamma_{|U_{i+1}}$ have the same orientation.

Let γ' be a minimal path homotopic to γ. We will show the existence of ϕ as in the lemma by induction on n. Since $\gamma \circ \gamma'^{-1}$ is not minimal, there is $\varepsilon > 0$ such that $\gamma'([0,\varepsilon]) \subset \bar{U}_1$. Consider ε maximal with this property.

Assume $\gamma'(\varepsilon) \notin X_{exc}$. We have $\varepsilon = 1$. Let $\varepsilon' \in (t_0, t_1]$ such that $\gamma(\varepsilon') = \gamma'(\varepsilon)$. The path $\gamma_{|[\varepsilon',1]}$ is homotopic to the identity, hence $n = 1$, $\varepsilon' = 1$ and $\gamma(t_1) = \gamma'(\varepsilon)$.

If $\gamma'(\varepsilon) \in X_{exc}$, then $\gamma'(\varepsilon) = \gamma(t_1)$ as well. In both cases, the paths $\gamma_{|[0,t_1]}$ and $\gamma'_{|[0,\varepsilon]}$ are injective and have the same image. So, there is a homeomorphism $\psi : [0,\varepsilon] \xrightarrow{\sim} [0,t_1]$ such that $\gamma'(t) = \gamma(\psi(t))$ for $t \in [0,\varepsilon]$ and the existence of ψ follows by induction.

□

Definition 7.1.19. *Let ζ be a non-identity homotopy class of paths in a 1-dimensional space X. We define the* support $\mathrm{supp}(\zeta)$ *of ζ to be the subspace $\gamma([0,1])$ of X, where γ is a minimal path in ζ.*

Lemma 7.1.18 ensures that the support is well defined. Note that $\mathrm{supp}(\zeta) = \bigcap_\gamma \gamma([0,1])$, where γ runs over paths with $[\gamma] = \zeta$.

Since a minimal path $[0,1] \to X$ is a morphism of 1-dimensional manifolds, it follows that the support of ζ is a compact connected 1-dimensional subspace of X.

We define the support of the identity homotopy class id_x at a point x to be $\{x\}$.

Lemma 7.1.20. *Let $f : X \to X'$ be a morphism of 1-dimensional spaces and let γ, γ' be two paths in X.*

- *γ is minimal if and only if $f(\gamma)$ is minimal. In particular,* $\mathrm{supp}([f(\gamma)]) = f(\mathrm{supp}([\gamma]))$.
- *If $f(\gamma) = f(\gamma')$, then $\gamma = \gamma'$ or γ and γ' are constant paths at two distinct points of X having the same image under f.*
- *If $[f(\gamma)] = [f(\gamma')]$, then $[\gamma] = [\gamma']$ or $[\gamma] = \mathrm{id}_{x_1}$ and $[\gamma'] = \mathrm{id}_{x_2}$ for some $x_1 \neq x_2 \in X$ with $f(x_1) = f(x_2)$.*

Proof. A minimal path is a locally injective path. Since every point of X has an open neighbourhood on which f is injective (cf Lemma 7.1.8), the image by f of a minimal path is a minimal path.

Consider the set $\Omega = \{t \in [0,1] \mid \gamma(t) \neq \gamma'(t)\}$, an open subset of $[0,1]$. Let I be a connected component of Ω. If $I = [0,1]$, then γ and γ' are constant paths at distinct points of X with the same image under f. Otherwise, let $s \in \overline{I} - I$. There is an open neighbourhood U of $\gamma(s) = \gamma'(s)$ such that $f_{|U}$ is injective. There is $t \in I$ such that $\gamma(t)$ and $\gamma'(t)$ are in U, hence $\gamma(t) = \gamma'(t)$, a contradiction. This shows the second assertion of the lemma.

Assume γ and γ' are minimal. Since $f(\gamma)$ and $f(\gamma')$ are minimal and homotopic, it follows from Lemma 7.1.18 that there is $\phi : [0,1] \xrightarrow{\sim} [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ such that $f(\gamma') = f(\gamma) \circ \phi = f(\gamma \circ \phi)$. It follows from the previous assertion of the lemma that $\gamma' = \gamma \circ \phi$.

Assume now γ is minimal. Since $f(\gamma)$ is minimal, it follows that $[f(\gamma')]$ is not the identity, hence $[\gamma']$ is not the identity. We deduce that the third assertion of the lemma holds when $[\gamma]$ and $[\gamma']$ are not both identities. The case where they are both identities is clear. ☐

7.1.5. *Tangential multiplicity.* Let X be a 1-dimensional space. Let $x \in X$ and U be a small open neighborhood of x.

Let $c \in C(x)$ and U_c be the connected component of $U - \{x\}$ corresponding to c. Given γ a path in X, let $I_c^+(\gamma)$ (resp. $I_c^-(\gamma)$) be the set of elements $t \in [0,1]$ such that $\gamma(t) = x$ and there is $\varepsilon > 0$ with $t + \varepsilon < 1$ and $\gamma((t, t+\varepsilon)) \subset U_c$ (resp. $t - \varepsilon > 0$ and $\gamma((t-\varepsilon, t)) \subset U_c$).

When γ is minimal, the set $I_c^\pm(\gamma)$ is finite and it follows from Lemma 7.1.18 that its cardinality depends only on the homotopy class $[\gamma]$. We put $m_c^\pm([\gamma]) = |I_c^\pm(\gamma)| \in \mathbf{Z}_{\geqslant 0}$ for γ minimal and $m_c([\gamma]) = m_c^+([\gamma]) - m_c^-([\gamma])$. Similarly, whether or not $0 \in I_c^+$ depends only on the homotopy class $[\gamma]$ (for γ minimal).

Lemma 7.1.21. *Let γ be a path in X such that $\gamma^{-1}(x)$ has finitely many connected components, none of which contain 0 or 1 in the closure of their interior.*

We have $\partial(\gamma^{-1}(x)) = \bigcup_{c \in C(x)} (I_c^+(\gamma) \cup I_c^-(\gamma))$ and $|I_c^+(\gamma)| - |I_c^-(\gamma)| = m_c([\gamma])$ for all $c \in C(x)$.

Proof. The first statement is clear. Let us now prove the second statement. That statement is clear if $\gamma((0,1)) \cap (X_{exc} \cup \{x\}) = \varnothing$.

The left side of the equality is additive under compositions of paths, and so is the right side by Lemma 7.1.22 below.

Assume now $\gamma^{-1}(X_{exc} \cup \{x\})$ is finite. The path γ is a (finite) composition of paths mapping $(0,1)$ into the complement of $X_{exc} \cup \{x\}$, hence the statement holds for γ.

Consider now the general case. The proof of Lemma 7.1.16 for $E = X_{exc} \cup \{x\}$ produces a path γ' homotopic to γ such that $\gamma'^{-1}(E)$ is finite and such that $|I_c^+(\gamma)| - |I_c^-(\gamma)| = |I_c^+(\gamma')| - |I_c^-(\gamma')|$. Since the statement holds for γ', it follows that it holds for γ. ☐

Let ζ be the homotopy class of a minimal path γ. Let $x = \zeta(0)$. There is a unique $c \in C(x)$ such that $0 \in I_c(\gamma)^+$ and we define $\zeta(0+) = \{c\}$. Similarly, we define $\zeta(1-) = \{c'\}$, where $c' \in C(\zeta(1))$ is unique such that $1 \in I_c(\gamma)^-$.

When ζ is the homotopy class of a constant path we put $\zeta(0+) = C(\zeta(0))$ and $\zeta(1-) = C(\zeta(1))$ and $m_c^\pm(\zeta) = m_c(\zeta) = 0$.

Given a category $\mathcal{C}$, we denote by $H_0(\mathcal{C})$ the abelian group generated by maps in $\mathcal{C}$ modulo the relation $f + g = f \circ g$ for any two composable maps f and g. We denote by $[\![f]\!]$ the class in $H_0(\mathcal{C})$ of a map f of $\mathcal{C}$. Note that if f is an identity map, then $[\![f]\!] = 0$.

Note that H_0 is left adjoint to the functor sending an abelian group to the category with one object with endomorphism monoid that abelian group.

Let $R(X) = H_0(\Pi(X))$. Note that $R(X)$ is generated by the set I of homotopy classes of paths γ such that γ is injective. It follows from the description of the composition of two minimal paths in §7.1.4 that $R(X)$ has a presentation with generating set the non-identity homotopy classes of paths and relations $[\gamma \circ \gamma'] = [\gamma] + [\gamma']$ if γ, γ' and $\gamma \circ \gamma'$ are minimal and $[\gamma] + [\gamma^{-1}] = 0$ for γ minimal. Note finally that every element of $R(X)$ is a linear combination of non-identity homotopy classes of paths such that the intersection between the supports of two distinct homotopy classes is finite.

Lemma 7.1.22. *Given $x \in X$, the map m_c induces a morphism of groups $R(X) \to \mathbf{Z}$.*

Proof. Consider γ and γ' two injective composable paths such that $\gamma \circ \gamma'$ is injective. We have $m_c^{\pm}([\gamma\gamma']) = m_c^{\pm}([\gamma]) + m_c^{\pm}([\gamma'])$.

Consider now γ a minimal path. We have $m_c^{\pm}([\gamma]) = m_c^{\mp}([\gamma^{-1}])$, hence $m_c([\gamma]) + m_c([\gamma^{-1}]) = 0 = m_c([\gamma^{-1} \circ \gamma])$. The lemma follows. ☐

The next lemma shows how to realize $R(X)$ as a subgroup of the group of maps $U \to \mathbf{Z}$, where U is a dense subset of X.

Lemma 7.1.23. *Let U be a dense subset of $X - (\partial X \cup X_{exc})$. Given $x \in U$, fix a group morphism $l_x : \mathbf{Z}^{C(x)} \to \mathbf{Z}$ that does not factor through the sum map.*

The morphism $(l_x \circ (m_c)_{c \in C(x)})_{x \in U} : R(X) \to \mathbf{Z}^U$ is injective.

Proof. Let L be a non-empty finite subset of I such that $\mathrm{supp}(\zeta) \cap \mathrm{supp}(\zeta')$ is finite for any two distinct elements ζ and ζ' in L. Let $r = \sum_{\zeta \in L} a_\zeta [\![\zeta]\!]$ where $a_\zeta \in \mathbf{Z} - \{0\}$ for $\zeta \in L$. Let $\zeta_0 \in L$. There is $x \in \mathrm{supp}(\zeta_0) \cap U$ with $x \notin \{\zeta_0(0), \zeta_0(1)\}$ and $x \notin \bigcup_{\zeta \in L - \{\zeta_0\}} \mathrm{supp}(\zeta)$. Let $c \in C(x)$ and $\iota(c)$ be the other element of $C(x)$. We have $m_c(\zeta_0) = -m_{\iota(c)}(\zeta_0) = \pm 1$, while $m_c(\zeta') = m_{\iota(c)}(\zeta') = 0$ for $\zeta' \in L - \{\zeta_0\}$. It follows that $m_c(r) = -m_{\iota(c)}(r) = \pm a_\gamma$. Consequently, $\big(l_x \circ (m_c, m_{\iota(c)})\big)(r) = \pm l_x(a_\gamma, -a_\gamma) \neq 0$. Since every non-zero element of $R(X)$ is of the form r as above, the lemma follows. ☐

Let $f : X \to X'$ be a morphism of 1-dimensional spaces. The next lemma follows from the injectivity statement of Lemma 7.1.8.

Lemma 7.1.24. *Given $x \in X$, $c \in C(X)$ and ζ a homotopy class of paths in X, we have $m_{f(c)}^{\pm}(f(\zeta)) = m_c^{\pm}(\zeta)$ and $m_{f(c)}(f(\zeta)) = m_c(\zeta)$.*

Note that f induces a morphism of groups $f : R(X) \to R(X')$.

Lemma 7.1.25. *Let H be the subgroup of $R(X')$ generated by classes $[\gamma]$ with $\mathrm{supp}(\gamma) \subset \overline{X' - f(X)}$.*

The composition $R(X) \xrightarrow{f} R(X') \xrightarrow{\mathrm{can}} R(X')/H$ is injective.

Proof. Let $U' = X' - (X'_f \cup X'_{exc} \cup \partial X')$, a dense subset of X'. Note that $U = f^{-1}(U')$ is a dense subset of $X - (X_{exc} \cup \partial X)$. Given $x' \in U'$, fix a morphism $l_{x'} : \mathbf{Z}^{C(x')} \to \mathbf{Z}$ that does not factor through the sum map. Given $x \in U$, let $l_x = l_{x'} \circ f : \mathbf{Z}^{C(x)} \to \mathbf{Z}$. Lemma 7.1.23 shows that $(l_x \circ (m_c)_{c \in C(x)})_{x \in U} : R(X) \to \mathbf{Z}^U$ is injective. This map is equal to the composition

$$R(X) \xrightarrow{f} R(X') \xrightarrow{(l_{x'} \circ (m_{c'})_{c' \in C(x')})_{x' \in U'}} \mathbf{Z}^{U'} \xrightarrow{f^*} \mathbf{Z}^U$$

since $m^{\pm}_{f(c)}(f(\zeta)) = m^{\pm}_c(\zeta)$ and $m_{f(c)}(f(\zeta)) = m_c(\zeta)$ for all $x \in X$, $c \in C(X)$ and all homotopy classes of paths ζ in X (Lemma 7.1.24). Since H is contained in the kernel of the composition

$$R(X') \xrightarrow{(l_{x'} \circ (m_{c'})_{c' \in C(x')})_{x' \in U'}} \mathbf{Z}^{U'} \xrightarrow{f^*} \mathbf{Z}^U,$$

it follows that the composite map of the lemma is injective. □

7.2. **Curves.**

7.2.1. *Definitions.* We consider now partially oriented 1-dimensional spaces. We build the theory so that the unoriented part is a manifold, and morphisms are injective on the unoriented part.

Definition 7.2.1. *We define a* curve *to be a* 1*-dimensional space Z endowed with*
- *an open subset Z_o containing Z_{exc}*
- *an orientation of $Z_o - Z_{exc}$ and*
- *a fixed-point free involution ι of $C_Z(z)$ for every $z \in Z_{exc}$*

satisfying the following conditions:
- $\partial Z = \varnothing$
- *$Z - Z_o$ has finitely many connected components, none of which are points*
- *given $z \in Z_{exc}$, given U a small open neighbourhood of z in Z_o, and given $L \in \pi_0(U - \{z\})$, then $L \cup \iota(L) \cup \{z\}$ has an orientation extending the given orientations on L and $\iota(L)$.*

We put $Z_u = Z - Z_o$. Note that $\partial Z_u = Z_u \cap \overline{Z_o}$. Given $z \in Z - Z_{exc}$, we have $|C(z)| = 2$ and we define ι as the unique non-trivial automorphism of $C(z)$.

We denote by Z^{opp} the *opposite* curve to Z all of whose data coincides with that of Z, except for $Z_o - Z_{exc}$, whose orientation is reversed.

Fix $n \geqslant 1$. The 1-dimensional space $Z = \mathrm{St}(2n)$ (cf §7.1.1) can be endowed with a structure of curve by giving $\mathbf{R}e^{i\pi r/n}$ the orientation of $\mathbf{R}$ for $0 \leqslant r < n$ and setting $Z_o = Z$. The involution ι is defined by $\iota(\mathbf{R}_{>0}e^{i\pi r/n}) = \mathbf{R}_{<0}e^{i\pi r/n}$.

7.2.2. *Morphisms and subcurves.*

Definition 7.2.2. *A* morphism of curves $f : Z \to Z'$ *is a morphism of* 1*-dimensional spaces such that*
- $f(Z_u) \subset Z'_u$
- *$f_{|f^{-1}(Z'_o - Z'_{exc})}$ is orientation-preserving*
- *given $z \in f^{-1}(Z'_{exc})$, the canonical map $C(f) : C_Z(z) \to C_{Z'}(f(z))$ is ι-equivariant.*

Note that a composition of morphisms of curves is a morphism of curves. Let $f : Z \to Z'$ be a morphism of curves. We have the following statements.

Properties 7.2.3.
- f is invertible if and only if it is a homeomorphism and $f(Z_o) \subset Z'_o$.
- $f(Z_{exc}) \subset Z'_{exc}$ and $C(f) : C_Z(z) \to C_{Z'}(f(z))$ is ι-equivariant for all $z \in Z$.
- f restricts to a homeomorphism from $f^{-1}(Z' - Z'_{exc})$ to the open subset $f(Z) \cap (Z' - Z'_{exc}) = f(Z - Z_{exc}) \cap (Z' - Z'_{exc})$ of Z', since $Z_f \subset f^{-1}(Z'_{exc})$. In particular, the restriction of f to Z_u is a homeomorphism $Z_u \xrightarrow{\sim} f(Z_u)$.

- If Z' is non-singular, then f is an open embedding.

We say that f is *strict* if $f(Z_u)$ is closed in Z'_u and $f(Z_o) \subset Z'_o$. Note that this implies that $f(Z_u)$ is also open in Z'_u.

Let Z be a curve.

Definition 7.2.4. *A* subcurve *of Z is a 1-dimensional subspace X of Z such that given $z \in X$, the image of $C_X(z)$ in $C_Z(z)$ is ι-stable.*

If X is a subcurve of Z, then X is a curve with $X_o = X \cap Z_o$, $X_{exc} \subset Z_{exc}$ and ι is defined on $C_X(z)$ as the restriction of ι on $C_Z(z)$, for $z \in X_{exc}$. Note that X_u is open in Z_u.

Equivalently, a subspace X of Z is a subcurve if it is a curve, $X_o = X \cap Z_o$ and the inclusion map $X \to Z$ is a morphism of curves.

We define an equivalence relation on connected components of $Z - Z_{exc}$: it is the relation generated by $T \sim T'$ if there is $z \in Z_{exc} \cap \overline{T} \cap \overline{T'}$, U a small open neighbourhood of z and $L \in \pi_0(U - \{z\})$ such that $L \subset T$ and $\iota(L) \subset T'$.

Let $\mathcal{E}$ be the set of equivalence classes of connected components of $Z - Z_{exc}$. Given $E \in \mathcal{E}$, let $Z_E = \bigcup_{T \in E} \overline{T}$. The subspaces Z_E of Z are called the *components* of Z.

A curve has only finitely many components, each of which is a closed subcurve.

If Z is non-singular, then its components are its connected components.

The local structure of a curve is described as follows. Let $z \in Z$. There is an open neighbourhood U of z that is a subcurve of Z and an isomorphism of curves $U \xrightarrow{\sim} X$, $z \mapsto 0$, where $X \subset \mathbf{C}$ is one of the following:

- $\mathbf{R}$ viewed as an unoriented manifold, if $z \in Z_u - \partial Z_u$
- $\mathbf{R}$ where $\mathbf{R}_{\geqslant 0}$ is unoriented and $\mathbf{R}_{<0}$ has either of its two orientations, if $z \in \partial Z_u$
- $\mathbf{R}$ viewed as an oriented manifold, if $z \in Z_o - Z_{exc}$
- $\mathrm{St}(n_z)$ if $z \in Z_{exc}$.

Remark 7.2.5. Let Z be a closed subspace of $\mathbf{R}^N$ for some $N > 0$. Assume there is a finite subset E of Z such that $Z - E$ is a 1-dimensional submanifold of $\mathbf{R}^N$ with no boundary and such that given $e \in E$, there is $n'_e > 1$ and a finite family $\{j_{e,i}\}_{1 \leqslant i \leqslant n'_e}$ of smooth embeddings $j_{e,i} : (-1,1) \to \mathbf{R}^N$ such that

- $j_{e,i}(0) = e$,
- $j_{e,i}((-1,0) \cup (0,1)) \subset Z - \{e\}$,
- $j_{e,i}((-1,1)) \cap j_{e,i'}((-1,1)) = \{e\}$ for $i \neq i'$
- $\mathbf{R}\frac{dj_{e,i}}{dt}(0) \neq \mathbf{R}\frac{dj_{e,i'}}{dt}(0)$ for $i \neq i'$ and
- $\bigcup_i j_{e,i}(-1,1)$ is an open neighborhood of e in Z.

Let us choose in addition an open subset Z_o of Z containing E and an orientation of the 1-dimensional manifold $Z_o - E$. We assume that $Z - Z_o$ has finitely many connected components, none of which are points. We assume furthermore that given $e \in E$ and $i \in \{1, \ldots, n'_e\}$, the orientation of $j_{e,i}^{-1}(Z_o - \{e\})$ extends to an orientation of $j_{e,i}^{-1}(Z_o)$.

Given $e \in E$, we denote by ι the involution of $C(e)$ that swaps $j_{e,i}((-1,0))$ and $j_{e,i}((0,1))$ for $1 \leqslant i \leqslant n'_e$. Note that $Z_{exc} = E$ and $n_e = 2n'_e$ for $e \in E$. This defines a structure of curve on Z that does not depend on the choice of the maps $j_{e,i}$.

We leave it to the reader to check that any curve is isomorphic to a curve obtained by such a construction.

7.2.3. *Quotients.* Let $(\tilde{Z}, \tilde{Z}_o, \tilde{\iota})$ be a curve.

Definition 7.2.6. *A* finite relation *on* $\tilde{Z}$ *is an equivalence relation* $\sim$ *such that the set of points that are not alone in their equivalence class is finite and contained in* $\tilde{Z}_o$.

Consider a finite relation $\sim$ on $\tilde{Z}$. We define a curve structure on the 1-dimensional space $Z = \tilde{Z}/\sim$.

Let $q : \tilde{Z} \to Z$ be the quotient map. We have $Z_{exc} = q(\tilde{Z}_{exc}) \cup \{z \in Z | \, |q^{-1}(z)| > 1\}$ (cf §7.1.3). Let $Z_o = q(\tilde{Z}_o)$. The map $q_{|\tilde{Z}_o - q^{-1}(Z_{exc})} : \tilde{Z}_o - q^{-1}(Z_{exc}) \to Z_o - Z_{exc}$ is a homeomorphism and we provide $Z_o - Z_{exc}$ with the orientation coming from $\tilde{Z}_o - q^{-1}(Z_{exc})$. Let $z \in Z_{exc}$. We define ι on $C(z)$ to make the canonical bijection $\coprod_{\tilde{z} \in q^{-1}(z)} C(\tilde{z}) \xrightarrow{\sim} C(z)$ ι-equivariant. This makes q into a strict morphism of curves.

Lemma 7.2.7. *Let* $f : Z \to Z'$ *be a morphism of curves.*

Define an equivalence relation on Z *by* $z_1 \sim z_2$ *if* $f(z_1) = f(z_2)$. *This is a finite relation on* Z *and* f *factors as a composition of morphisms of curves* $Z \xrightarrow{f_1} Z/\sim \xrightarrow{f_2} Z'$ *where* f_1 *is the quotient map and* f_2 *is injective.*

Proof. We have $Z_f \subset f^{-1}(Z'_o) \subset Z_o$. It follows that $\sim$ is a finite relation on Z and the lemma follows from Lemma 7.1.13. $\square$

We define the category of non-singular curves with a finite relation as the category with objects pairs $(Z, \sim)$ where Z is a non-singular curve and $\sim$ is a finite relation on Z, and where $\mathrm{Hom}((Z, \sim), (Z', \sim'))$ is the set of morphisms of curves $f : Z \to Z'$ such that if $z_1 \sim z_2$, then $f(z_1) \sim' f(z_2)$.

The next proposition shows that curves can be viewed as non-singular curves with a finite relation.

Proposition 7.2.8. *The quotient construction defines an equivalence from the category of non-singular curves with a finite relation to the category of curves.*

Proof. Let $(\tilde{Z}, \sim)$ and $(\tilde{Z}', \sim')$ be two non-singular curves with finite relations and let $q : \tilde{Z} \to Z = \tilde{Z}/\sim$ and $q' : \tilde{Z}' \to Z' = \tilde{Z}'/\sim'$ be the quotient maps.

A morphism of curves $f : \tilde{Z} \to \tilde{Z}'$ such that $z_1 \sim z_2$ implies $f(z_1) \sim' f(z_2)$ induces a morphism of curves $Z \to Z'$. So, the quotient functor induces indeed a functor as claimed. Consider $f' : \tilde{Z} \to \tilde{Z}'$ such that $z_1 \sim z_2$ implies $f'(z_1) \sim' f'(z_2)$. If $q' \circ f = q' \circ f'$, then f and f' coincide outside a finite set of points, hence $f = f'$. So, the quotient functor is faithful.

Consider now a morphism of curves $g : Z \to Z'$. Let E' be the finite subset of $\tilde{Z}'$ of points that are not alone in their equivalence class and $E = q^{-1}(g^{-1}(q'(E')))$. Consider the composition of continuous maps

$$f : \tilde{Z} - E \xrightarrow{q} Z - q(E) \xrightarrow{g} Z' - q'(E') \xrightarrow{(q'_{|\tilde{Z}' - E'})^{-1}} \tilde{Z}' - E'.$$

Given $z \in E$, the ι-equivariance of $C(g) : C_Z(q(z)) \to C_{Z'}(g(q(z)))$ ensures that f extends to a continous map at z. So, f extends (uniquely) to a continous map $\tilde{Z} \to \tilde{Z}'$, and that map is a morphism of 1-dimensional spaces.

We have $\tilde{Z}_u \subset \tilde{Z} - E$ and $f(\tilde{Z}_u) \subset \tilde{Z}'_u$. Since $g_{|g^{-1}(\tilde{Z}'_o)-E}$ is orientation-preserving, it follows that $f_{|f^{-1}(\tilde{Z}'_o-E')}$ is orientation-preserving. So, $f : \tilde{Z} \to \tilde{Z}'$ is a morphism of curves and it is compatible with the relations. This shows that the quotient functor is fully faithful.

Let now Z be a curve. Let $z \in Z_{exc}$ and $U_z \subset Z_o$ be a small open neighbourhood of z. Fix an isomorphism of curves $f_z : U_z \xrightarrow{\sim} \mathrm{St}(n_z)$, $z \mapsto 0$. The equivalence relation on $\pi_0(U_z - \{z\})$ whose equivalence classes are the orbits of ι defines via f_z the equivalence relation on $\{e^{i\pi r/2n_z}\}_{0\leqslant r<2n_z}$ given by $\zeta \sim \zeta'$ if and only if $\zeta' = \zeta^{\pm 1}$.

The proof of Lemma 7.1.14 provides us a non-singular curve $\hat{Z}$ with a finite relation. Indeed, with the notations of the proof of Lemma 7.1.14, we have $\hat{U}_z = \coprod_{0\leqslant r<n_z} \mathbf{R}e^{i\pi r/n_z}$. Note that $\hat{Z}_o$ is the subspace of $\hat{Z}$ obtained by adding to $Z_o - Z_{exc}$ the point 0 of $\mathbf{R}e^{i\pi r/n_z}$ for each $r \in \{0, \ldots, n_z - 1\}$ and each $z \in Z_{exc}$.

This gives $\hat{Z}$ a structure of non-singular curve. As in the proof of Lemma 7.1.14, we obtain a finite relation on $\hat{Z}$ and an isomorphism of curves $Z \xrightarrow{\sim} \hat{Z}/\sim$. This shows that the quotient functor is essentially surjective. □

Definition 7.2.9. *Given Z a curve, the* non-singular cover *of Z is a non-singular curve $\hat{Z}$, together with a finite relation $\sim$ and an isomorphism $\hat{Z}/\sim \xrightarrow{\sim} Z$.*

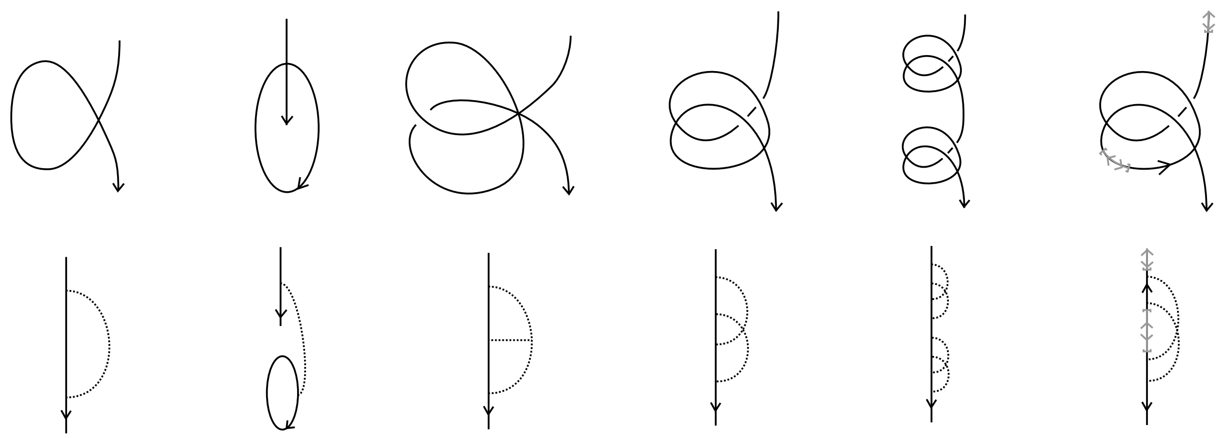
Note that $Z_{exc} = Z_q$ where $q : \hat{Z} \to Z$ is the canonical map. Proposition 7.2.8 shows that non-singular covers exist and are unique up to a unique isomorphism. The following proposition makes this more precise.

Proposition 7.2.10. *The functor sending a curve Z to its non-singular cover is right adjoint to the embedding of the category of non-singular curves in the category of curves.*

Proof. Let Z' be a non-singular curve. We have a map $h : \mathrm{Hom}(Z', \hat{Z}) \to \mathrm{Hom}(Z', Z)$, $g \mapsto q\circ g$. Since Z_q is finite, it follows that h is injective.

Consider now a morphism of curves $f : Z' \to Z$. We factor f as $Z' \xrightarrow{f_1} Z'/\sim \xrightarrow{f_2} Z$ as in Lemma 7.2.7. By Proposition 7.2.8, there is a morphism $\hat{f} : Z' \to \hat{Z}$ such that $q \circ \hat{f} = f$, hence $h(\hat{f}) = f$. So h is surjective. □

Example 7.2.11. Let us provide some examples of curves and non-singular covers. The dotted lines link the points in the same equivalence class. The grey part corresponds to Z_u.



7.2.4. *Chord diagrams as singular curves.* We describe here the relation between singular curves and chord (or arc) diagrams.

We define a *chord diagram* to be to be a triple $(\mathcal{Z}, \mathbf{a}, \mu)$ where

- $\mathcal{Z}$ is a closed oriented 1-dimensional manifold (i.e., a finite disjoint union of copies of S^1 and $[0,1]$)
- $\mathbf{a} = (a_1, \ldots, a_{2k})$ is a collection of distinct points of $\mathring{\mathcal{Z}}$
- $\mu : \mathbf{a} \to \{1, \ldots, k\}$ is a 2-to-1 map.

A chord diagram gives rise to a smooth oriented curve $\tilde{Z} = \mathring{\mathcal{Z}}$ with the following relation: given $z \neq z'$, we have $z \sim z'$ if there is j such that $\mu^{-1}(j) = \{z, z'\}$. We obtain an oriented curve $Z = \tilde{Z}/\sim$.

Up to suitable isomorphism, this defines a bijection from chord diagrams to oriented singular curves with $n_z \in \{2, 4\}$ for all z.

Convention 7.2.12. We will use the above bijection composed with the reversal of all orientations when identifying chord diagrams with certain singular curves. This orientation reversal is related to the usual direction reversal between arrows in a quiver and morphisms in the corresponding path category, and to the time-reversal of graphs mentioned in Example 7.3.8 below.

When $\mathcal{Z}$ is a union of intervals, we recover the notion of (possibly degenerate) arc diagram due to Zarev [Za, Definition 2.1] (compare Example 7.2.11 and [Za, Figures 3 and 4]).

The chord diagrams such that the singular curve Z is connected and $k > 0$ correspond to the chord diagrams of [AnChePeReiSu].

Zarev's definition generalizes that of pointed matched circles due to Lipshitz, Ozsváth and Thurston [LiOzTh1, §3.2]: they correspond to the case where $\mathcal{Z}$ is a single interval ($\mathring{\mathcal{Z}}$ is obtained from the circle considered in [LiOzTh1] by removing its basepoint).

7.2.5. *Sutured surfaces and topological field theories.* We define a *sutured surface* to be a quadruple (F, Λ, S^+, S^-) where F is a compact oriented surface, $\Lambda \subset \partial F$ is a compact 0-manifold, and S^+ and S^- are unions of components of $\partial F - \Lambda$ such that $\Lambda \subset \overline{S^+} \cap \overline{S^-}$ (this is [Za, Definition 1.2] without the topological restrictions). A sutured surface is representable by a chord diagram (as we define it) if and only if each component of F (not ∂F) intersects S^+ and S^- nontrivially.

Let $(\mathcal{Z}, \mathbf{a}, \mu)$ be a chord diagram. We define a sutured surface (F, Λ, S^+, S^-):

- the oriented surface F is obtained from $\coprod_i (Z_i \times [0,1])$ by adding 1-handles at $\mu^{-1}(j) \times \{0\}$ for all j
- $\Lambda = \partial\mathcal{Z} \times \frac{1}{2}$
- $S^+ = \big(\mathcal{Z} \times \{1\}\big) \cup \big(\partial\mathcal{Z} \times (\frac{1}{2}, 1]\big)$
- S^- is obtained from surgery (corresponding to the handle addition) on $\big(\mathcal{Z} \times \{0\}\big) \cup \big(\partial\mathcal{Z} \times [0, \frac{1}{2})\big)$.

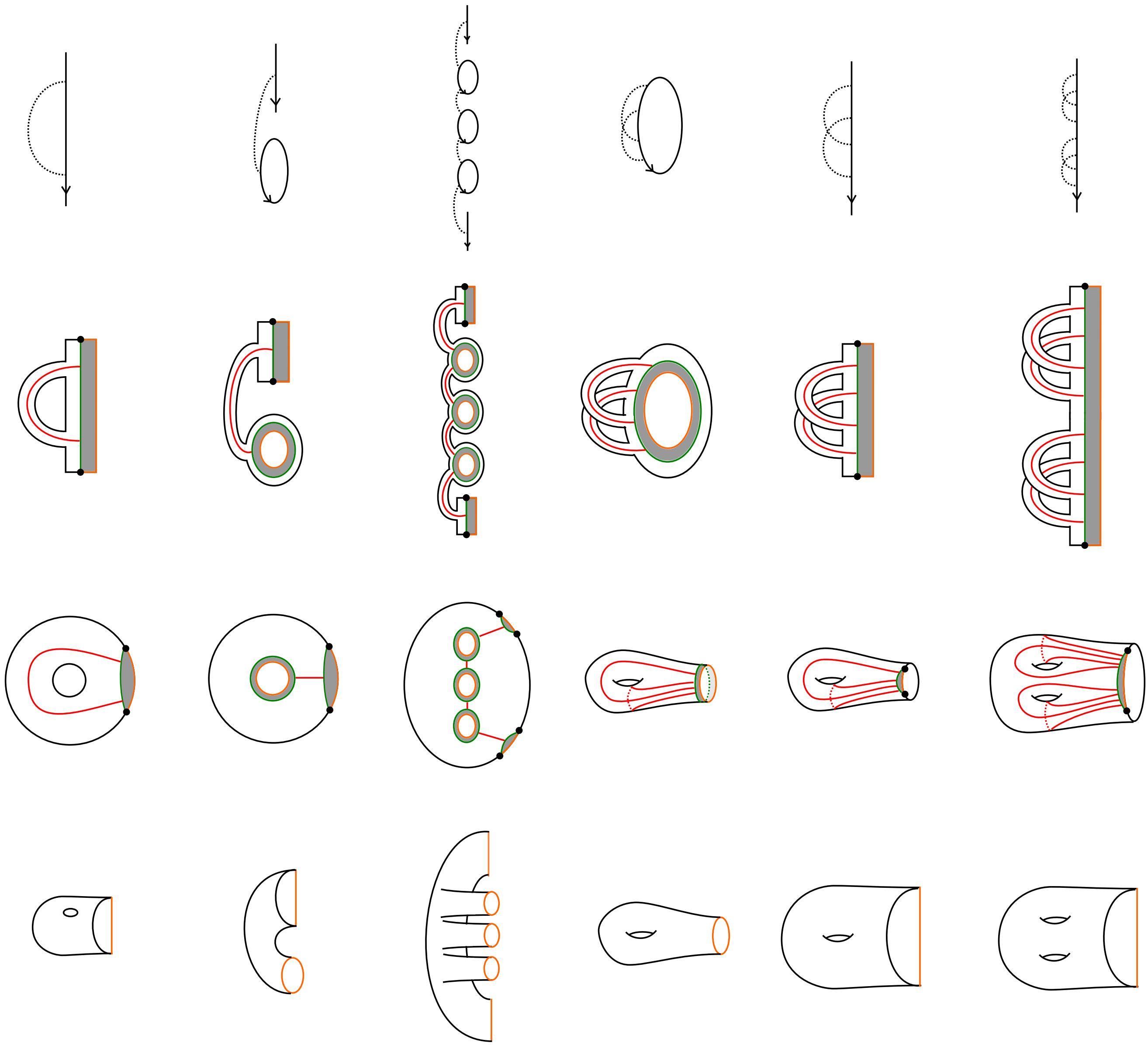
When $\mathcal{Z}$ is a union of intervals, this is Zarev's construction [Za, §2.1].

Example 7.2.13. In the table below, the first row depicts some chord diagrams. The second and third rows show the corresponding sutured surfaces; the second row applies the above construction directly, and the third row gives an alternate perspective. The fourth row shows the sutured surfaces as open-closed cobordisms (with empty source and with target colored in orange); this interpretation is discussed in §7.2.5.

Under the strands algebra construction of §8.1, the first and second columns give rise to simple 2-representations of $\mathcal{U}$, categorifying the vector representation and its dual. Tensor powers of the algebra of the first column give algebras very similar to the one considered by Tian [Ti]; in fact, Tian's algebras were an important early clue in the development of the present work. Tensor powers of the algebra of the second column will be studied from the Heegaard Floer perspective by the first-named author in work in preparation.

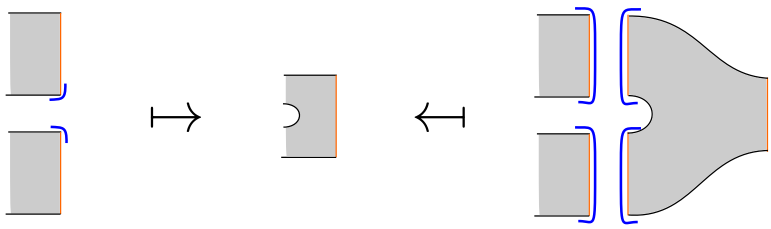
The algebra of the third column is the $n = 3$ case of a family of algebras considered in [ManMarWi, LePo]. For general n, these are isomorphic to the algebras $\mathcal{B}(n) = \oplus_{k=0}^{n}\mathcal{B}(n,k)$ used by Ozsváth and Szabó in their theory of bordered knot Floer homology [OsSz2, OsSz3, OsSz4] (their notation is slightly different). The middle summand of the algebra of the fourth column is the undeformed version of a curved A_∞-algebra used by Lipshitz-Ozsváth-Thurston (in preparation) to define bordered HF^- for 3-manifolds with torus boundary. The middle summand of the algebra of the fifth column is the well-known "torus algebra" from bordered Floer homology. The fifth and sixth columns together illustrate our perspective on cornered Floer homology; following Zarev's ideas, we view the cornered Floer gluing theorem as recovering the algebra of two matched intervals glued end-to-end, rather than as the invariants of two matched intervals with distinguished endpoints being glued to form a pointed matched circle.

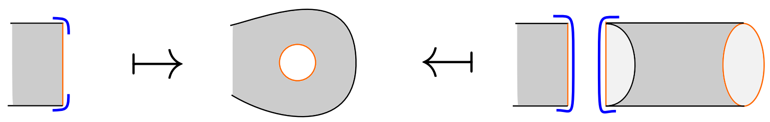
The first, fifth, and sixth columns give algebras that are among Zarev's strands algebras $\mathcal{A}(\mathcal{Z})$, although the first diagram is degenerate (equivalently, its sutured surface has closed circles in S^-). The second, third, and fourth columns do not satisfy the restrictions that Zarev imposes. As far as we are aware, our strands categories below give the first detailed description of strands algebras associated to general chord diagrams with circles as well as intervals; less formal descriptions have appeared previously, cf. [Au2, Proposition 11]. As indicated by Lipshitz-Ozsváth-Thurston's work in preparation, curved A_∞-deformations of the algebras appear necessary in the general setting when defining modules and bimodules for 3-manifolds with boundary, although in special cases like Ozsváth-Szabó's bordered knot Floer homology (third column) this complication should be avoidable.



A sutured surface can be viewed as a morphism in the 2d open-closed cobordism category with empty source; if (F, Λ, S^+, S^-) is a sutured surface, the corresponding open-closed cobordism has target given by S^+ and non-gluing boundary given by S^-. See the bottom row of the figure in Example 7.2.13; the targets of these open-closed cobordisms are shown in orange and the non-gluing boundary is shown in black.

Let us consider how the end-to-end gluings of chord diagrams covered by our results in §8 can be viewed in terms of open-closed cobordisms. When gluing two distinct intervals of a chord diagram end-to-end, the corresponding sutured surface gets glued as in the top-left picture below: the two intervals marked in blue are glued together to form the top-middle picture. However, we can also consider the top-middle picture as arising from the top-right picture; in this latter case the gluing is an instance of composition (with an open pair of pants) in the open-closed cobordism category. Similarly, when self-gluing the two endpoints of an interval of a chord diagram, the sutured surface gets glued as in the bottom-left picture below, producing the bottom-middle picture; we can also think of the bottom-middle picture as arising from the bottom-right picture, which is another instance of composition in the open-closed category.





One could try to view our constructions as giving part of the structure of an open-closed 2d TQFT valued in a category whose objects are dg 2-categories and whose morphisms are certain dg 2-functors. In particular, this hypothetical open-closed TQFT would assign a dg 2-category of 2-representations of $\mathcal{U}$ to an interval. To an open-closed cobordism with empty source, the open-closed TQFT would assign an object of the dg 2-category of the target, encoding the data of a lax multi-2-action of $\mathcal{U}$ for the interval components of the target. Our approach doesn't quite realize that. We associate 2-representations of $\mathcal{U}$ to chord diagrams or singular curves rather than directly to surfaces.

One can also consider the extent to which such a theory would extend to a point. Things are considerably simpler for the decategorified version of the theory, where one sees many relationships with other work on 3d TQFTs; this will be addressed in more detail in a follow-up paper [Man].

7.3. **Paths.**

7.3.1. *Admissible paths.* Let Z be a curve.

Definition 7.3.1. *An* oriented path γ *in* Z *is defined to be a path whose restriction to* $\gamma^{-1}(Z_o - Z_{exc})$ *is compatible (non strictly) with the orientation.*

Let us note some basics facts about oriented paths.

Properties 7.3.2. Let γ be a non-constant oriented path in Z.

(1) We have $\gamma([0,1]) \cap Z_o = \mathrm{supp}([\gamma]) \cap Z_o$ and $\gamma([0,1]) \cap Z_u$ is contained in the union of the connected components of Z_u that have a non-empty intersection with $\mathrm{supp}([\gamma])$.

(2) If γ is homotopic to a constant path, then it is contained in Z_u (as $\gamma([0,1])$ is contractible).

(3) There are unique real numbers $0 = t_0 < t_1 < \cdots < t_r = 1$ such that
   - given $0 \leqslant i < r$, there are $\{j,k\} = \{i, i+1\}$ with the property that $\gamma([t_j, t_{j+1}]) \subseteq Z_u$ (if $j < r$) and $\gamma([t_k, t_{k+1}]) \subseteq \bar{Z}_o$ (if $k < r$) (cf Lemma 7.1.16 for $E = Z_u \cap \bar{Z}_o$).

– given $0 < i < r$ and $\varepsilon > 0$ such that $\gamma([t_i, t_i+\varepsilon]) \subset Z_u \cap \bar{Z}_o$, we have $\gamma([t_i, t_{i+1}]) \not\subseteq \bar{Z}_o$
– given $0 < i < r$ and $\varepsilon > 0$ such that $\gamma([t_i-\varepsilon, t_i]) \subset Z_u \cap \bar{Z}_o$, we have $\gamma([t_{i-1}, t_i]) \not\subseteq \bar{Z}_o$.

The sequence $[\gamma_{|[t_0,t_1]}], \ldots, [\gamma_{|[t_{r-1},t_r]}]$ depends only on $[\gamma]$.

(4) Consider homotopy classes of oriented paths ζ_1, ζ_2 and ζ_3 with $[\gamma] = \zeta_3 \circ \zeta_2 \circ \zeta_1$. If $\operatorname{supp}(\zeta_2)$ is contained in $\overline{Z_o}$ but not in Z_u, then there are $0 \leqslant t_1 \leqslant t_2 \leqslant 1$ such that $[\gamma_{|[0,t_1]}] = \zeta_1$, $[\gamma_{|[t_1,t_2]}] = \zeta_2$ and $[\gamma_{|[t_2,1]}] = \zeta_3$.

Lemma 7.3.3. *Let γ be a path in Z. The following conditions are equivalent:*

(i) *γ lifts to a path in the non-singular cover of Z*
(ii) *given $z \in Z_{exc}$, given a small open neighbourhood U of z in Z_o and given K a connected component of $\gamma^{-1}(z)$, the set of $L \in \pi_0(U - \{z\})$ with $K \cap \overline{\gamma^{-1}(L)} \neq \varnothing$ is contained in an orbit of ι.*

Proof. Let $\hat{Z}$ be the non-singular cover of Z and $q : \hat{Z} \to Z$ be the quotient map.

Assume (i). Consider z, U, K as in the lemma and let $\hat{\gamma}$ be a lift of γ. Consider $L_i \in \pi_0(U - \{z\})$ with $K \cap \overline{\gamma^{-1}(L_i)} \neq \varnothing$ for $i \in \{1, 2\}$. We have $\hat{\gamma}(K) \subset \overline{q^{-1}(L_i)}$. Consequently, we have $\overline{q^{-1}(L_1)} \cap \overline{q^{-1}(L_2)} \neq \varnothing$. If L_1 and L_2 are not in the same ι-orbit, then $\overline{q^{-1}(L_1)}$ and $\overline{q^{-1}(L_2)}$ are in distinct connected components of $\overline{q^{-1}(U)}$, a contradiction. So, (ii) holds.

Assume (ii). Since lifts of non-identity paths are unique if they exist (Lemma 7.1.20), it is enough to show the existence of lifts locally on Z. This is clear for a small open neighbourhood of a point of $Z - Z_{exc}$. Consider now $z \in Z_{exc}$ and a small open neighbourhood U of z in Z_o. Let K be a connected component of $\gamma^{-1}(z)$ and let W be the connected component of $\gamma^{-1}(U)$ containing K. There is $L \in \pi_0(U - \{z\})$ such that $\gamma(W) \subset L \cup \{z\} \cup \iota(L)$. Since q splits over $L \cup \{z\} \cup \iota(L)$, it follows that the restriction of γ to W lifts to $\hat{Z}$. □

Definition 7.3.4. *We say that a path γ in Z is* smooth *if it satisfies the equivalent conditions of Lemma 7.3.3.*

We say that a path γ in Z is admissible *if it is oriented and smooth.*

We say that a homotopy class of paths is smooth (resp. admissible, resp. oriented) if it contains a smooth (resp. an admissible, resp. an oriented) path.

Let us note some basic properties of smooth and admissible paths and classes.

Properties 7.3.5.

(1) A path is smooth if and only if its inverse is smooth.
(2) A smooth path is contained in a component of Z.
(3) Every admissible path γ is homotopic to a minimal admissible path via a homotopy involving only admissible paths contained in the support of γ (cf Lemma 7.1.18).
(4) A minimal path in a smooth (resp. admissible) homotopy class is smooth (resp. admissible).
(5) An oriented path is admissible if and only if its homotopy class is admissible (Lemma 7.1.16 provides a minimal oriented path γ_{min} homotopic to a given oriented path γ with the property that γ is admissible if γ_{min} is admissible, hence we obtain the desired equivalence by (4) above).
(6) Given two oriented homotopy classes of paths α and β with $\alpha \circ \beta$ admissible, then α and β are admissible (cf (5) above).

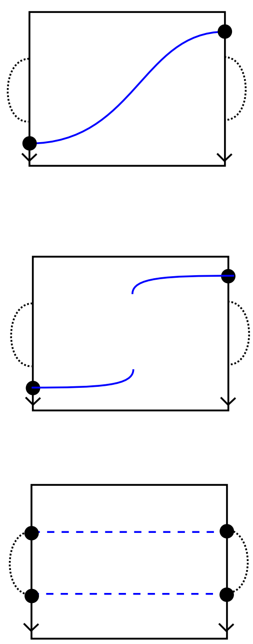
Definition 7.3.6. *Given two smooth non-identity homotopy classes of paths ζ_1 and ζ_2 contained in the same component of Z, there is a unique $\varepsilon \in \{\pm 1\}$ such that there is a minimal smooth path γ in Z with the property that ζ_1 and ζ_2^ε are equal to the classes of restrictions of γ. We say that ζ_1 and ζ_2 have the* same orientation *(resp.* opposite orientation*) if $\varepsilon = 1$ (resp. $\varepsilon = -1$).*

Note that Z^{opp} and Z have the same smooth paths. Note also that the notion of "opposite orientation" does not depend on the orientation of Z.

Remark 7.3.7. Assume $X \subset \mathbf{R}^N$ is obtained by the construction of Remark 7.2.5. A homotopy class of paths in X is smooth if and only if it contains a path γ such that the composition $[0,1] \xrightarrow{\gamma} X \hookrightarrow \mathbf{R}^N$ is a smooth immersion.

Example 7.3.8. We give below some examples of paths. The top and bottom paths are admissible, while the middle one is not. The left and middle columns describe the path in the singular curve, while the right column describes the lifted path (if it exists) in the non-singular cover.

In the middle and right columns, and throughout the paper, we depict paths γ using their time-reversed graphs, so that $\gamma(0)$ is on the right and $\gamma(1)$ is on the left.



7.3.2. *Pointed category of admissible paths.* We now define a category associated with admissible paths.

Definition 7.3.9. *We define $\mathcal{S}^\bullet(Z,1)$ to be the pointed category with object set Z, with*

$$\mathrm{Hom}_{\mathcal{S}^\bullet(Z,1)}(x,y) = \{0\} \sqcup \{\textit{admissible homotopy classes of paths } x \to y\}$$

and

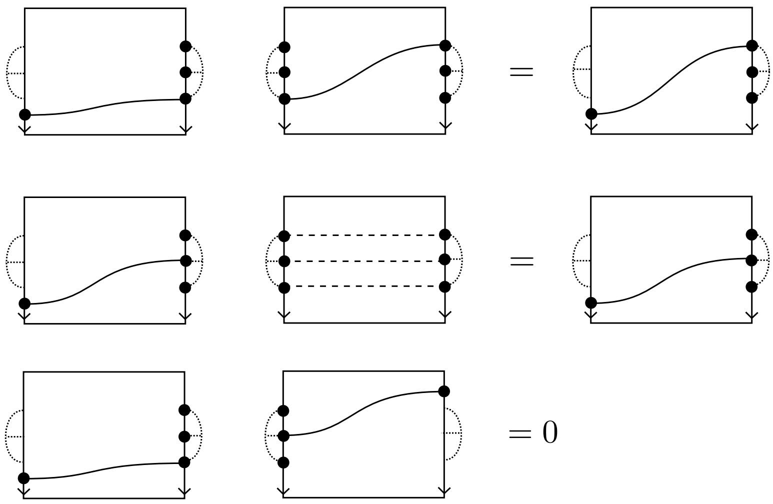
$$\alpha\beta = \begin{cases} \alpha \circ \beta & \textit{if } \alpha \circ \beta \textit{ is admissible} \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 7.3.10. Consider $\Pi_o(Z)$ the category with objects the points of Z and arrows the oriented homotopy classes of paths, a subcategory of $\Pi(Z)$. We define a $\mathbf{Z}_{\geqslant 0}$-filtration on $\Pi_o(Z)$ by defining a class ζ to have degree $\leqslant d$ if it is the product of $d+1$ admissible homotopy classes of paths. The category $\mathcal{S}^\bullet(Z,1)$ is isomorphic to the degree 0 part of $\mathrm{gr}\Pi_o(Z)$.

Note finally that if Z is non-singular, then $\mathcal{S}^\bullet(Z,1)$ is the pointed category associated to $\Pi_o(Z)$.

We put $\mathcal{S}(Z,1) = \mathbf{F}_2[\mathcal{S}^\bullet(Z,1)]$.

Example 7.3.11. We describe below some examples of products in $\mathcal{S}^\bullet(Z,1)$. Here Z is the third singular curve of example 7.2.11 and the paths are drawn in the smooth cover.



7.3.3. *Central extension.* Let $L(Z) = (\bigoplus_{c\in T(Z)} \mathbf{Z}e_c)/(\bigoplus_{c\in T(Z)} \mathbf{Z}(e_c+e_{\iota(c)}))$. We define a bilinear map $\langle -,-\rangle : R(Z)\times R(Z) \to L(Z)$ by

$$\langle \alpha,\beta\rangle = \frac{1}{2}\sum_{c\in T(Z)} (m_{\iota(c)} - m_c)(\alpha)\cdot (m_c + m_{\iota(c)})(\beta) e_c.$$

Note that $(m_c + m_{\iota(c)})(\beta) = 0$ for all but finitely many c's, hence the sum above is finite. More precisely, let ζ be a non-identity homotopy class of paths in Z. We have $\zeta(0+) \neq \zeta(1-)$ and

$$(m_c + m_{\iota(c)})(\zeta) = \begin{cases} 1 & \text{if } c \in \zeta(0+)\cup\iota(\zeta(0+)) \text{ and } c\notin \zeta(1-)\cup\iota(\zeta(1-)) \\ -1 & \text{if } c\in\zeta(1-)\cup\iota(\zeta(1-)) \text{ and } c\notin\zeta(0+)\cup\iota(\zeta(0+)) \\ 0 & \text{otherwise.}\end{cases} \tag{7.3.1}$$

If ζ is admissible and non-identity, then $\zeta(1-)\neq\zeta(0+)$, hence

$$\langle \alpha, [\![\zeta]\!]\rangle = (m_{\iota(\zeta(0+)} - m_{\zeta(0+)})(\alpha) e_{\zeta(0+)} - (m_{\iota(\zeta(1-)} - m_{\zeta(1-)})(\alpha) e_{\zeta(1-)}.$$

We define a group $\Gamma'(Z)$, a central extension of $R(Z)$ by $L(Z)$. The set of elements of $\Gamma'(Z)$ is $L(Z)\times R(Z)$ and the multiplication is given by

$$(m,\alpha)(n,\beta) = (m+n+\langle\alpha,\beta\rangle, \alpha+\beta).$$

We put $\Gamma(Z) = (\bigoplus_{\Omega\in\pi_0(Z)} \frac{1}{2}\mathbf{Z}e_\Omega)\times\Gamma'(Z)$.

Note that $L(Z)$, $\Gamma'(Z)$ and $\Gamma(Z)$ depend only on the 1-dimensional space underlying Z and on ι.

Let D be a subset of $T(Z)$ such that $D \cap \iota(D) = \emptyset$. We denote by $\Gamma_D(Z)$ the quotient of $\Gamma(Z)$ by the central subgroup generated by $\{e_c - \frac{1}{2}e_\Omega\}$, where $c \in D$ and Ω is the connected component of Z containing c. The canonical map $\bigoplus_{\Omega\in\pi_0(Z)} \frac{1}{2}\mathbf{Z}e_\Omega \to \Gamma_D(Z)$ is injective and we identify $(\frac{1}{2}\mathbf{Z})^{\pi_0(Z)}$ with its image.

We put a partial order on $\Gamma_D(Z)$ by setting $g_1 \geqslant g_2$ if $g_1g_2^{-1} \in (\frac{1}{2}\mathbf{Z}_{\geqslant 0})^{\pi_0(Z)}$.

We define $\bar{\Gamma}_D(Z)$ to be the quotient of $\Gamma_D(Z)$ by the central subgroup generated by $\frac{1}{2}e_\Omega - \frac{1}{2}e_{\Omega'}$ for $\Omega, \Omega' \in \pi_0(Z)$.

The image of $(\frac{1}{2}\mathbf{Z}_{\geqslant 0})^{\pi_0(Z)}$ in $\bar{\Gamma}_D(Z)$ is $\frac{1}{2}\mathbf{Z}$ (where $\frac{1}{2}e_\Omega \mapsto \frac{1}{2}$). Let $r \in \frac{1}{2}\mathbf{Z}$. We still denote by r the image of r in $\bar{\Gamma}_D(Z)$. Given $x \in \bar{\Gamma}_D(Z)$, we put $x + r = x \cdot r = r \cdot x$.

We define a partial order on $\bar{\Gamma}_D(Z)$ by setting $g_1 \geqslant g_2$ if $g_1g_2^{-1} \in \frac{1}{2}\mathbf{Z}_{\geqslant 0}$.

Given $z \in Z_o$, we denote by $C(z)^+$ the set of $c \in C(z)$ such that there is an oriented path γ in Z with $m_c^+(\gamma) = 1$. Note that $C(z) = C(z)^+ \coprod \iota(C(z)^+)$. Note also that given ζ an oriented homotopy class of paths in Z, we have

$$m_c(\zeta)e_c + m_{\iota(c)}(\zeta)e_{\iota(c)} = (m_c^+(\zeta) + m_{\iota(c)}^-(\zeta))e_c \text{ for } z \in Z_o \text{ and } c \in C(z)^+. \tag{7.3.2}$$

Given E a subset of Z_o, we put $E^+ = \coprod_{z\in E} C(z)^+$.

Remark 7.3.12. Fix an orientation of each component of Z (forgetting about the already given orientation of Z_o) and define $Z^+ \subset T(Z)$ to be the set of pairs (z, c) such that there is an oriented path γ in Z (for the given new orientation) with $m_c^+(\gamma) = 1$.

There is a quotient map $L(Z) \to \mathbf{Z}^{\pi_0(Z)}$ given by $e_c \mapsto e_\Omega$ for all $s \in \Omega$ and $(z, c) \in Z^+$. Let us show that the bilinear form $R(Z) \times R(Z) \to \mathbf{Z}^{\pi_0(Z)}$ obtained by composing $\langle -, -\rangle$ with this quotient map is antisymmetric. Let γ and γ' be two injective oriented paths in Z (for the given new orientation). If the supports of γ and γ' are disjoint, then $\langle [\![\gamma]\!], [\![\gamma']\!]\rangle = 0$. We have $\langle [\![\gamma]\!], [\![\gamma]\!]\rangle = -e_{\gamma(0+)} - e_{\gamma(1-)}$. If $\gamma([0, 1]) \cap \gamma'([0, 1]) = \{\gamma(1)\}$, then

$$\langle [\![\gamma]\!], [\![\gamma']\!]\rangle = -e_{\gamma'(0+)} \text{ and } \langle [\![\gamma]\!], [\![\gamma']\!]\rangle = -e_{\gamma(1-)}.$$

We deduce the antisymmetry statement.

7.3.4. *Functoriality.* Let $f : Z \to Z'$ be a morphism of curves.

Lemma 7.3.13. *Let ζ be a homotopy class of paths in Z. The class $f(\zeta)$ is smooth if and only if ζ is smooth. If ζ is admissible, then $f(\zeta)$ is admissible.*

Proof. Given γ an oriented path in Z, the path $f(\gamma)$ is oriented. It is smooth if and only $f(\gamma)$ is smooth. This shows that if ζ is a smooth (resp. admissible) homotopy class of paths in Z, then $f(\zeta)$ is smooth (resp. admissible).

Consider now ζ a homotopy class of paths in Z such that $f(\zeta)$ is smooth. Given γ a minimal path in ζ, then $f(\gamma)$ is minimal (Lemma 7.1.20). Since $f(\zeta)$ is smooth, it follows that $f(\gamma)$ is smooth (Properties 7.3.5(4)), hence γ is smooth and finally ζ is smooth. ☐

It follows from the previous lemma that the morphism f induces a functor $f : \mathcal{S}^\bullet(Z, 1) \to \mathcal{S}^\bullet(Z', 1)$. We have constructed a functor $\mathcal{S}^\bullet(-, 1)$ from the category of curves to the category of pointed categories.

Let us state a version of Lemma 7.1.20 for morphisms of curves.

Lemma 7.3.14. *Let γ, γ' be two admissible paths in Z. If $[f(\gamma)] = [f(\gamma')] \neq \mathrm{id}$, then $[\gamma] = [\gamma']$. The functor $f : \mathcal{S}^\bullet(Z, 1) \to \mathcal{S}^\bullet(Z', 1)$ is faithful.*

Note that f induces an injective morphism of groups $f : L(Z) \to L(Z')$ and a map $f : \pi_0(Z) \to \pi_0(Z')$.

The next lemma is an immediate consequence of Lemma 7.1.24.

Lemma 7.3.15. *Given $\alpha, \beta \in R(Z)$, we have $\langle f(\alpha), f(\beta)\rangle = f(\langle \alpha, \beta\rangle)$.*

It follows from Lemmas 7.3.15 and 7.1.25 that we have a morphism of groups $f : \Gamma(Z) \to \Gamma(Z')$, $(r, (m, \alpha)) \mapsto (f(r), (f(m), f(\alpha))$ which restricts to an injective morphism of groups $\Gamma'(Z) \to \Gamma'(Z')$.

Let D be a subset of $T(Z)$ such that given $z \in \mathrm{pt}(D)$, the composition $D \cap \mathrm{pt}^{-1}(z) \to C(z) \to C(z)/\iota$ is bijective. The morphism $f : \Gamma(Z) \to \Gamma(Z')$ induces a morphism $f : \Gamma_D(Z) \to \Gamma_{f(D)}(Z')$. Let $g, h \in \Gamma_D(Z)$. If $g < h$, then $f(g) < f(h)$. If $f : \pi_0(Z) \to \pi_0(Z')$ is injective and $f(g) < f(h)$, then $g < h$.

Finally, the morphism $f : \Gamma_D(Z) \to \Gamma_{f(D)}(Z')$ induces a morphism $f : \bar{\Gamma}_D(Z) \to \bar{\Gamma}_{f(D)}(Z')$. Given $g, h \in \bar{\Gamma}_D(Z)$, we have $g < h$ if and only if $f(g) < f(h)$.

Let $Z_1, \ldots, Z_r$ be the connected components of Z. There are isomorphisms of groups $R(Z_1) \times \cdots \times R(Z_r) \xrightarrow{\sim} R(Z)$ and $L(Z_1) \times \cdots \times L(Z_r) \xrightarrow{\sim} L(Z)$ given by the inclusions $Z_i \hookrightarrow Z$. They induce an isomorphism of groups

$$\Gamma(Z_1) \times \cdots \times \Gamma(Z_r) \xrightarrow{\sim} \Gamma(Z). \tag{7.3.3}$$

The inclusions $Z_i \hookrightarrow Z$ induce pointed functors $\mathcal{S}^\bullet(Z_i, 1) \to \mathcal{S}^\bullet(Z, 1)$ and give rise to an isomorphism of pointed categories

$$\mathcal{S}^\bullet(Z_1, 1) \vee \cdots \vee \mathcal{S}^\bullet(Z_r, 1) \xrightarrow{\sim} \mathcal{S}^\bullet(Z, 1). \tag{7.3.4}$$

7.3.5. *Pullback.* Let $f : Z \to Z'$ be a morphism of curves. We define a non-multiplicative "functor" $f^\# : \mathrm{add}(\mathcal{S}(Z', 1)) \to \mathrm{add}(\mathcal{S}(Z, 1))$. It commutes with coproducts but is not a functor, i.e., it is not compatible with composition for a general f. We put $f^\#(z') = \coprod_{z \in f^{-1}(z')} z$. Given $\zeta' \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z',1)}(z'_1, z'_2)$ non-zero, we define $f^\#(\zeta')$ to be

- id if $\zeta' = \mathrm{id}$
- 0 if ζ' does not lift to an admissible class of paths in Z
- the composition
$$\coprod_{z \in f^{-1}(z'_1)} z \xrightarrow{\text{projection}} z_1 \xrightarrow{\zeta} z_2 \xrightarrow{\text{inclusion}} \coprod_{z \in f^{-1}(z'_2)} z$$
where $\zeta : z_1 \to z_2$ is the unique lift of ζ', otherwise (cf Lemma 7.3.14).

We denote by $f^{-1}(\zeta')$ the set of admissible lifts of ζ'. We have $f^\#(\zeta') = \sum_{\zeta \in f^{-1}(\zeta')} \zeta$.

Given $\zeta'_1 \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z',1)}(z'_1, z'_2)$ and $\zeta'_2 \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z',1)}(z'_2, z'_3)$ such that $f^\#(\zeta'_1) \neq 0$ and $f^\#(\zeta'_2) \neq 0$, we have $f^\#(\zeta'_2) f^\#(\zeta'_1) = f^\#(\zeta'_2\zeta'_1)$ (cf Lemma 7.3.13).

Given $f' : Z' \to Z''$ a morphism of curves, we have $(f'f)^\# = f^\# f'^\#$.

Lemma 7.3.16. *Let γ' be a smooth path in Z'. Consider the following assertions:*

(1) *γ' lifts to a smooth path in Z*
(2) *$\gamma'([0,1]) \subset f(Z)$.*
(3) *$[\gamma']$ lifts to a smooth homotopy class in Z*
(4) *$\mathrm{supp}([\gamma']) \subset f(Z)$.*

We have (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4).

Assume f is strict. Then (3) $\Rightarrow$ (2). Furthermore, if γ' is admissible and it lifts to a smooth path in Z, then that path is admissible.

Proof. The implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear. We can assume that γ' is not constant, for otherwise the other implications are trivial.

Assume (2). Let $\hat{f} : \hat{Z} \to \hat{Z}'$ be the map between non-singular covers corresponding to f. Since γ' is smooth, it lifts uniquely to a path $\hat{\gamma}'$ on $\hat{Z}'$ and $\hat{\gamma}'([0,1]) \subset \hat{f}(\hat{Z})$. Since $\hat{f}$ is an open embedding, it follows that $\hat{\gamma}'$ is the image of a path of $\hat{Z}$. Its image in Z is a smooth path that lifts γ', hence (1) holds.

Assume (4). Let γ'_0 be a minimal smooth path homotopic to γ' (cf Properties 7.3.5(4)). We have $\gamma'_0([0,1]) = \mathrm{supp}([\gamma']) \subset f(Z)$, hence γ'_0 lifts to a smooth path in Z. So (3) holds.

Assume (3) and f is strict. Note that $\gamma'([0,1]) \cap Z'_o = \mathrm{supp}([\gamma']) \cap Z'_o$ and $\gamma'([0,1]) \cap Z'_u$ is contained in the union of the connected components of Z'_u that have a non-empty intersection with $\mathrm{supp}([\gamma'])$ (Properties 7.3.2(1)). Since $f(Z_u)$ is open and closed in Z'_u, it follows that $\gamma'([0,1]) \subset f(Z)$, so (2) holds.

Assume γ' is admissible and lifts to Z. Since f is strict, it follows that the lift is oriented. $\square$

Since quotient maps are strict, we have the following consequence of Lemma 7.3.16.

Lemma 7.3.17. *Assume f is the quotient map of Z by a finite relation. Every non-constant admissible path in Z' lifts uniquely to a path in Z and that lift is admissible.*

Proposition 7.3.18. *If f is strict, then $f^\# : \mathrm{add}(\mathcal{S}(Z',1)) \to \mathrm{add}(\mathcal{S}(Z,1))$ is a functor.*

Proof. We need to check that $f^\#$ is compatible with composition. This is clear if Z and Z' are non-singular. In general, consider two maps ζ'_1 and ζ'_2 in $\mathcal{S}(Z',1)$ such that $f^\#(\zeta'_2 \circ \zeta'_1) \neq 0$. Let $\hat{f} : \hat{Z} \to \hat{Z}'$ be the map corresponding to f between non-singular covers $q : \hat{Z} \to Z$ and $q' : \hat{Z}' \to Z'$.

We have

$$\begin{aligned} q^\# f^\#(\zeta'_2 \circ \zeta'_1) &= \hat{f}^\# q'^\#(\zeta'_2 \circ \zeta'_1) = \hat{f}^\#\big(q'^\#(\zeta'_2) \circ q'^\#(\zeta'_1)\big) = \big(\hat{f}^\# q'^\#(\zeta'_2)\big) \circ \big(\hat{f}^\# q'^\#(\zeta'_1)\big) \\ &= q^\# f^\#(\zeta'_2) \circ q^\# f^\#(\zeta'_1), \end{aligned}$$

hence $f^\#(\zeta'_1) \neq 0$ and $f^\#(\zeta'_2) \neq 0$ since $q^\# f^\#(\zeta'_2 \circ \zeta'_1) \neq 0$ by Lemma 7.3.17. It follows that $f^\#(\zeta'_2 \circ \zeta'_1) = f^\#(\zeta'_2) \circ f^\#(\zeta'_1)$. $\square$

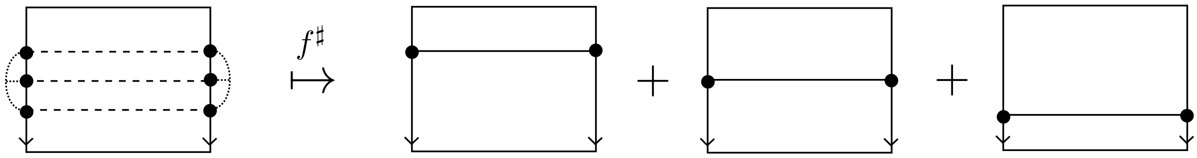
The construction $Z \mapsto \mathrm{add}(\mathcal{S}(Z,1))$ and $f \mapsto f^\#$ defines a contravariant functor from the category of curves with strict morphisms to the category of $\mathbf{F}_2$-linear categories.

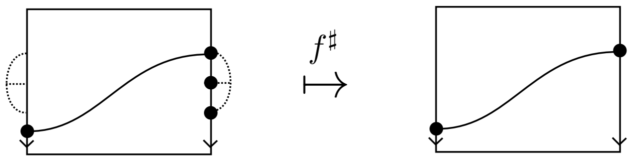
Lemma 7.3.17 and Proposition 7.3.18 have the following consequence.

Proposition 7.3.19. *Let Z be a curve with an admissible relation $\sim$ and let $q : Z \to Z/{\sim}$ be the quotient map. The functor $q^\# : \mathrm{add}(\mathcal{S}(Z/{\sim},1)) \to \mathrm{add}(\mathcal{S}(Z,1))$ is faithful.*

Note that Proposition 7.3.19 provides an identification of $\mathcal{S}(Z/{\sim},1)$ with a (non-full) subcategory of $\mathrm{add}(\mathcal{S}(Z,1))$.

Example 7.3.20. We describe the image by the map $f^{\#}$ of two paths, the first of which is the constant path at the singular point of Z_{exc} (we draw the lifts in the non-singular cover).





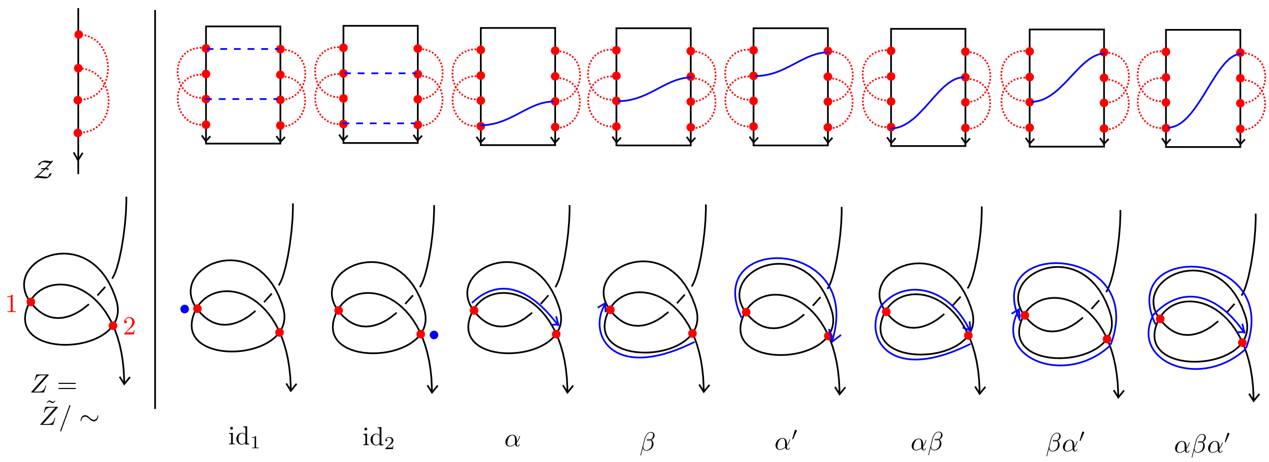
7.3.6. *One strand bordered algebras.* Consider an chord diagram $(\mathcal{Z},\mathbf{a},\mu)$ as in §7.2.4 with associated singular curve Z. Proposition 7.3.19 shows the category $\mathcal{S}(Z,1)$ has a full subcategory corresponding to Zarev's algebra $A(\mathcal{Z},1)$ [Za, Definition 2.6] (this will be explained for the more general algebras $\mathcal{S}(Z,i)$ in §7.4.11):

$$A(\mathcal{Z},1)\xrightarrow{\sim}\mathrm{End}_{\mathrm{add}(\mathcal{S}(Z,1))}\Big(\bigoplus_{z\in Z_{exc}} z\Big).$$

- Consider the chord diagram $(\mathbf{R},\{1,2,3,4\},\mu:\begin{matrix}1,3\mapsto 1\\ 2,4\mapsto 2\end{matrix})$.

The associated singular curve Z is the quotient of oriented $\mathbf{R}$ by the relation whose non trivial equivalence classes are $1=\{1,3\}$ and $2=\{2,4\}$.

The full pointed subcategory of $\mathcal{S}(Z,1)$ with object set $\{1,2\}$ is generated by $\alpha,\alpha':1\to 2$ and $\beta:2\to 1$ with relations $\beta\alpha=\alpha'\beta=0$. This corresponds to the well-known "torus-algebra" in bordered Floer homology.



- Consider the chord diagram $(S^1,\{1,i,-1,-i\},\mu:\begin{matrix}\pm 1\mapsto 1\\ \pm i\mapsto 2\end{matrix})$.

The associated singular curve Z is the quotient of oriented S^1 by the relation whose non trivial equivalence classes are $1 = \{\pm 1\}$ and $2 = \{\pm i\}$.

The full pointed subcategory of $\mathcal{S}(Z,1)$ with object set $\{1,2\}$ is generated by $\alpha, \alpha' : 1 \to 2$ and $\beta, \beta' : 2 \to 1$ with relations $\beta\alpha = \alpha'\beta = \alpha\beta' = \beta'\alpha' = 0$. A curved A_∞-deformation of this subcategory appears in a work in preparation of Lipshitz, Ozsváth and Thurston [LiOzTh2].

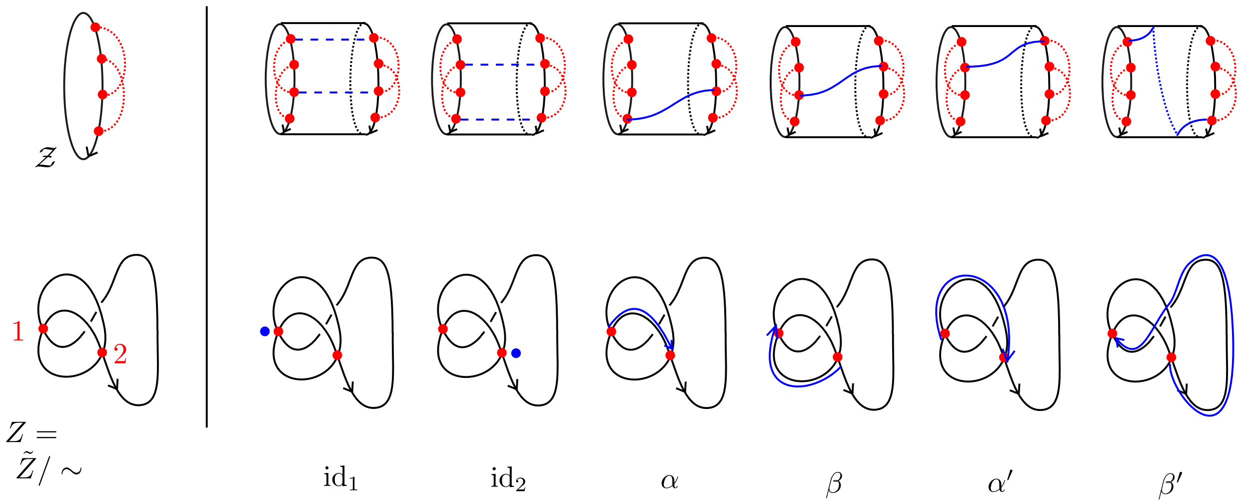
We have

$$\mathrm{End}_{\mathcal{S}(Z,1)}(1) = \{\mathrm{id}\} \sqcup \{(\beta'\alpha\beta\alpha')^n\}_{n\geqslant 1} \sqcup \{(\beta'\alpha\beta\alpha')^n\beta'\alpha\}_{n\geqslant 0} \sqcup \{\beta\alpha'(\beta'\alpha\beta\alpha')^n)\}_{n\geqslant 0} \sqcup \{(\beta\alpha'\beta'\alpha)^n)\}_{n\geqslant 1}$$

$$\mathrm{End}_{\mathcal{S}(Z,1)}(2) = \{\mathrm{id}\} \sqcup \{(\alpha'\beta'\alpha\beta)^n\}_{n\geqslant 1} \sqcup \{(\alpha'\beta'\alpha\beta)^n\alpha'\beta'\}_{n\geqslant 0} \sqcup \{\alpha\beta(\alpha'\beta'\alpha\beta)^n)\}_{n\geqslant 0} \sqcup \{(\alpha\beta\alpha'\beta')^n)\}_{n\geqslant 1}$$

$$\mathrm{Hom}_{\mathcal{S}(Z,1)}(1,2) = \{\alpha'(\beta'\alpha\beta\alpha')^n\}_{n\geqslant 0} \sqcup \{\alpha\beta\alpha'(\beta'\alpha\beta\alpha')^n\}_{n\geqslant 0} \sqcup \{\alpha'\beta'\alpha(\beta\alpha'\beta'\alpha)^n)\}_{n\geqslant 0} \sqcup \{\alpha(\beta\alpha'\beta'\alpha)^n)\}_{n\geqslant 0}$$

$$\mathrm{Hom}_{\mathcal{S}(Z,1)}(2,1) = \{(\beta'\alpha\beta\alpha')^n\beta'\}_{n\geqslant 0} \sqcup \{(\beta'\alpha\beta\alpha')^n\beta'\alpha\beta'\}_{n\geqslant 0} \sqcup \{(\beta\alpha'\beta'\alpha)^n)\beta\}_{n\geqslant 0} \sqcup \{(\beta\alpha'\beta'\alpha)^n)\beta\alpha'\beta'\}_{n\geqslant 0}$$



7.3.7. *Intersection multiplicity.* Let γ and γ' be two paths in Z. We consider the number of intersection points between the graphs of γ and γ'

$$i(\gamma, \gamma') = |\{t \in [0,1] \mid \gamma(t) = \gamma'(t)\}| \in \mathbf{Z}_{\geqslant 0} \cup \{\infty\}.$$

Note that $i(\gamma_1 \circ \gamma_2, \gamma_1' \circ \gamma_2') = i(\gamma_1, \gamma_1') + i(\gamma_2, \gamma_2') - \delta_{\gamma_1(0) = \gamma_1'(0)}$.

Given ζ and ζ' two admissible homotopy classes of paths in Z, we put

$$i(\zeta, \zeta') = \min_{\gamma, \gamma'} i(\gamma, \gamma'),$$

where γ (resp. γ') runs over admissible paths in $[\zeta]$ (resp. in $[\zeta']$). Note that $i(\zeta_1\zeta_2, \zeta_1'\zeta_2') \leqslant i(\zeta_1, \zeta_1') + i(\zeta_2, \zeta_2') - \delta_{\zeta_1(0) = \zeta_1'(0)}$.

The next lemma relates the intersection multiplicity with a constant path and tangential multiplicities.

Lemma 7.3.21. *Let γ_0 be a minimal admissible path in Z and let $z \in Z$. We have*

$$i([\gamma_0], \mathrm{id}_z) = \min_{\substack{\gamma \text{ admiss.} \\ [\gamma] = [\gamma_0]}} i(\gamma, \mathrm{id}_z) = i(\gamma_0, \mathrm{id}_z) = \frac{1}{2}\Big(\sum_{c \in C(z)} (m_c^+([\gamma_0]) + m_c^-([\gamma_0])) + \delta_{\gamma_0(0) = z} + \delta_{\gamma_0(1) = z}\Big).$$

If $z \in Z_o$, then we have

$$i([\gamma_0], \mathrm{id}_z) = \frac{1}{2}\Big(\sum_{c \in C(z)^+} (m_c([\gamma_0]) - m_{\iota(c)}([\gamma_0])) + \delta_{\gamma_0(0)=z} + \delta_{\gamma_0(1)=z}\Big).$$

Proof. Note that

$$i([\gamma_0], \mathrm{id}_z) \leqslant \min_{\substack{\gamma \text{ admiss.} \\ [\gamma]=[\gamma_0]}} i(\gamma, \mathrm{id}_z) \leqslant i(\gamma_0, \mathrm{id}_z).$$

The third equality of the lemma follows from Lemma 7.1.21.

When $Z = S^1$ unoriented, the lemma follows from Lemma 6.2.3.

When Z is a connected non-singular curve, there is an injective morphism of curves $f : Z \to S^1$. We have $i([\gamma_0], \mathrm{id}_z) \geqslant i(f([\gamma_0]), \mathrm{id}_{f(z)}) = i(f(\gamma_0), \mathrm{id}_{f(z)}) = i(\gamma_0, \mathrm{id}_z)$, hence the first two equalities of the lemma hold for Z. It follows that they hold for any non-singular curve.

Consider now a general Z and let $q : \hat{Z} \to Z$ be the non-singular cover. Let $\hat{\gamma}_0$ be the lift of γ_0 to $\hat{Z}$. We have

$$i(\gamma_0, \mathrm{id}_z) = \sum_{\hat{z} \in f^{-1}(z)} i(\hat{\gamma}_0, \mathrm{id}_{\hat{z}}) = \sum_{\hat{z} \in f^{-1}(z)} ([\hat{\gamma}_0], \mathrm{id}_{\hat{z}}) \leqslant i([\gamma_0], \mathrm{id}_z).$$

We deduce that the first two equalities of the lemma hold.

The last equality of the lemma follows from (7.3.2). ☐

Let us now state some basic properties of intersection counts.

Lemma 7.3.22. *Let ζ_1 and ζ_2 be two admissible homotopy classes of paths in Z. Assume $\zeta_1(t) \neq \zeta_2(t)$ for $t \in \{0, 1\}$.*

(1) *We have $i(\zeta_1, \zeta_2) < \infty$.*
(2) *There are minimal or identity admissible paths γ_1 in ζ_1 and γ_2 in ζ_2 such that $i(\zeta_1, \zeta_2) = i(\gamma_1, \gamma_2)$.*
(3) *Given $f : Z' \to Z$ a morphism of curves such that ζ_1 and ζ_2 are images of admissible homotopy classes of paths in Z', we have $i(\zeta_1, \zeta_2) = \sum_{\zeta_i' \in f^{-1}(\zeta_i)} i(\zeta_1', \zeta_2')$.*

Proof. • Assume ζ_1 or ζ_2 is an identity. In that case, (1) and (2) follow from Lemma 7.3.21 and (3) follows from Lemmas 7.1.24 and 7.3.21.

From now on, we assume that neither ζ_1 nor ζ_2 is an identity.

• Let $f : Z \to Z'$ be an injective morphism of curves and assume $f(\zeta_1)$ and $f(\zeta_2)$ satisfy (1) and (2). We have $i(f(\zeta_1), f(\zeta_2)) \leqslant i(\zeta_1, \zeta_2)$. There are minimal admissible paths γ_i' in $f(\zeta_i)$ for $i \in \{1, 2\}$ such that $i(f(\zeta_1), f(\zeta_2)) = i(\gamma_1', \gamma_2')$. There are admissible paths γ_i of Z such that $\gamma_i' = f(\gamma_i)$ for $i \in \{1, 2\}$. It follows that $i(\zeta_1, \zeta_2) \geqslant i(\gamma_1', \gamma_2') = i(\gamma_1, \gamma_2)$, hence $i(f(\zeta_1), f(\zeta_2)) = i(\zeta_1, \zeta_2)$. We deduce also that (1) and (2) hold for ζ_1 and ζ_2.

• Assume $Z = S^1$ unoriented. The assertions (1) and (2) follow from Lemma 6.2.3.

• Assume Z is non-singular and connected. There is an injective map $f : Z \to S^1$. It follows that Z satisfies (1) and (2). This shows that (1) and (2) hold for a general non-singular curve.

• Let Z' be an arbitrary curve and let $f : Z \to Z'$ be the non-singular cover of Z'. Assume $f(\zeta_1(t)) \neq f(\zeta_2(t))$ for $t \in \{0, 1\}$. Since all admissible paths in Z' lift to Z, it follows that $i(\zeta_1, \zeta_2) \leqslant i(f(\zeta_1), f(\zeta_2))$.

Consider two minimal admissible paths γ_1 and γ_2 in ζ_1 and ζ_2 such that $i(\gamma_1,\gamma_2)=i(\zeta_1,\zeta_2)$. We assume that given $\rho_1,\rho_2:[0,1]\xrightarrow{\sim}[0,1]$ any two homeomorphisms fixing 0 and 1 and such that $i(\gamma_1\circ\rho_1,\gamma_2\circ\rho_2)=i(\zeta_1,\zeta_2)$, we have $i(f(\gamma_1),f(\gamma_2))\leqslant i(f(\gamma_1\circ\rho_1),f(\gamma_2\circ\rho_2))$. Let $t_0\in(0,1)$ such that $\gamma_1(t_0)\neq\gamma_2(t_0)$ but $f(\gamma_1(t_0))=f(\gamma_2(t_0))$. There is a small open neighbourhood U of $z'=f(\gamma_1(t_0))$ homeomorphic to $\mathrm{St}(n_{z'})$ and with $U\cap f(Z_f)=\{z'\}$ and there are $0\leqslant t_1<t_0<t_2\leqslant 1$ such that $f(\gamma_1)([t_1,t_2])\subset U$ and $f(\gamma_2)([t_1,t_2])\subset U$. The paths $(\gamma_1)_{|[t_1,t_2]}$ and $(\gamma_2)_{|[t_1,t_2]}$ are contained in disjoint connected components of $f^{-1}(U)$, hence $f(\gamma_1)([t_1,t_2])\cap f(\gamma_2)([t_1,t_2])=\{z'\}$. So, by reparametrizing $f(\gamma_1)$ and $f(\gamma_2)$ in the interval $[t_1,t_2]$, we can assume they do not have a common value in that interval. This contradicts the minimality of $i(f(\gamma_1),f(\gamma_2))$. It follows that

$$i(\zeta_1,\zeta_2)=i(\gamma_1,\gamma_2)=i(f(\gamma_1),f(\gamma_2))\geqslant i(f(\zeta_1),f(\zeta_2)),$$

hence $i(\zeta_1,\zeta_2)=i(f(\zeta_1),f(\zeta_2))$. This shows that (1) and (2) hold for $f(\gamma_1)$ and $f(\gamma_2)$. We deduce that (1) and (2) hold in full generality. It follows also that (3) holds when f is injective.

- Consider now a morphism of curves $f:Z\to Z'$. Consider the map $\hat{f}:\hat{Z}\to\hat{Z}'$ between non-singular covers corresponding to f. Let $\hat{\zeta}_i$ be the lift of ζ_i to $\hat{Z}$. Since $\hat{f}$ is injective, it follows that $i(\hat{f}(\hat{\zeta}_1),\hat{f}(\hat{\zeta}_2))=i(\hat{\zeta}_1,\hat{\zeta}_2)$. The study above shows that $i(\hat{f}(\hat{\zeta}_1),\hat{f}(\hat{\zeta}_2))=i(f(\zeta_1),f(\zeta_2))$ and $i(\hat{\zeta}_1,\hat{\zeta}_2)=i(\zeta_1,\zeta_2)$. It follows that $i(f(\zeta_1),f(\zeta_2))=i(\zeta_1,\zeta_2)$. This completes the proof of the lemma. ☐

We provide now an upper bound for intersections involving a composition of paths.

Lemma 7.3.23. *Consider ζ, ζ_1 and ζ_2 three homotopy classes of admissible paths in Z. Assume ζ is not an identity, $\zeta_2(1)=\zeta_1(0)$, $\zeta(0)\neq\zeta_2(0)$ and $\zeta(1)\neq\zeta_1(1)$. We have*

$$i(\zeta,\zeta_1\circ\zeta_2)\leqslant\min(m^+_{\zeta(0+)}(\zeta_2)+i(\zeta,\zeta_1),m^-_{\zeta(1-)}(\zeta_1)+i(\zeta,\zeta_2)).$$

Proof. Let ζ' and ζ'' be homotopy classes of admissible paths such that $\zeta=\zeta'\circ\zeta''$. We have

$$i(\zeta,\zeta_1\circ\zeta_2)\leqslant i(\zeta',\zeta_1)+i(\zeta'',\zeta_2).$$

Let γ be a minimal path in ζ and let $t\in(0,1)$. We have $m_{\zeta(0)^+}(\zeta_2)=i(\gamma_{|[0,t]},\zeta_2)$ for t small enough. Since $i([\gamma_{[t,1]}],\zeta_1)\leqslant i(\zeta,\zeta_1)$, it follows that

$$i(\zeta,\zeta_1\circ\zeta_2)\leqslant i(\zeta,\zeta_1)+m_{\zeta(0)^+}(\zeta_2).$$

The second inequality follows from the first one by replacing Z by Z^{opp}. ☐

Recall that we denote by $\Pi(Z)$ the fundamental groupoid of Z. Consider ζ_1,ζ_2 two admissible homotopy classes of paths in Z with $\zeta_1(t)\neq\zeta_2(t)$ for $t\in\{0,1\}$.

Let $I(\zeta_1,\zeta_2)$ be the set of non-identity classes $\zeta\in\mathrm{Hom}_{\Pi(Z)}(\zeta_1(0),\zeta_2(0))$ such that

(i) ζ, $\zeta_2\circ\zeta$ and $\zeta\circ\zeta_1^{-1}$ are smooth
(ii) ζ and $\bar{\zeta}:=\zeta_2\circ\zeta\circ\zeta_1^{-1}$ have opposite orientations (cf Definition 7.3.6).

Note that there are bijections

$$\mathrm{inv}:I(\zeta_1,\zeta_2)\xrightarrow{\sim}I(\zeta_2,\zeta_1),\ \zeta\mapsto\zeta^{-1}\text{ and }I(\zeta_1,\zeta_2)\xrightarrow{\sim}I(\zeta_1^{-1},\zeta_2^{-1}),\ \zeta\mapsto\bar{\zeta}.$$

If Z is non-singular, then the condition (i) in the definition of $I(\zeta_1,\zeta_2)$ is automatically satisfied.

Let $f : Z \to Z'$ be a morphism of curves. If $f(\zeta_1(t)) \neq f(\zeta_2(t))$ for $t \in \{0,1\}$, then the map f induces an injection $I(\zeta_1,\zeta_2) \hookrightarrow I(f(\zeta_1), f(\zeta_2))$ with image $f\big(\mathrm{Hom}_{\Pi(Z)}(\zeta_1(0),\zeta_2(0))\big) \cap I(f(\zeta_1), f(\zeta_2))$.

The next lemma is immediate.

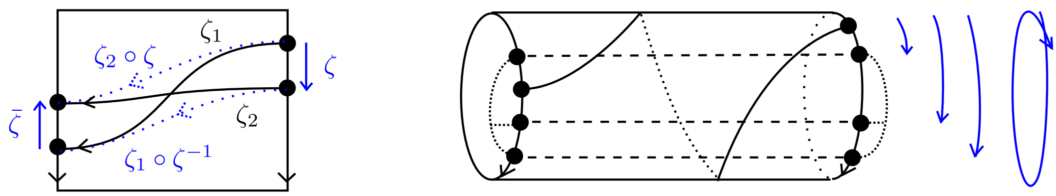
Lemma 7.3.24. *Let $q : \hat{Z} \to Z$ be the non-singular cover of Z. The map q induces a bijection*

$$\coprod_{\hat{\zeta}_i \in q^{-1}(\zeta_i)} I(\hat{\zeta}_1,\hat{\zeta}_2) \xrightarrow{\sim} I(\zeta_1,\zeta_2).$$

Lemma 7.3.25. *If $\zeta \in I(\zeta_1,\zeta_2)$, then* $\mathrm{supp}(\zeta) \subset \mathrm{supp}(\zeta_1) \cup \mathrm{supp}(\zeta_2)$.

Proof. Consider three non-identity homotopy classes of paths ζ, ζ_1 and ζ_2 in $\mathbf{R}$ with $\zeta(0) = \zeta_1(0) \neq \zeta(1) = \zeta_2(0)$. If ζ and $\zeta_2 \circ \zeta \circ \zeta_1^{-1}$ have opposite orientations, then $\mathrm{supp}(\zeta) \subset \mathrm{supp}(\zeta_1) \cup \mathrm{supp}(\zeta_2)$. We deduce that the lemma holds for $Z = S^1$ by using the universal cover of Z. As a consequence, the lemma holds when Z is connected and smooth by embedding it in S^1, hence it holds for Z smooth. Lemma 7.3.24 shows that the lemma holds for any Z, since it holds for the non-singular cover of Z. ☐

Example 7.3.26. In the two examples below, we describe the set $I(\zeta_1,\zeta_2)$. In the second example, ζ_2 is the identity at the singular point.



7.4. **Strands.**

7.4.1. *Braids.* Let Z be a curve. Let I and J be two finite subsets of Z.

Definition 7.4.1. *A* parametrized braid $I \to J$ *is a family $\vartheta = (\vartheta_s)_{s\in I}$ where ϑ_s is an admissible path in Z with $\vartheta_s(0) = s$ and such that $s \mapsto \vartheta_s(1)$ defines a bijection $\chi(\vartheta) : I \xrightarrow{\sim} J$. A* braid *$I \to J$ is a homotopy class of parametrized braids,* i.e., *a family of admissible homotopy classes of paths.*

Definition 7.4.2. *We define the pre-strand category $\mathcal{P}^\bullet(Z) = S(\mathcal{S}^\bullet(Z,1))$ (cf §2.4).*

The objects of this pointed category are the finite subsets of Z and $\mathrm{Hom}_{\mathcal{P}^\bullet(Z)}(I,J)$ is the set of braids $I \to J$, together with a 0-element. Given $\theta : I \to J$ and $\theta' : J \to K$ two braids, we have $\theta' \circ \theta = (\theta'_{\theta_s(1)} \circ \theta_s)_{s\in I}$ if $\theta'_{\theta_s(1)} \circ \theta_s$ is admissible for all $s \in I$, and we have $\theta' \circ \theta = 0$ otherwise. If $\theta' \circ \theta \neq 0$, we have $\chi(\theta' \circ \theta) = \chi(\theta') \circ \chi(\theta)$.

We put $\mathcal{P}(Z) = \mathbf{F}_2[\mathcal{P}^\bullet(Z)]$.

Note that there is a decomposition $\mathcal{P}^\bullet(Z) = \bigvee_{n\geqslant 0} \mathcal{P}^\bullet(Z,n)$, where $\mathcal{P}^\bullet(Z,n)$ is the full subcategory of $\mathcal{P}^\bullet(Z)$ with objects subsets with n elements. We have $\mathcal{P}^\bullet(Z,1) = \mathcal{S}^\bullet(Z,1)$.

Given M a subset of Z, we denote by $\mathcal{P}^\bullet_M(Z)$ the full subcategory of $\mathcal{P}^\bullet(Z)$ with objects the finite subsets of M.

Given $\theta : I \to J$ a braid and I' a subset of I, we denote by $\theta_{|I'}$ the braid $(\theta_s)_{s\in I'}$.

Let $f : Z \to Z'$ be a morphism of curves. We denote by $\mathcal{P}^\bullet_f(Z)$ the full subcategory of $\mathcal{P}^\bullet(Z)$ with objects those finite subsets I of Z such that $|f(I)| = |I|$.

The next proposition follows immediately from Lemma 7.3.14 and §2.4.

Proposition 7.4.3. *The functor $f : \mathcal{S}^\bullet(Z,1) \to \mathcal{S}^\bullet(Z',1)$ defines a faithful pointed functor*

$$f : \mathcal{P}^\bullet_f(Z) \to \mathcal{P}^\bullet(Z'),\ I \mapsto f(I),\ \theta \mapsto (f(\theta_s))_{f(s)}.$$

In particular if $f : Z \to Z'$ is injective then we have a faithful pointed functor $f : \mathcal{P}^\bullet(Z) \to \mathcal{P}^\bullet(Z')$.

We define a non-multiplicative $f^\# : \mathrm{add}(\mathcal{P}(Z')) \to \mathrm{add}(\mathcal{P}(Z))$ that commutes with coproduct. Given I' a finite subset of Z', we put

$$f^\#(I') = \coprod_{p:I'\to Z,\ fp=\mathrm{id}_{I'}} p(I').$$

Consider now $\theta' \in \mathrm{Hom}_{\mathcal{P}^\bullet(Z')}(I', J')$ non-zero. Given $s' \in I'$, we have a decomposition $f^\#(\theta'_{s'}) = \sum_{s\in f^{-1}(s')} f^\#(\theta'_{s'})_s$ along the decomposition $f^\#(s') = \bigoplus_{s\in f^{-1}(s')} s$ (cf §7.3.4). Given $p : I' \to Z$ with $fp = \mathrm{id}_{I'}$, we put $f^\#_p(\theta') = \big(f^\#(\theta'_{f(s)})_s\big)_{s\in p(I')}$, a map in $\mathcal{P}(Z)$ with source $p(I')$.

We define

$$f^\#(\theta') = \sum_{p:I'\to Z,\ fp=\mathrm{id}_{I'}} f^\#_p(\theta').$$

Note that $f^\#(\theta') = \sum_{\theta\in f^{-1}(\theta')} \theta$, where $f^{-1}(\theta')$ is the set of braids in Z lifting θ.

Given $f' : Z' \to Z''$ a morphism of curves, we have $(f'f)^\# = f^\# f'^\#$.

The next two propositions are immediate consequences of Propositions 7.3.18 and 7.3.19 (cf §2.4).

Proposition 7.4.4. *If f is strict, then $f^\#$ defines a functor $\mathrm{add}(\mathcal{P}(Z')) \to \mathrm{add}(\mathcal{P}_f(Z))$ commuting with coproducts.*

Proposition 7.4.5. *Let Z be a curve with a finite admissible relation $\sim$ and let $q : Z \to Z/{\sim}$ be the quotient map. The functor $q^\# : \mathrm{add}(\mathcal{P}(Z/{\sim})) \to \mathrm{add}(\mathcal{P}_q(Z))$ is faithful and every map in $\mathcal{P}^\bullet(Z/{\sim})$ is in the image of the functor $q : \mathcal{P}^\bullet_q(Z) \to \mathcal{P}^\bullet(Z/{\sim})$.*

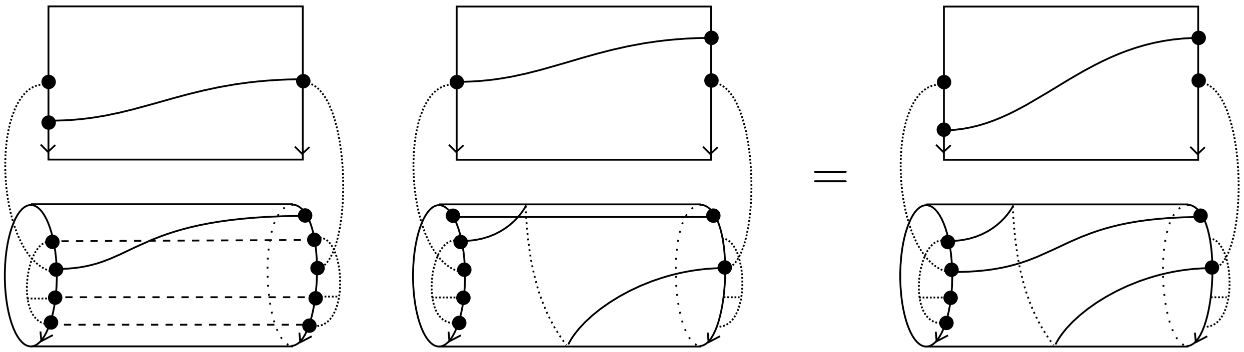
Note that the construction $Z \mapsto \mathrm{add}(\mathcal{P}(Z))$ and $f \mapsto f^\#$ defines a contravariant functor from the category of curves with strict morphisms to the category of $\mathbf{F}_2$-linear categories.

Let $Z_1, \ldots, Z_r$ be the connected components of Z. The isomorphism (7.3.4) induces an isomorphism of pointed categories

$$\mathcal{P}^\bullet(Z_1) \wedge \cdots \wedge \mathcal{P}^\bullet(Z_r) \xrightarrow{\sim} \mathcal{P}^\bullet(Z). \tag{7.4.1}$$

Note that the inverse functor sends a braid $\theta : I \to J$ in Z to $(\theta_1, \ldots, \theta_r)$, where θ_i be the restriction of θ to $I \cap Z_i$.

Example 7.4.6. We describe below an example of product in $\mathcal{P}^\bullet(Z)$.



7.4.2. *Degree.* Consider $\theta : I \to J$ a braid. We put

$$i(\theta) = \frac{1}{2} \sum_{\Omega \in \pi_0(Z)} \sum_{s \neq s' \in I \cap \Omega} i(\theta_s, \theta_{s'}) e_\Omega \in (\mathbf{Z}_{\geqslant 0})^{\pi_0(Z)}$$

We define $[\![\theta]\!] = \sum_{s \in I} [\![\theta_s]\!] \in R(Z)$ and

$$m(\theta) = \sum_{s \in I} \sum_{c \in \theta_s(0+) \cup \iota(\theta_s(0+))} m_c([\![\theta]\!]) e_c \in L(Z).$$

Finally, we define $\deg'(\theta) \in \Gamma(Z)$ by

$$\deg'(\theta) = (i(\theta), (-m(\theta), [\![\theta]\!])).$$

Given $D \subset T(Z)$ with $D \cap \iota(D) = \varnothing$, we denote by $\deg_D(\theta)$ the image of $\deg'(\theta)$ in $\Gamma_D(Z)$. Note that if $D' \subset D$, then $\deg_D(\theta)$ is the image of $\deg_{D'}(\theta) \in \Gamma_{D'}(Z)$ in $\Gamma_D(Z)$.

We put $\deg(\theta) = \deg_{Z^+_{exc}}(\theta)$ and we denote by $\overline{\deg}(\theta)$ (resp. $\overline{\deg}_D(\theta)$) the image of $\deg(\theta)$ (resp. $\deg_D(\theta)$) in $\bar{\Gamma}_{Z^+_{exc}}(Z)$ (resp. $\bar{\Gamma}_D(Z)$).

Lemma 7.4.7. *Let $\theta : I \to J$ be a braid in Z. Let E be a subset of $\{s \in I \cap Z_o \mid \theta_s = \mathrm{id}_s\}$ and let $\bar{\theta} = (\theta_s)_{s \in I - E}$. We have* $\deg_{E^+}(\theta) = \deg_{E^+}(\bar{\theta})$.

Proof. Note that $[\![\theta]\!] = [\![\bar{\theta}]\!]$. Let $s \in E$. We have

$$\sum_{c \in C(s)} m_c([\![\theta]\!]) e_c = \sum_{c \in C(s)^+} \sum_{s' \in I,\ s' \neq s} (m_c - m_{\iota(c)})([\![\theta_{s'}]\!]) e_c \xrightarrow{e_c \to 1} 2 \sum_{s' \in I,\ s' \neq s} i(\mathrm{id}_s, \theta_{s'})$$

by Lemma 7.3.21. The lemma follows. ☐

Remark 7.4.8. Note that $i(\theta) = \sum_{I' \subset I,\ |I'|=2} i(\theta_{|I'})$.

The next lemma shows that the failure of multiplicativity of deg and i coincide up to terms involving points in Z_{exc}.

Lemma 7.4.9. *Let $\theta : I \to J$ and $\theta' : I' \to I$ be two braids such that $\theta \circ \theta'$ is a braid. The element* $\deg(\theta) \cdot \deg(\theta') \cdot \deg(\theta \circ \theta')^{-1}$ *of* $\Gamma_{Z^+_{exc}}(Z)$ *is in* $\bigoplus_\Omega \frac{1}{2}\mathbf{Z}e_\Omega$ *and it is equal to*

$$i(\theta) + i(\theta') - i(\theta \circ \theta') \\ - \frac{1}{2} \sum_{\substack{\Omega,\ s' \in I' \cap Z_{exc} \cap \Omega \\ \theta'_{s'} = \mathrm{id},\ \theta_{s'} \neq \mathrm{id} \\ c' \in C(s')^+ \setminus \theta_{s'}(0+)}} (m_{c'} - m_{\iota(c')})([\![\theta']\!])e_\Omega - \frac{1}{2} \sum_{\substack{\Omega,\ s' \in I' \cap \Omega \\ \theta'_{s'} \neq \mathrm{id},\ \theta_{\theta'_{s'}(1)} = \mathrm{id} \\ c \in C(\theta'_{s'}(1))^+ \setminus \iota(\theta'_{s'}(1-)) \\ \theta'_{s'}(1) \in Z_{exc}}} (m_c - m_{\iota(c)})([\![\theta]\!])e_\Omega.$$

and is also equal to

$$\frac{1}{2} \sum_{\substack{\Omega, (s'_1, s'_2) \in (I' \cap \Omega)^2 \\ (s'_1, s'_2) \notin E \cup E'}} \big(i(\theta_{s_1}, \theta_{s_2}) + i(\theta'_{s'_1}, \theta'_{s'_2}) - i(\theta_{s_1} \circ \theta'_{s'_1}, \theta_{s_2} \circ \theta'_{s'_2})\big)e_\Omega + \\ + \sum_{\Omega,\ (s'_1, s'_2) \in E \cap \Omega} \big(i(\theta_{s_1}, \theta_{s_2}) + m^+_{\theta_{s_1}(0+)}(\theta'_{s'_2}) - i(\theta_{s_1}, \theta_{s_2} \circ \theta'_{s'_2})\big)e_\Omega + \\ + \sum_{\Omega,\ (s'_1, s'_2) \in E' \cap \Omega} \big(i(\theta'_{s'_1}, \theta'_{s'_2}) + m^-_{\theta'_{s'_1}(1-)}(\theta_{s_2}) - i(\theta'_{s'_1}, \theta_{s_2} \circ \theta'_{s'_2})\big)e_\Omega$$

where

- *given* $(s'_1, s'_2) \in I'^2$, *we put* $s_i = \theta'_{s'_i}(1)$
- E *is the set of pairs* $(s'_1, s'_2) \in I' \times I'$ *with* $s'_1 \in Z_{exc}$, $\theta'_{s'_1} = \mathrm{id}$, $\theta_{s'_1} \neq \mathrm{id}$, $\theta'_{s'_2} \neq \mathrm{id}$
- E' *is the set of pairs* $(s'_1, s'_2) \in I' \times I'$ *with* $s_1 \in Z_{exc}$, $\theta'_{s'_1} \neq \mathrm{id}$, $\theta_{s_1} = \mathrm{id}$ *and* $\theta_{s_2} \neq \mathrm{id}$.

Proof. Given $s' \in I'$ and $s = \theta'_{s'}(1)$, the class $\theta_s \circ \theta'_{s'}$ is admissible, hence $\theta_s(0+) \cup \iota(\theta_s(0+)) = \theta'_{s'}(1-) \cup \iota(\theta'_{s'}(1-))$ unless $s \in Z_{exc}$ and one of θ_s and $\theta_{s'}$ is the identity, but not the other.

Given $c \in T(Z)$, we put

$$v_c = (m_c - m_{\iota(c)})([\![\theta]\!])e_c = m_c([\![\theta]\!])e_c + m_{\iota(c)}([\![\theta]\!])e_{\iota(c)} = v_{\iota(c)}.$$

Let

$$a = \sum_{\substack{s' \in I' \cap Z_{exc} \\ \theta'_{s'} = \mathrm{id},\ \theta_{s'} \neq \mathrm{id} \\ c' \in C(s') \setminus (\theta_{s'}(0+) \cup \iota(\theta_{s'}(0+))}} m_{c'}([\![\theta']\!])e_{c'} = \sum_{\substack{s' \in I' \cap Z_{exc} \\ \theta'_{s'} = \mathrm{id},\ \theta_{s'} \neq \mathrm{id} \\ c' \in C(s')^+ \setminus \theta_{s'}(0+)}} (m_{c'} - m_{\iota(c')})([\![\theta']\!])e_{c'}.$$

We have

$$m(\theta \circ \theta') - m(\theta) - m(\theta') = \\ = \sum_{\substack{s' \in I' \\ c' \in (\theta \circ \theta')_{s'}(0+) \cup \iota((\theta \circ \theta')_{s'}(0+))}} m_{c'}([\![\theta]\!])e_{c'} - \sum_{\substack{s \in I \\ c \in \theta_s(0+) \cup \iota(\theta_s(0+))}} m_c([\![\theta]\!])e_c - a \\ = \sum_{\substack{s' \in I' \\ \theta'_{s'} \neq \mathrm{id}}} v_{\theta'_{s'}(0+)} - \sum_{\substack{s' \in I' \\ \theta'_{s'} \neq \mathrm{id} \\ \theta_{\theta'_{s'}(1)} \neq \mathrm{id}}} v_{\theta'_{s'}(1-)} - \frac{1}{2} \sum_{\substack{s' \in I' \\ \theta'_{s'} \neq \mathrm{id} \\ \theta_{\theta'_{s'}(1)} = \mathrm{id} \\ c \in C(\theta'_{s'}(1))}} v_c - a$$

Using (7.3.1), we find

$$\begin{aligned}\langle [\![\theta]\!], [\![\theta']\!]\rangle &= -\frac{1}{2}\sum_{\substack{s'\in I'\\ c'\in\theta'_{s'}(0+)\cup\iota(\theta'_{s'}(0+))}} v_{c'} + \frac{1}{2}\sum_{\substack{s'\in I'\\ c\in\theta'_{s'}(1-)\cup\iota(\theta'_{s'}(1-))}} v_c\\ &= -\sum_{\substack{s'\in I'\\ \theta'_{s'}\neq\mathrm{id}}} v_{\theta'_{s'}(0+)} + \sum_{\substack{s'\in I'\\ \theta'_{s'}\neq\mathrm{id}}} v_{\theta'_{s'}(1-)}.\end{aligned}$$

We deduce that

$$\langle [\![\theta]\!], [\![\theta']\!]\rangle + m(\theta\circ\theta') - m(\theta) - m(\theta') = -\sum_{\substack{s'\in I'\\ \theta'_{s'}\neq\mathrm{id}\\ \theta_{\theta'_{s'}(1)}=\mathrm{id}\\ c\in C(\theta'_{s'}(1))^+\setminus\iota(\theta'_{s'}(1-))\\ \theta'_{s'}(1)\in Z_{exc}}} (m_c - m_{\iota(c)})([\![\theta]\!])e_c - a$$

and the first equality of the lemma follows.

Consider $s'_1 \neq s'_2$ in I'.

If $s'_1 \in Z_{exc}$, $\theta'_{s'_1} = \mathrm{id}_{s'_1}$ and $\theta_{s_1} \neq \mathrm{id}_{s_1}$, it follows from Lemma 7.3.21 that

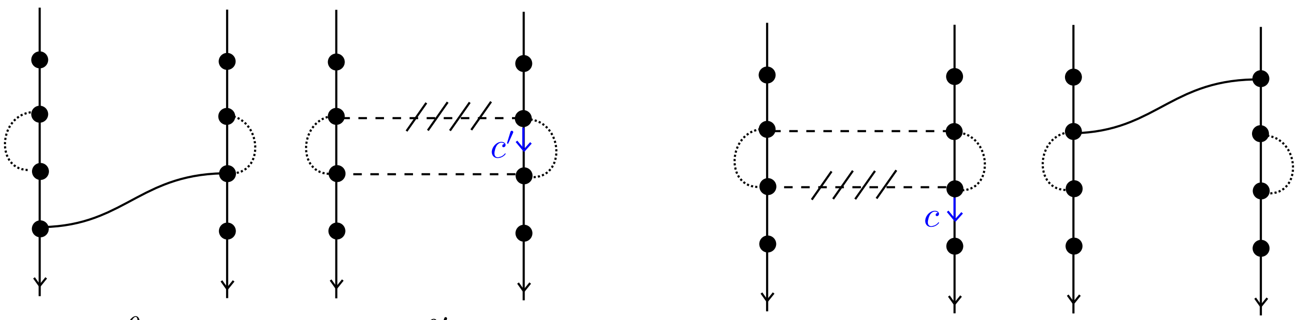
$$\sum_{\substack{s'_2\in I'\\ \theta'_{s'_2}\neq\mathrm{id}}} i(\mathrm{id}_{s'_1}, \theta'_{s'_2}) = \frac{1}{2}\sum_{\substack{s'_2\in I'\\ \theta'_{s'_2}\neq\mathrm{id}_{s'_2}\\ c'\in C(s'_1)^+}} (m_{c'} - m_{\iota(c')})(\theta'_{s'_2}) = \frac{1}{2}\sum_{c'\in C(s'_1)^+} (m_{c'} - m_{\iota(c')})([\![\theta']\!]).$$

Similarly, if $s_1 \in Z_{exc}$, $\theta'_{s'_1} \neq \mathrm{id}$ and $\theta_{s_1} = \mathrm{id}$, we have

$$\sum_{\substack{s'_2\in I'\\ \theta_{s_2}\neq\mathrm{id}}} i(\mathrm{id}_{s_1}, \theta_{s_2}) = \frac{1}{2}\sum_{c\in C(s_1)^+} (m_c - m_{\iota(c)})([\![\theta]\!]).$$

The second equality of the lemma follows. $\square$

Example 7.4.10. The left (respectively second) side of the diagram below shows a typical instance where the left (respectively right) sum of Lemma 7.4.9 is nonzero.



θ θ' θ θ'

Remark 7.4.11. Let $\theta : I \to J$ and $\theta' : I' \to I$ be two braids such that $\theta\circ\theta'$ is a braid. By Lemma 7.3.23, the terms $i(\theta_{s_1}, \theta_{s_2}) + i(\theta'_{s'_1}, \theta'_{s'_2}) - i(\theta_{s_1}\circ\theta'_{s'_1}, \theta_{s_2}\circ\theta'_{s'_2})$, $i(\theta_{s_1}, \theta_{s_2}) + m^+_{\theta_{s_1}(0+)}(\theta'_{s'_2}) - i(\theta_{s_1}, \theta_{s_2}\circ\theta'_{s'_2})$ and $i(\theta'_{s'_1}, \theta'_{s'_2}) + m^-_{\theta'_{s'_1}(1-)}(\theta_{s_2}) - i(\theta'_{s'_1}, \theta_{s_2}\circ\theta'_{s'_2})$ in Lemma 7.4.9 are all non-negative.

We deduce that the following assertions are equivalent:

- $\deg(\theta)\cdot\deg(\theta')=\deg(\theta\circ\theta')$
- $\deg(\theta_{|E})\cdot\deg(\theta'_{|E'})=\deg(\theta_{|E}\circ\theta'_{|E'})$ for any two-element subset $E'\subset I'$, where $E=\chi(\theta')(E')$.

If given $s\in I'$ with $\theta'_s=\mathrm{id}$ or $\theta_{\chi(\theta')(s)}=\mathrm{id}$, we have $s\notin Z_{exc}$, then $\deg(\theta)\cdot\deg(\theta')=\deg(\theta\circ\theta')$ if and only if $i(\theta_{\theta'_s(1)},\theta_{\theta'_{s'}(1)})+i(\theta'_s,\theta'_{s'})=i((\theta\circ\theta')_s,(\theta\circ\theta')_{s'})$ for all $s\neq s'$ in I'.

Lemma 7.4.12. *Let $f:Z\to Z'$ be a morphism of curves. Let I and J be two finite subsets of Z such that $|f(I)|=|f(J)|=|I|=|J|$. Let $\theta:I\to J$ be a braid in Z. Let $E=\{s\in I\cap Z_f\mid \theta_s=\mathrm{id}_s\}$.*

We have $f(\deg_{f^{-1}(f(E))^+}(\theta))=\deg_{f(E)^+}(f(\theta))$.

Proof. Assume first $E=\varnothing$. Given $s\in I$ with $\theta_s=\mathrm{id}_s$, we have a bijection $C(s)\xrightarrow{\sim}C(f(s))$. It follows that

$$\begin{aligned}f(m(\theta))&=\sum_{\substack{s\in I\\ \theta_s\neq\mathrm{id}_s}}\ \sum_{c\in\theta_s(0+)\cup\iota(\theta_s(0+))}m_c(\llbracket\theta\rrbracket)f(e_c)+\sum_{\substack{s\in I\\ \theta_s=\mathrm{id}_s}}\ \sum_{c\in C(s)}m_c(\llbracket\theta\rrbracket)f(e_c)\\&=\sum_{\substack{s'\in f(I)\\ f(\theta)_{s'}\neq\mathrm{id}_{s'}}}\ \sum_{c'\in f(\theta)_{s'}(0+)\cup\iota(f(\theta)_{s'}(0+))}m_{c'}(\llbracket f(\theta)\rrbracket)e_{c'}+\sum_{\substack{s'\in f(I)\\ f(\theta)_{s'}=\mathrm{id}_{s'}}}\ \sum_{c'\in C(s')}m_{c'}(\llbracket f(\theta)\rrbracket)e_{c'}\\&=m(f(\theta))\end{aligned}$$

by Lemma 7.1.24.

Given $s'\in f(I)$ such that $f(\theta)_{s'}=\mathrm{id}_{s'}$, we have $s'\notin Z'_f$. We deduce that $i(\theta_s,\theta_t)=i(f(\theta)_{f(s)},f(\theta)_{f(t)})$ for all $s\neq t\in I$ by Lemma 7.3.22. So $f(i(\theta))=i(f(\theta))$. We deduce that the lemma holds for θ.

Consider now the case where $E\neq\varnothing$. Let $\bar\theta=(\theta_s)_{s\in I-E}$. We have $\deg_{E^+}(\theta)=\deg_{E^+}(\bar\theta)$ by Lemma 7.4.7; taking quotients, we obtain $\deg_{f^{-1}(f(E))^+}(\theta)=\deg_{f^{-1}(f(E))^+}(\bar\theta)$. Since $f(\bar\theta)=(f(\theta)_t)_{t\in f(I)-f(E)}$, it follows again from Lemma 7.4.7 that $\deg_{f(E)^+}(f(\theta))=\deg_{f(E)^+}(f(\bar\theta))$. Since the lemma holds for $\bar\theta$, we deduce that the lemma holds for θ. ☐

As a consequence of Lemma 7.4.12, we have the following result.

Proposition 7.4.13. *Let $f:Z\to Z'$ be a morphism of curves and let θ' be a non-zero map in $\mathcal{P}^\bullet(Z')$. Then $f^\#(\theta')$ is a sum of maps θ such that $f(\deg_{Z_f^+}(\theta))=\deg_{f(Z_f)^+}(\theta')$.*

Let $Z_1,\ldots,Z_r$ be the connected components of Z. The isomorphism (7.4.1) is compatible with the degree function in the following sense. Given $\theta:I\to J$ a braid in Z, let θ_i be the restriction of θ to $I\cap Z_i$. The image of $(\deg(\theta_1),\ldots,\deg(\theta_r))$ in $\Gamma(Z)$ by the map of (7.3.3) is $\deg(\theta)$.

Let I and J be two finite subsets of Z and let $\theta:I\to J$ be a braid in Z. We define

$$L(\theta)=\coprod_{i_1\neq i_2\in I}I(\theta_{i_1},\theta_{i_2}).$$

Note that $\zeta\mapsto\zeta^{-1}$ induces a fixed-point free involution inv on $L(\theta)$.

Let $\zeta\in L(\theta)$. Put $i_1=\zeta(0)$ and $i_2=\zeta(1)$. We define θ^ζ by $(\theta^\zeta)_i=\theta_i$ if $i\in I-\{i_1,i_2\}$, $(\theta^\zeta)_{i_1}=\theta_{i_2}\circ\zeta=\bar\zeta\circ\theta_{i_1}$ and $(\theta^\zeta)_{i_2}=\theta_{i_1}\circ\zeta^{-1}=\bar\zeta^{-1}\circ\theta_{i_2}$. Note that $\theta^{\zeta^{-1}}=\theta^\zeta$.

Let $D(\theta)$ be the set of classes ζ in $L(\theta)$ such that

(a) given a class of smooth paths $\zeta' : \zeta(0) \to \zeta(1)$ such that $\zeta \circ \zeta'^{-1}$ and $\zeta'^{-1} \circ \zeta$ are smooth and have the same orientation as ζ and ζ', and given a class of smooth paths $\zeta'' : \bar{\zeta}(0) \to \bar{\zeta}(1)$ such that $\bar{\zeta} \circ \zeta''^{-1}$ and $\zeta''^{-1} \circ \bar{\zeta}$ are smooth and have the same orientation as $\bar{\zeta}$ and ζ'', then $\zeta' = \zeta$ or $\zeta'' = \bar{\zeta}$.

(b) given ζ' and ζ'' in $L(\theta)$ with $\zeta = \zeta' \circ \zeta''$, then ζ' and ζ'' have opposite orientations.

Remark 7.4.14. Condition (a) above is automatically satisfied if the component of the support of ζ is not isomorphic to S^1.

The subset $D(\theta)$ of $L(\theta)$ is stable under the involution inv.

The next lemma restricts the cases where condition (b) above needs to be checked.

Lemma 7.4.15. *Let $\zeta, \zeta', \zeta'' \in L(\theta)$ such that $\zeta = \zeta' \circ \zeta''$. If $\zeta'(0) \in Z_o$ and $\theta_{\zeta'(0)} = \mathrm{id}$, then ζ' and ζ'' have opposite orientations.*

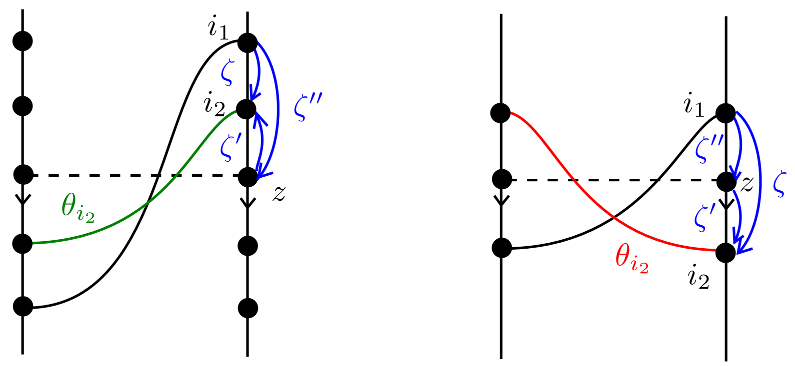
Proof. Let $z = \zeta'(0) = \zeta''(1)$. We have $\zeta' \in I(\mathrm{id}_z, \theta_{\zeta(1)})$. Since $\bar{\zeta}' = \theta_{\zeta(1)} \circ \zeta'$ is smooth and has opposite orientation to ζ', it follows that $\zeta'(0+) \in \iota(C(z)^+)$. Similarly, $\zeta''(1-) \in \iota(C(z)^+)$. We deduce that ζ' and ζ'' have opposite orientations. □

Lemma 7.4.16. *Let I' be a subset of I such that $I - I' \subset Z_o$ and $\theta_i = \mathrm{id}$ for $i \in I - I'$.*

We have $D(\theta_{|I'}) \subset D(\theta)$.

Proof. We have $L(\theta_{|I'}) \subset L(\theta)$ and Lemma 7.4.15 shows that $D(\theta_{|I'}) \subset D(\theta)$. □

Example 7.4.17. In the picture below, the left side shows a valid braid θ, for which the conclusion of Lemma 7.4.15 holds. For contrast, the right side shows a braid θ that is disallowed since θ_{i_2} is not oriented, and the conclusion of Lemma 7.4.15 fails.



7.4.3. *Strands on S^1.* Let $Z = S^1$, viewed as an unoriented manifold. Fix a family $\mathbf{a} = \{a_1, \ldots, a_n\}$ of cyclically ordered points on S^1, *i.e.*, $a_j = e^{ie_j}$ for some real numbers $e_1 < \cdots < e_n$ with $e_n - e_1 < 2\pi$.

Fix $r', r \in \{1, \ldots, n\}$. There is a bijection

$$F_{r',r} : r' - r + n\mathbf{Z} \xrightarrow{\sim} \mathrm{Hom}_{\Pi(S^1)}(a_r, a_{r'}) :$$

it sends l to the homotopy class of paths going in the positive direction and winding $\lfloor \frac{l}{n} \rfloor$ times around S^1, if $l \geqslant 0$, and to the homotopy class of paths going in the negative direction and winding $\lfloor \frac{-l}{n} \rfloor$ times around S^1, otherwise.

We put

$$F_r = \sum_{r'} F_{r',r} : \mathbf{Z} \xrightarrow{\sim} \coprod_{r'} \mathrm{Hom}_{\Pi(S^1)}(a_r, a_{r'}).$$

Given $r, r' \in \{1, \ldots, n\}$ and $l, l' \in \mathbf{Z}$ with $r' - r = l \pmod n$, we have $F_r(l + l') = F_{r'}(l') \circ F_r(l)$. Note also that given $j \in \{1, \ldots n\}$ and $j' \in \mathbf{Z}$, we have

$$\mathrm{supp}(F_j(j'-j)) = \begin{cases} S^1 & \text{if } |j'-j| \geqslant n \\ \{e^{iu} \mid e_j \leqslant u \leqslant e_{j''} + 2\pi\delta_{j'>n} & \text{if } j'-j \in \{0, \ldots, n-1\} \\ \{e^{iu} \mid e_{j''} - 2\pi\delta_{j'\leqslant 0} \leqslant u \leqslant e_j & \text{if } j-j' \in \{0, \ldots, n-1\} \end{cases}$$

where $j'' \in \{1, \ldots, n\}$ and $j'' - j' \in n\mathbf{Z}$.

We denote by $\vec{S}^1$ the oriented curve S^1. Fix $z = e^{ix} \in S^1$ with $x < e_1$ and $e_n - x < 2\pi$ and Ω a connected open neighbourhood of z in S^1 containing no a_i. Let $I = S^1 - \{z\}$ unoriented and $\vec{I} = S^1 - \{z\}$ oriented. We define $\dot{S}^1$ to be the curve S^1 with $(\dot{S}^1)_o = \Omega$ with its standard orientation.

Proposition 7.4.18. *There is an isomorphism of pointed categories $F : (\mathcal{S}_n)_+ \xrightarrow{\sim} \mathcal{P}^\bullet_{\mathbf{a}}(S^1)$ given by $F(J) = \{a_j\}_{j \in \tilde{J} \cap [1,n]}$ and $F(\sigma)_{a_j} = F_j(\sigma(j) - j)$ for σ a map of $\mathcal{S}_n$.*

It restricts to isomorphisms of pointed categories

$$(\mathcal{S}_n^+)_+ \xrightarrow{\sim} \mathcal{P}^\bullet_{\mathbf{a}}(\dot{S}^1),\ (\mathcal{S}_n^{++})_+ \xrightarrow{\sim} \mathcal{P}^\bullet_{\mathbf{a}}(\vec{S}^1),\ (\mathcal{S}_n^f)_+ \xrightarrow{\sim} \mathcal{P}^\bullet_{\mathbf{a}}(I) \text{ and } (\mathcal{S}_n^{f++})_+ \xrightarrow{\sim} \mathcal{P}^\bullet_{\mathbf{a}}(\vec{I}).$$

Proof. Consider $J, J' \subset \mathbf{Z}/n$. We have an injective map $f : \mathrm{Hom}_{\mathcal{S}_n}(J, J') \to \mathbf{Z}^J,\ \sigma \mapsto (\sigma(j) - j))_b$, where $j \in \{1, \ldots, n\}$ and $b = j + n\mathbf{Z}$. The image of that map is the set of those $c \in \mathbf{Z}^J$ such that $\{c_b + b\}_b = J'$ and we obtain a bijection

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}_n}(J, J') &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{P}^\bullet_{\mathbf{a}}(\vec{S}^1)}(\{a_j\}_{j \in \tilde{J} \cap [1,n]}, \{a_{j'}\}_{j' \in \tilde{J}' \cap [1,n]}) \\ \sigma &\mapsto \big(F_{j',j}(f(\sigma)_{j+n\mathbf{Z}})\big)_{j \in \tilde{J} \cap [1,n],\ j' \in \tilde{J}' \cap [1,n],\ \sigma(j) - j' \in n\mathbf{Z}}. \end{aligned}$$

We deduce that F induces a bijection on pointed Hom-sets. Consider now $\sigma : J \to J'$ and $\sigma' : J' \to J''$ two maps in $\mathcal{S}_n$. Given $j \in \tilde{J} \cap [1, n]$, we have

$$F(\sigma'\sigma)_{a_j} = F_j(\sigma'\sigma(j) - j) = F_j(\sigma'(\sigma(j)) - \sigma(j) + \sigma(j) - j) = F_{\sigma(j)}(\sigma')_{a_{\sigma(j)}} \circ F_j(\sigma)_{a_j}.$$

We deduce that F is a functor and the first statement of the proposition follows.

Consider now $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(J, J')$.

The map $F(\sigma)$ is in $\mathcal{P}^\bullet_{\mathbf{a}}(\dot{S}^1)$ if and only if $\sigma(j) \geqslant 0$ for all $j \in [1, n] \cap \tilde{J}$, hence if and only if σ is in $\mathcal{S}_n^+$.

The map $F(\sigma)$ is in $\mathcal{P}^\bullet_{\mathbf{a}}(\vec{S}^1)$ if and only if $\sigma(j) - j \geqslant 0$ for all $j \in \tilde{J}$, hence if and only if σ is in $\mathcal{S}_n^{++}$.

The map $F(\sigma)$ is in $\mathcal{P}^\bullet_{\mathbf{a}}(I)$ if and only if $\sigma(j) \in [1, n]$ for all $j \in \tilde{J} \cap [1, n]$, hence if and only if σ is in $\mathcal{S}_n^f$.

The proposition follows. □

There are morphisms of groups $F_R : R_n \to R(S^1),\ \alpha_{j+n\mathbf{Z}} \mapsto [\![F_j(1)]\!]$ and $F_L : L_n \to L(S^1),\ \varepsilon_{j+n\mathbf{Z}} \mapsto e_{c_j}$, where $j \in \{1, \ldots, n\},\ c_j = (a_j, a_j e^{iu}) \in C(a_j)$ and $u \in \mathbf{R}_{>0}$ is small enough.

Lemma 7.4.19. *Given $\alpha, \beta \in R_n$, we have $F_L(\langle\alpha, \beta\rangle) = \langle F_R(\alpha), F_R(\beta)\rangle$ and there is an injective morphism of groups $F_\Gamma : \Gamma_n \to \Gamma(S^1)$, $(r, (l, \alpha)) \mapsto (r, (F_L(l), F_R(\alpha)))$.*

Let D be a subset of $\{1, \ldots, n\} \times \{\pm 1\}$ that embeds in its projection on $\{1, \ldots, n\}$. Define $\partial : D \to T(S^1)$ by $\partial((i, \nu_i)) = c_i$ if $\nu_i = 1$ and $\partial((i, \nu_i)) = \iota(c_i)$ otherwise. The morphism F_Γ induces an injective morphism of groups $F_D : \Gamma_D \to \Gamma_{\partial(D)}(S^1)$. We have $u < u'$ if and only if $F_D(u) < F_D(u')$.

Let σ be a map in $\mathcal{S}_n$. We have $F_R([\![\sigma]\!]) = [\![F(\sigma)]\!]$, $m(F(\sigma)) = F_L(m(\sigma))$, $i(F(\sigma)) = \ell(\sigma)$ and $\deg(F(\sigma)) = F_\Gamma(\deg(\sigma))$.

Proof. Let $r, j \in \{1, \ldots, n\}$ and let $j' \in \mathbf{Z}$. We have

$$m_{c_r}([\![F_j(j'-j)]\!]) = |\{i \in r + n\mathbf{Z} \mid j \leqslant i < j'\}| - |\{i \in r + n\mathbf{Z} \mid j > i \geqslant j'\}|$$

and

$$m_{\iota(c_r)}([\![F_j(j'-j)]\!]) = -|\{i \in r + n\mathbf{Z} \mid j < i \leqslant j'\}| + |\{i \in r + n\mathbf{Z} \mid j \geqslant i > j'\}|$$

In particular, $m_{c_r}([\![F_j(1)]\!]) = \delta_{r,j}$ and $m_{\iota(c_r)}([\![F_j(1)]\!]) = -\delta_{r,j+1}$. This shows that F_R is injective. This shows also that given $i \in \{1, \ldots, n\}$, we have

$$\langle [\![F_i(1)]\!], [\![F_j(1)]\!]\rangle = (\delta_{i,j+1} + \delta_{i,j})F_L(\varepsilon_{j+1+n\mathbf{Z}}) - (\delta_{i,j} + \delta_{i+1,j})F_L(\varepsilon_{j+n\mathbf{Z}}) = F_L(\langle\alpha_{i+n\mathbf{Z}}, \alpha_{j+n\mathbf{Z}}\rangle).$$

This shows the first equality and this shows that F_R induces an injective morphism of groups F_Γ.

Taking quotients, we obtain an injective morphisms of groups $F_D : \Gamma_D \to \Gamma_{\partial(D)}(S^1)$ compatible with the order.

Consider $\sigma \in \mathrm{Hom}_{\mathcal{S}_n}(I, J)$. Given $d \in \mathbf{Z}$, we have $[\![F_r(d)]\!] = F_R(\alpha_{r,r+d})$, hence $F_R([\![\sigma]\!]) = [\![F(\sigma)]\!]$.

We have

$$m(F(\sigma)) = \sum_{r,j \in \tilde{I} \cap [1.n]} (m_{c_r} - m_{\iota(c_r)})([\![F_j(\sigma(j) - j)]\!]) e_{c_r}$$

and

$$m(\sigma) = \sum_{r,j \in \tilde{I} \cap [1,n]} \alpha_{j,\sigma(j)} \cdot \varepsilon_{r+n\mathbf{Z}}.$$

Since

$$\alpha_{j,\sigma(j)} \cdot \varepsilon_{r+nZ} = (m_{c_r} - m_{\iota(c_r)})(F(\sigma))\varepsilon_{r+nZ},$$

it follows that $m(F(\sigma)) = F_L(m(\sigma))$.

Consider $i_1, i_2 \in \tilde{I}$ with $0 \leqslant i_1 < i_2 < n$. We have $i(F(\sigma)_{a_{i_1}}, F(\sigma)_{a_{i_2}}) = i(\gamma_1, \gamma_2)$ for some minimal paths γ_l in $F(\sigma)_{a_{i_l}}$ by Lemma 7.3.22. Lemma 6.2.3 shows that $i(F(\sigma)_{a_{i_1}}, F(\sigma)_{a_{i_2}}) = \left|\lfloor \frac{\sigma(i_2) - \sigma(i_1)}{n} \rfloor\right|$. Lemma 6.2.2 shows now that $i(F(\sigma)) = \ell(\sigma)$. ☐

Given $i_1, i_2 \in \mathbf{Z}$ with $i_2 \notin i_1 + n\mathbf{Z}$, we put $\lambda(i_1, i_2) = F_{i'_1}(i_2 - i_1)$, where $i'_1 \in [1, n] \cap (i_1 + n\mathbf{Z})$.

Lemma 7.4.20. *Let σ be a map in $\mathcal{S}_n$. Given (i_1, i_2) in $L(\sigma)$ (resp. $D(\sigma)$), the class $\lambda(i_1, i_2)$ is in $L(F(\sigma))$ (resp. $D(F(\sigma))$) and $F(\sigma)^{\lambda(i_1,i_2)} = F(\sigma^{i_1,i_2})$. Furthermore, λ induces bijections*

$$L(\sigma)/n\mathbf{Z} \xrightarrow{\sim} L(F(\sigma))/\mathrm{inv} \text{ and } D(\sigma)/n\mathbf{Z} \xrightarrow{\sim} D(F(\sigma))/\mathrm{inv}.$$

Proof. Note first that, given i_1' and i_2' two distinct elements of $\{1,\ldots,n\}$, then λ induces a bijection
$$\big((i_1'+n\mathbf{Z})\times(i_2'+n\mathbf{Z})\big)/n\mathbf{Z}\xrightarrow{\sim}\mathrm{Hom}_{\Pi(S^1)}(a_{i_1'},a_{i_2'}).$$

Consider $\sigma\in\mathrm{Hom}_{\mathcal{S}_n}(I,J)$ and $i_1,i_2\in\tilde{I}$ with $i_2\notin i_1+n\mathbf{Z}$. Note that $\lambda(i_1,i_2)=\lambda(i_2,i_1)^{-1}$ for any i_1,i_2.

Let $\zeta_r=F(\sigma)_{a_{i_r}}$ and $\zeta=\lambda(i_1,i_2)$. We have $\bar{\zeta}=\lambda(\sigma(i_1),\sigma(i_2))$. So, $\zeta\in L(F(\sigma))$ if and only if i_1-i_2 and $\sigma(i_1)-\sigma(i_2)$ have opposite signs. On the other hand, $(i_1,i_2)\in L(\sigma)$ if and only if $i_1<i_2$ and $\sigma(i_2)<\sigma(i_1)$. This shows that $\lambda(L(\sigma))\subset L(F(\sigma))$ and λ induces a bijection $L(\sigma)/n\mathbf{Z}\xrightarrow{\sim}L(F(\sigma))/\mathrm{inv}$.

Consider $(i_1,i_2)\in L(\sigma)$. Let $r=\lfloor\frac{i_2-i_1}{n}\rfloor$ and $s=\lfloor\frac{\sigma(i_1)-\sigma(i_2)}{n}\rfloor$. We have $r>0$ if and only if $\mathrm{supp}(\lambda(i_1,i_2-rn))\subsetneq\mathrm{supp}(\lambda(i_1,i_2))$ and $s>0$ if and only if $\mathrm{supp}(\lambda(i_1,i_2+sn))\subsetneq\mathrm{supp}(\lambda(i_1,i_2))$. There is i such that (i_1,i) and (i,i_2) are in $L(\sigma)$ if and only if there are ζ' and ζ'' with the same orientations in $L(F(\sigma))$ such that $\lambda(i_1,i_2)=\zeta''\circ\zeta'$. We have $i_2-i_1>n$ if and only if there is ζ' such that ζ, ζ' and $\zeta\circ\zeta'^{-1}$ have the same orientation. We have $\sigma(i_1)-\sigma(i_2)>n$ if and only if there is ζ'' such that $\bar{\zeta}$, ζ'' and $\bar{\zeta}\circ\zeta''^{-1}$ have the same orientation.

We deduce that $(i_1,i_2)\in D(\sigma)$ if and only if $\lambda(i_1,i_2)\in D(F(\sigma))$.

Assume now $(i_1,i_2)\in L(\sigma)$. Let $i_r'\in[1,n]\cap(i_r+n\mathbf{Z})$ for $r\in\{1,2\}$. We have
$$(F(\sigma)^{\lambda(i_1,i_2)})_{a_{i_1'}}=F_{i_2'}(\sigma(i_2)-i_2)\circ F_{i_1'}(i_2-i_1)=F_{i_1'}(\sigma(i_2)-i_1)=(F(\sigma^{i_1,i_2}))_{a_{i_1}}.$$
Similarly, $(F(\sigma)^{\lambda(i_1,i_2)})_{a_{i_2'}}=(F(\sigma^{i_1,i_2}))_{a_{i_2}}$. It follows that $F(\sigma)^{\lambda(i_1,i_2)}=F(\sigma^{i_1,i_2})$. This completes the proof of the lemma. ☐

7.4.4. *Strand category.* Let Z be a curve.

We have a positivity result in the setting of Lemma 7.4.9.

Lemma 7.4.21. *Let θ and θ' be two braids such that $\theta\circ\theta'$ is a braid. We have* $\deg(\theta)\cdot\deg(\theta')\geqslant\deg(\theta\circ\theta')$.

Given D a subset of $T(Z)$ containing Z^+_{exc} and such that $D\cap\iota(D)=\varnothing$, the following assertions are equivalent:

- $\deg(\theta)\cdot\deg(\theta')=\deg(\theta\circ\theta')$
- $\deg(\theta)\cdot\deg(\theta')$ *and* $\deg(\theta\circ\theta')$ *have the same image in* $\Gamma_D(Z)$
- $\deg(\theta)\cdot\deg(\theta')$ *and* $\deg(\theta\circ\theta')$ *have the same image in* $\bar{\Gamma}_D(Z)$.

Proof. Assume $Z=S^1$ unoriented. Let $\mathbf{a}$ be a family as in §7.4.3. Assume $\mathbf{a}$ contains $\theta_s(r)$ and $\theta_s'(r)$ for $r\in\{0,1\}$ and all s. Proposition 7.4.18 and Lemma 7.4.19 show that the inequality follows from the corresponding inequality for maps in $\mathcal{S}_n$, which is given by Lemmas 6.2.1 and 6.2.5.

Given Z a non-singular connected curve, there is an injective morphism of curves $Z\to S^1$, and the lemma follows from Proposition 7.4.3 and Lemma 7.4.12. We deduce that the inequality holds for any non-singular curve Z.

Consider now a general curve Z and let $q:\hat{Z}\to Z$ be the non-singular cover. Since the functor $q^{\#}:\mathrm{add}(\mathcal{P}(Z))\to\mathrm{add}(\mathcal{P}(\hat{Z}))$ is compatible with degrees (Proposition 7.4.13), if follows that the inequality holds for Z.

The equivalence of the three assertions follows from the fact that an element of $(\frac{1}{2}\mathbf{Z}_{\geqslant 0})^{\pi_0(Z)}\subset\Gamma_{Z^+_{exc}}(Z)$ is zero if and only if its image in $\frac{1}{2}\mathbf{Z}_{\geqslant 0}\subset\bar{\Gamma}_D(Z)$ is zero. ☐

By Lemma 7.4.21, the degree function gives a $\Gamma_{Z^+_{exc}}(Z)$-filtration on the category $\mathcal{P}^\bullet(Z)$.

Definition 7.4.22. *We define the* strand category $\mathcal{S}^\bullet(Z)$ *as the* $\Gamma_{Z^+_{exc}}(Z)$*-graded pointed category associated with the filtered pointed category* $\mathcal{P}^\bullet(Z)$ *(cf §2.3.3).*

The category $\mathcal{S}^\bullet(Z)$ has the same objects and the same maps as the category $\mathcal{P}^\bullet(Z)$. It is a pointed category with objects the finite subsets of Z and with $\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(I,J)$ the set of braids $I \to J$, together with a 0-element.

The product of two braids $\theta : I \to J$ with $\theta' : J \to K$ is defined as follows:

$$\theta' \cdot \theta = \begin{cases} \theta' \circ \theta & \text{if } \deg(\theta' \circ \theta) = \deg(\theta') \cdot \deg(\theta) \\ 0 & \text{otherwise.} \end{cases}$$

Note that the strand category decomposes as a disjoint union $\mathcal{S}^\bullet(Z) = \coprod_{n \geqslant 0} \mathcal{S}^\bullet(Z,n)$, where $\mathcal{S}^\bullet(Z,n)$ is the full subcategory with objects subsets with n elements.

It follows from Lemma 7.4.21 that given D a subset of $T(Z)$ containing Z^+_{exc} and such that $D \cap \iota(D) = \varnothing$, the structure of $\Gamma_D(Z)$-graded (resp. $\bar{\Gamma}_D(Z)$-graded) category on $\mathcal{S}(Z)$ obtained from the quotient morphism $f : \Gamma_{Z^+_{exc}}(Z) \to \Gamma_D(Z)$ (resp. $f : \Gamma_{Z^+_{exc}}(Z) \to \bar{\Gamma}_D(Z)$) is the same as the graded category obtained from the structure of $\Gamma_D(Z)$-filtered (resp. $\bar{\Gamma}_D(Z)$-filtered) category on $\mathcal{P}^\bullet(Z)$ that is deduced from the structure of $\Gamma_{Z^+_{exc}}(Z)$-filtered category via f.

Remark 7.4.23. We leave to the reader to check the following alternate definition of the product in the strand category.

We have $\theta' \cdot \theta \neq 0$ if and only if there are parametrized braids ϑ, ϑ' with $\theta = [\vartheta]$ and $\theta' = [\vartheta']$ and there are $\alpha : I' \to I$ and $\alpha' : K \to K'$ two parametrized braids with $I', K' \subset Z \backslash Z_{exc}$ such that $i(\alpha) = i(\alpha') = 0$, $\alpha'_{\vartheta' \circ \vartheta \circ \alpha_s(1)} \circ \vartheta'_{\vartheta \circ \alpha_s(1)} \circ \vartheta_{\alpha_s(1)} \circ \alpha_s$ is admissible for all $s \in I$ and $i(\alpha' \circ \theta' \circ \theta \circ \alpha) = i(\alpha' \circ \theta') + i(\theta \circ \alpha)$.

Let $\mathcal{S}(Z) = \mathbf{F}_2[\mathcal{S}^\bullet(Z)]$, a $\Gamma_{Z^+_{exc}}(Z)$-graded $\mathbf{F}_2$-linear category.

Let $f : Z \to Z'$ be a morphism of curves.

Let $\mathcal{S}^\bullet_f(Z)$ be the full subcategory of $\mathcal{S}^\bullet(Z)$ with objects those finite subsets I of Z such that $|f(I)| = |I|$. We deduce from Proposition 7.4.3 and Lemma 7.4.12 a faithful $\Gamma_{Z'^+_{exc}}(Z')$-graded pointed functor $f : \mathcal{S}^\bullet_f(Z) \to \mathcal{S}^\bullet(Z')$. Here, the $\Gamma_{Z'^+_{exc}}(Z')$-grading on $\mathcal{S}^\bullet_f(Z)$ comes from the $\Gamma_{f^{-1}(Z'_{exc})^+}(Z)$-grading via the morphism $\Gamma(f)$.

Assume f is strict. Propositions 7.4.4 and 7.4.13 provide an additive $\mathbf{F}_2$-linear $\Gamma_{Z'^+_{exc}}(Z')$-graded functor $f^\# : \mathrm{add}(\mathcal{S}(Z')) \to \mathrm{add}(\mathcal{S}_f(Z))$, where the $\Gamma_{Z'^+_{exc}}(Z')$-grading on $\mathcal{S}_f(Z)$ is deduced from the $\Gamma_{f^{-1}(Z'_{exc})^+}(Z)$-grading via the morphism $\Gamma(f)$.

If f is a quotient morphism, it follows from Proposition 7.4.5 that $f^\#$ is faithful.

Given M a subset of Z, we denote by $\mathcal{S}^\bullet_M(Z)$ the full subcategory of $\mathcal{S}^\bullet(Z)$ whose objects are the finite subsets of M. We denote by $\mathcal{S}^\bullet_{M,f}(Z)$ the full subcategory of $\mathcal{S}^\bullet_f(Z)$ with objects subsets contained in M.

We put $\mathcal{A}^\bullet(Z) = \mathcal{S}^\bullet_{Z_{exc}}(Z)$.

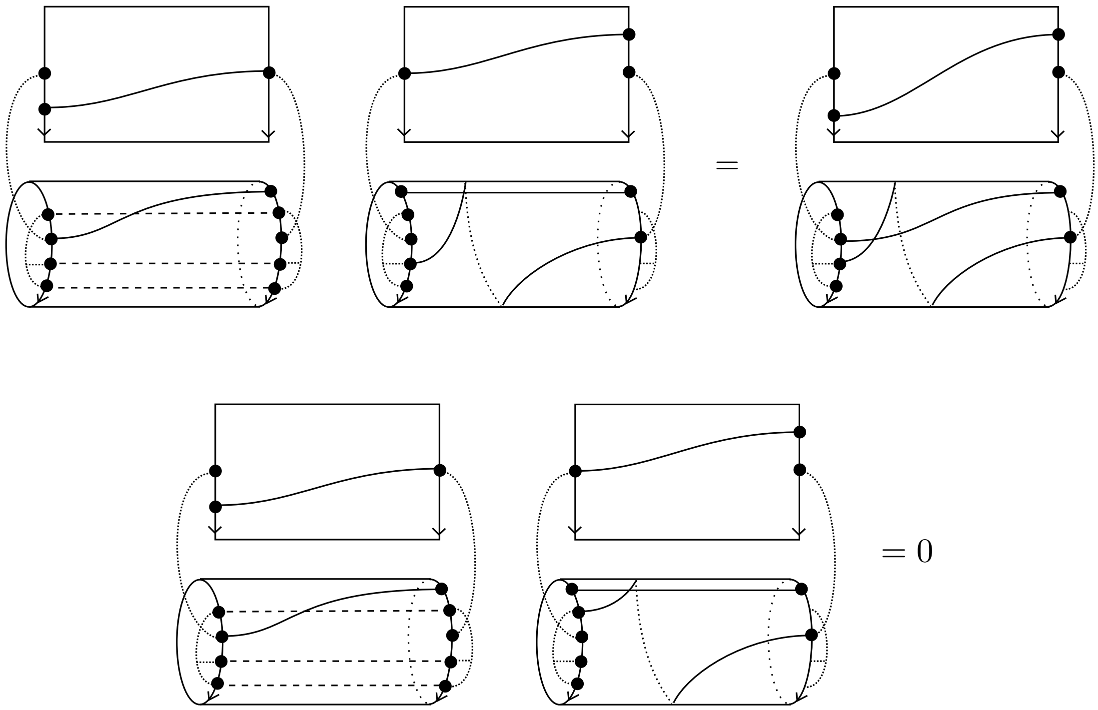
Let $Z_1, \ldots, Z_r$ be the connected components of Z. The isomorphism (7.4.1) induces an isomorphism of $\Gamma_{Z^+_{exc}}(Z)$-graded pointed categories

$$\mathcal{S}^\bullet(Z_1) \wedge \cdots \wedge \mathcal{S}^\bullet(Z_r) \xrightarrow{\sim} \mathcal{S}^\bullet(Z) \tag{7.4.2}$$

where the grading on the left hand term is deduced from the the $\left(\prod_{i=1}^{r}\Gamma_{(Z_i)^+_{exc}}(Z_i)\right)$-grading via (7.3.3) and an isomorphism of $\mathbf{F}_2$-linear categories

$$\mathcal{S}(Z_1)\otimes\cdots\otimes\mathcal{S}(Z_r)\xrightarrow{\sim}\mathcal{S}(Z). \tag{7.4.3}$$

Example 7.4.24. In the example below the first row is the product in $\mathcal{P}^\bullet(Z)$, while the second row is the product in $\mathcal{S}^\bullet(Z)$.



7.4.5. *Generation.* We equip Z with a metric. Given ξ a path in Z, we denote by $|\xi|$ its length. Given ζ a homotopy class of paths in Z, we put $|\zeta|=|\xi|$, where ξ is a minimal path in ζ. Given $\theta: I\to J$ a braid in Z, we put $|\theta|=\sum_{s\in I}|\theta_s|$.

Let M be a finite subset of Z.

Lemma 7.4.25. *Let $\theta\in\mathrm{Hom}_{\mathcal{S}^\bullet_M(Z)}(I,J)$ and $\theta'\in\mathrm{Hom}_{\mathcal{S}^\bullet_M(Z)}(I',I)$ such that $\theta\circ\theta'$ is a braid. Let I_0 be a finite subset of $M\backslash(I\cup I'\cup J)$.*

If $|\theta|+|\theta'|=|\theta\circ\theta'|$, then $(\theta\boxtimes\mathrm{id}_{I_0})\cdot(\theta'\boxtimes\mathrm{id}_{I_0})=(\theta\cdot\theta')\boxtimes\mathrm{id}_{I_0}$.

Proof. Let $s'\in I'$. Since $|\theta_{\theta'(s')}|+|\theta'_{s'}|=|\theta_{\theta'(s')}\circ\theta'_{s'}|$, it follows that $i(\theta_{\theta'(s')},\mathrm{id}_i)+i(\theta'_{s'},\mathrm{id}_i)=i(\theta_{\theta'(s')}\circ\theta'_{s'},\mathrm{id}_i)$ for all $i\in I_0$. As a consequence,

$$i((\theta\circ\theta')\boxtimes\mathrm{id}_{I_0})-i(\theta\boxtimes\mathrm{id}_{I_0})-i(\theta'\boxtimes\mathrm{id}_{I_0})=i(\theta\circ\theta')-i(\theta)-i(\theta').$$

The lemma follows now from Lemma 7.4.9. $\square$

Let $\theta\in\mathrm{Hom}_{\mathcal{S}^\bullet_M(Z)}(I,J)$ be a non-zero braid.

Let $I_0=\{i\in I\mid\theta_i=\mathrm{id}_i\}$. Let $i\in I\backslash I_0$. There is a (unique) decomposition $\theta_i=\beta^i\cdot\alpha^i$ in $\mathcal{S}^\bullet(Z,1)$ with

- $\alpha^i(1)\in M\backslash I_0$
- $|\theta_i|=|\alpha^i|+|\beta^i|$
- given a minimal path ξ in α^i, we have $\xi((0,1))\cap M\subset I_0$.

We define a quiver $\Gamma(\theta)$ with vertex set $I \backslash I_0$. There is an arrow $i \to i'$ if $\alpha^i(1) = i'$. Note that there is at most one arrow with a given source (that arrow can be a loop).

Lemma 7.4.26. *Let I' be a non-empty finite subset of $I \backslash I_0$ such that*

- *if there is an arrow $i \to i'$ in $\Gamma(\theta)$ with $i \in I'$, then $i' \in I'$*
- *given $i \neq i' \in I'$, we have $\alpha^i(1) \neq \alpha^{i'}(1)$.*

There is a (unique) decomposition $\theta = \theta^u \cdot u$ in $\mathcal{S}^\bullet_M(Z)$, where $|\theta| = |\theta^u| + |u|$ and

$$u_i = \begin{cases} \alpha^i & \textit{if } i \in I' \\ \mathrm{id}_i & \textit{otherwise.} \end{cases}$$

Proof. Note that the second assumption on I' show that the full subquiver of $\Gamma(\theta)$ with vertex set I' is a disjoint union of oriented lines and oriented circles.

Let $i_1 \neq i_2 \in I$. If $i_1, i_2 \in I \backslash I'$, then $u(i_1) \neq u(i_2)$. Assume now $i_1 \in I'$ and $i_2 \in I \backslash I'$. Since $i_1 \to i_2$ is not an arrow of the quiver, we have $u(i_1) \neq i_2$, hence $u(i_1) \neq u(i_2)$. Finally if $i_1, i_2 \in I'$, then $u(i_1) \neq u(i_2)$. We have shown that u is a braid.

Note that there is a (unique) decomposition $\theta = \theta^u \circ u$ with $|\theta| = |\theta^u| + |u|$. In order to show that $\theta^u \cdot u \neq 0$, we can replace θ by $\theta_{|I \backslash I_0}$ and M by $M \backslash I_0$, thanks to Lemma 7.4.25. So, we assume now that $I_0 = \emptyset$.

Let $q : \tilde{Z} \to Z$ be a non-singular cover of Z. Let $\tilde{M} = q^{-1}(M)$. Let $\tilde{\theta} : \tilde{I} \to \tilde{J}$ be the unique lift of θ to $\tilde{Z}$. We have a decomposition $\tilde{\theta}_i = \tilde{\beta}^i \cdot \tilde{\alpha}^i$ for $i \in \tilde{I}$ and $q(\tilde{\alpha}^i) = \alpha^{q(i)}$.

Let $\tilde{I}' = q^{-1}(I') \cap \tilde{I}$. Note that q induces a morphism of quivers $\Gamma(\tilde{\theta}) \to \Gamma(\theta)$, hence $\tilde{I}'$ satisfies the assumptions of the lemma and we have a decomposition $\tilde{\theta} = \tilde{\theta}^{\tilde{u}} \circ \tilde{u}$. Since $q(\tilde{u}) = u$, it follows that if the lemma holds for $\tilde{\theta}$, then it holds for θ.

We assume now that Z is non-singular. If the lemma holds for connected components of Z, it will hold for Z, hence it is enough to prove the lemma for Z connected. Assume now Z is connected. There is an injective morphism of curves $f : Z \to S^1$, where S^1 is unoriented. It the lemma holds for S^1, it holds for Z.

We assume finally that $Z = S^1$ unoriented. Let $i_1 \neq i_2 \in I'$ such that $i(u_{i_1}, u_{i_2}) \neq 0$. Note that u_{i_1} and u_{i_2} have opposite directions and $i(u_{i_1}, u_{i_2}) = 1$. Furthermore, θ_{i_r} has the same direction as u_{i_r}, hence $i(\theta_{i_1}, \theta_{i_2}) = i((\theta^u)_{i_1}, (\theta^u)_{i_2}) + 1$. Given $i_1 \neq i_2 \in I$ with $i_1 \notin I'$, we have $i(u_{i_1}, u_{i_2}) = 0$. It follows from Remark 7.4.11 that $\theta^u \cdot u \neq 0$. This completes the proof of the lemma. $\square$

Note that the length of a map in $\mathcal{S}^\bullet_M(Z)$ takes value in a finitely generated submonoid of $\mathbf{R}_{\geqslant 0}$. So, a repeated application of the previous lemma provides a decomposition of any map θ of $\mathcal{S}^\bullet_M(Z)$ as a product $\theta = u_n \cdots u_1$, where u_i is a map u as in the lemma.

7.4.6. *Decomposition at a point.* Let $z_0 \in Z_o$ with $z_0 \notin M$.

Given ζ a homotopy class of admissible paths in Z with $\zeta \neq \mathrm{id}_{z_0}$, we put $\mu(\zeta) = i(\zeta, \mathrm{id}_{z_0})$.

Assume $\mu(\zeta) \geqslant 1$. There is a unique decomposition $\zeta = \zeta^{r-} \cdot \zeta^r$ in $\mathcal{S}^\bullet(Z, 1)$ such that $\zeta^r(1) = z_0$ and $\mu(\zeta^r) = 1$.

Given $\theta \in \mathrm{Hom}_{\mathcal{S}^\bullet_M(Z)}(I, J)$, we put $\mu(\theta) = \sum_{s \in I} \mu(\theta_s)$. Given $\theta' \in \mathrm{Hom}_{\mathcal{S}^\bullet_M(Z)}(I', I)$ with $\theta \cdot \theta' \neq 0$, we have $\mu(\theta \cdot \theta') = \mu(\theta) + \mu(\theta')$.

Lemma 7.4.27. *Let $\theta \in \mathrm{Hom}_{\mathcal{S}^\bullet_M(Z)}(I,J)$ with $\mu(\theta) \geqslant 2$.*

There exists a decomposition $\theta = r'(\theta)\cdot r(\theta)$ in $\mathcal{S}_M(Z)$ with $\mu(r(\theta)) = 1$ and with the following property.

Let $s \in I$ such that $\mu(r(\theta)_s) = 1$. Given $s' \in I$ such that $\mu(\theta_{s'}) \geqslant 1$ and $\mathrm{supp}(\theta^r_{s'}) \subset \mathrm{supp}(\theta^r_s)$, then $s' = s$.

Proof. We prove the lemma by induction on $|\theta|$. Assume there is a set I' satisfying the assumptions of Lemma 7.4.26 and such that $\mu(u) = 0$. By induction, there is a decomposition $\theta^u = r'(\theta^u)\cdot r(\theta^u)$ as in the lemma. Now $r(\theta) = r(\theta^u)\cdot u$ and $r'(\theta) = r'(\theta^u)$ satisfy the requirements of the lemma.

Assume now that given any set I' satisfying the assumptions of Lemma 7.4.26, we have $\mu(u) \geqslant 1$.

Let $s \in I$ with $\mu(\theta_s) \geqslant 1$ such that given $s' \in I$ with $\mu(\theta_{s'}) \geqslant 1$, we have $\mathrm{supp}(\theta^r_s) \subset \mathrm{supp}(\theta^r_{s'})$. Given $s' \in I\backslash\{s\}$, we have $\mu(\alpha^{s'}) = 0$ (notations of §7.4.5).

Let I' be the set of $s' \in I$ such that there is a sequence $s_0 = s, s_1, \ldots, s_r = s'$ of elements of I such that $s_i \to s_{i+1}$ is an arrow of $\Gamma(\theta)$ for $0 \leqslant i < r$. Assume there exist $s_1, \ldots, s_d$ in $I'\backslash\{s_0\}$ such that $s_d = s_1$ and $s_i \to s_{i+1}$ is an arrow of $\Gamma(\theta)$ for $1 \leqslant i < d$. Then $I'' = \{s_1, \ldots, s_d\}$ satisfies the assumptions of Lemma 7.4.26. On the other hand, we have $\mu(\alpha^{s'}) = 0$ for $s' \in I''$, hence we get a contradiction. It follows that I' is a cycle or a line and it satisfies the assumptions of Lemma 7.4.26. The braids $r'(\theta) = \theta^u$ and $r(\theta) = u$ of Lemma 7.4.26. satisfy the requirements of the lemma. ☐

7.4.7. *Differential.* Let us start with a description of $i(\theta)$ in terms of $L(\theta)$, using our previous analysis of S^1.

Let $f : Z \to Z'$ be a morphism of curves. Given $\theta \in \mathrm{Hom}_{\mathcal{P}^\bullet_f(Z)}(I,J)$, the map f induces an injection $f : L(\theta) \hookrightarrow L(f(\theta))$ by the discussion above Lemma 7.3.24.

Lemma 7.4.28. *Given $\theta' \in f(\mathrm{Hom}_{\mathcal{P}^\bullet_f(Z)}(I,J))$, the map f induces a bijection $\bigcup_{\theta\in f^{-1}(\theta')} L(\theta) \xrightarrow{\sim} L(\theta')$. It restricts to a bijection $\bigcup_{\theta\in f^{-1}(\theta')} D(\theta) \xrightarrow{\sim} D(\theta')$.*

Proof. Assume first f is a non-singular cover of Z'.

Let $\zeta' \in L(\theta')$. There are $i'_1 \neq i'_2 \in f(I)$ such that $\zeta' \in I(\theta'_{i'_i}, \theta'_{i'_2})$. By Lemma 7.3.24, there are elements $\zeta_r \in f^{-1}(\theta'_{i_r})$ and $\zeta \in I(\zeta_1,\zeta_2)$ such that $\zeta' = f(\zeta)$. We define $\theta \in f^{-1}(\theta')$ by setting $\theta_{\zeta_r(0)} = \zeta_r$ and by setting θ_i to be any lift of $\theta'_{f(i)}$ for all $f(i) \notin \{i'_1, i'_2\}$. This shows the surjectivity part of the first statement of the lemma.

Consider now θ and $\hat\theta$ maps in $\mathcal{P}^\bullet_f(Z)$ such that $f(\theta) = f(\hat\theta) = \theta'$. Let $\zeta \in L(\theta)$ and $\hat\zeta \in L(\hat\theta)$ such that $f(\zeta) = f(\hat\zeta) = \zeta'$. There are $i'_1 \neq i'_2 \in f(I)$ such that $\zeta' \in I(\theta'_{i'_1}, \theta'_{i'_2})$. We have $\hat\theta_{\hat\zeta(t)}, \theta_{\zeta(t)} \in f^{-1}(\theta'_{i'_r})$ for $t \in \{0,1\}$. It follows from Lemma 7.3.24 that $\zeta = \hat\zeta$. So, the first statement of the lemma holds.

Assume now f is an open embedding. The injectivity of the first map of the lemma is clear, while the surjectivity follows from Lemma 7.3.25.

We deduce the first part of the lemma for Z and Z' non-singular and the general case follows now by taking non-singular covers of Z and Z' and the lift of f.

Let us prove now the second statement of the lemma about $D(\theta)$.

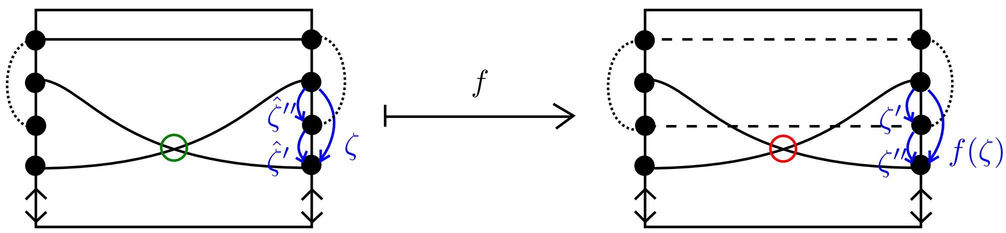
Consider $\theta \in f^{-1}(\theta')$ and $\zeta \in L(\theta)$. It is clear that if $f(\zeta) \in D(\theta')$, then $\zeta \in D(\theta)$.

Assume now $\zeta \in D(\theta)$. Fix $i_1 \neq i_2 \in I$ so that $\zeta \in I(\theta_{i_1}, \theta_{i_2})$.

Let $\zeta', \zeta'' \in L(\theta')$ such that $f(\zeta) = \zeta' \circ \zeta''$. Let $z = \zeta'(0) = \zeta''(1)$. If $|f^{-1}(\theta'_z)| > 1$, then $z \in Z_o$ and $\theta'_z = \mathrm{id}$, hence ζ' and ζ'' have opposite orientations by Lemma 7.4.15. Assume now θ'_z has a unique lift. Let $\hat{\zeta}'$ and $\hat{\zeta}''$ be the unique lifts of ζ' and ζ'' (first part of the lemma). By unicity of lifts, we have $\zeta = \hat{\zeta}' \circ \hat{\zeta}''$. We have $\hat{\zeta}', \hat{\zeta}'' \in L(\theta)$, hence $\hat{\zeta}'$ and $\hat{\zeta}''$ have opposite orientations. It follows that ζ' and ζ'' have opposite orientations as well.

Consider now $\zeta' : f(i_1) \to f(i_2)$ a smooth homotopy class of paths such that $f(\zeta) \circ \zeta'^{-1}$ and $\zeta'^{-1} \circ f(\zeta)$ are smooth and have the same orientation as $f(\zeta)$ and ζ'. Let $\hat{\zeta}'$ be the unique lift of ζ'. Since $f(\zeta) \circ \zeta'^{-1}$ is smooth, it follows that $\hat{\zeta}'(0) = i_1$ and $\zeta \circ \hat{\zeta}'^{-1}$ is smooth and has the same orientation as ζ. Similarly, $\hat{\zeta}'(1) = i_2$ and $\hat{\zeta}'^{-1} \circ \zeta$ is smooth and has the same orientation as ζ. A similar statement holds for ζ replaced by $\bar{\zeta}$. We deduce that $f(\zeta) \in D(\theta')$. $\square$

Remark 7.4.29. The picture below shows what would go wrong in Lemma 7.4.28 if we allowed unoriented points in Z_{exc}. In the proof, we need $\hat{\zeta}'$ and $\hat{\zeta}''$ to be in $L(\theta)$, which would not be true if this example were valid.



Proposition 7.4.30. *Let $\theta \in \mathrm{Hom}_{\mathcal{P}^\bullet(Z)}(I,J)$. We have $i(\theta) = \sum_{\Omega \in \pi_0(Z)} |(L(\theta) \cap \Omega)/\mathrm{inv}| e_\Omega$. In particular, $L(\theta)$ is finite.*

Proof. The statement is true for $Z = S^1$ unoriented by Lemmas 3.2.3, 7.4.19 and 7.4.20. It follows from Lemmas 7.4.28 and 7.3.22 that it holds for any connected non-singular Z, by embedding it in S^1. So, the lemma holds for any non-singular Z. By realizing an arbitrary Z as a quotient of its non-singular cover, we deduce from Lemmas 7.4.28 and 7.3.22 that the lemma holds for any Z. $\square$

Given $f : Z \to Z'$ a morphism of curves, given $\theta \in \mathrm{Hom}_{\mathcal{P}^\bullet_f(Z)}(I,J)$ and given $\zeta \in L(\theta)$, we have $f(\theta^\zeta) = f(\theta)^{f(\zeta)}$.

Lemma 7.4.31. *Given $\theta \in \mathrm{Hom}_{\mathcal{P}^\bullet(Z)}(I,J)$ and $\zeta \in L(\theta)$, we have $\theta^\zeta \in \mathrm{Hom}_{\mathcal{P}^\bullet(Z)}(I,J)$. We have $\zeta \in D(\theta)$ if and only if $\overline{\deg}_D(\theta^\zeta) = \overline{\deg}_D(\theta) - 1$ for some (or equivalently, any) finite subset D of $T(Z)$ such that $D \cap \iota(D) = \varnothing$.*

Proof. Let us shows the first statement. We can assume $\theta^\zeta_{\zeta(0)} \neq \mathrm{id}$.

Assume $\theta^\zeta_{\zeta(0)}$ has the same orientation as ζ^{-1}. We have $\theta_{\zeta(1)} = \theta^\zeta_{\zeta(0)} \circ \zeta^{-1}$. If γ and γ' are minimal paths in $\theta^\zeta_{\zeta(0)}$ and ζ^{-1}, then $\gamma \circ \gamma'$ is a minimal path in $\theta_{\zeta(1)}$. Since $\gamma \circ \gamma'$ is admissible, it follows that γ is admissible, hence $\theta^\zeta_{\zeta(0)}$ is admissible.

Otherwise, $\theta^\zeta_{\zeta(0)}$ has the same orientation as $\bar{\zeta}^{-1}$ and $\theta_{\zeta(0)} = \bar{\zeta}^{-1} \circ \theta^\zeta_{\zeta(0)}$, hence we deduce as above that $\theta^\zeta_{\zeta(0)}$ is admissible.

Similarly, $\theta^\zeta_{\zeta(1)}$ is admissible and we deduce that θ^ζ is a braid.

Let us prove the second part of the lemma. When $Z = S^1$ unoriented, this holds by Lemmas 7.4.20, 6.2.8 and 7.4.19 and Proposition 7.4.18. We deduce that the lemma holds when Z is a connected non-singular curve, by embedding Z in S^1. So, it holds when Z is a non-singular curve (since $\mathrm{supp}(\zeta)$ is contained in a connected component of Z).

Consider now a general Z and the non-singular cover $q : \hat{Z} \to Z$. There is a braid $\hat{\theta}$ in $\hat{Z}$ with $q(\hat{\theta}) = \theta$ (Lemma 7.4.5) and there is $\hat{\zeta} \in L(\hat{\theta})$ such that $\zeta = q(\hat{\zeta})$ (Lemma 7.4.28). The considerations above show that $\hat{\theta}^{\hat{\zeta}}$ is a braid in $\hat{Z}$, hence $\theta^{\zeta} = q(\hat{\theta}^{\hat{\zeta}})$ is a braid in Z. The statement on degrees follows from Lemmas 7.4.28 and 7.4.12. □

Given $\theta \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(I, J)$, we put

$$d(\theta) = \sum_{\zeta \in D(\theta)/\mathrm{inv}} \theta^{\zeta} \in \mathrm{Hom}_{\mathcal{S}(Z)}(I, J).$$

Note that the set $D(\theta)$ is finite by Proposition 7.4.30.

Theorem 7.4.32. *The map d equips $\mathcal{S}(Z)$ with a structure of differential $\mathbf{F}_2$-linear category and $\mathcal{S}^\bullet(Z)$ with a structure of differential pointed catgory.*

Let $f : Z \to Z'$ be a morphism of curves.

- *The functor $f : \mathcal{S}^\bullet_f(Z) \to \mathcal{S}^\bullet(Z')$ is a faithful pointed functor and its restriction to $\mathcal{S}^\bullet_{\{z \in Z \mid |f^{-1}f(z)|=1\}}(Z)$ is a differential pointed functor.*
- *If f is strict, then $f^\# : \mathrm{add}(\mathcal{S}(Z')) \to \mathrm{add}(\mathcal{S}_f(Z))$ is a differential functor commuting with coproducts.*
- *If f is a quotient morphism, then $f^\#$ is faithful and every map in $\mathcal{S}^\bullet(Z')$ is in the image by f of a map of $\mathcal{S}^\bullet_f(Z)$.*

Proof. Lemma 7.4.28 shows that $d(f^\#(\theta')) = f^\#(d(\theta'))$ for any θ' and that $d(f(\theta)) = f(d(\theta))$ if $|f^{-1}f(\theta)| = 1$.

Assume $Z = S^1$ (unoriented) and consider a finite subset $\mathbf{a}$ of Z as in §7.4.3. We use the notations of that section. It follows from Lemma 7.4.19 that the isomorphism F of Proposition 7.4.18 induces an isomorphism of $\mathbf{F}_2$-linear categories $F : \mathbf{F}_2[\mathcal{H}_n] \xrightarrow{\sim} \mathcal{S}_{\mathbf{a}}(Z)$. It follows now from Lemma 7.4.20 that this isomorphism commutes with d. In particular, d is a differential on $\mathcal{S}_{\mathbf{a}}(Z)$. Since this holds for any finite subset $\mathbf{a}$ of Z, we deduce that d is a differential on $\mathcal{S}(Z)$.

Consider now a non-singular connected Z and an injective morphism of curves $f : Z \hookrightarrow S^1$. Since f induces a faithful $\mathbf{F}_2$-linear functor $\mathcal{S}(Z) \to \mathcal{S}(S^1)$ commuting with d, we deduce that d is a differential on $\mathcal{S}(Z)$.

The decomposition (7.4.3) is compatible with d, hence d is a differential on $\mathcal{S}(Z)$ for any non-singular Z.

Consider now a general Z and $q : \hat{Z} \to Z$ its non-singular cover. Since the additive $\mathbf{F}_2$-linear functor $q^\#$ commutes with d, it follows that d is a differential on $\mathcal{S}(Z)$.

The last statement of the theorem follows from Lemma 7.3.17. □

There is an isomorphism of differential pointed categories

$$\mathcal{S}^\bullet(Z^{\mathrm{opp}}) \xrightarrow{\sim} \mathcal{S}^\bullet(Z)^{\mathrm{opp}},\ I \mapsto I,\ \theta \mapsto (\theta_s^{-1})_s. \tag{7.4.4}$$

Note that the construction $Z \mapsto \mathrm{add}(\mathcal{S}(Z))$ and $f \mapsto f^{\#}$ defines a contravariant functor from the category of curves with strict morphisms to the category of differential categories.

7.4.8. *Strands on non-singular curves.* We consider as in §7.4.3 a family $\mathbf{a} = \{a_1, \ldots, a_n\}$ of points on S^1 and $z \in S^1 - \mathbf{a}$ such that $a_1, \ldots, a_n, z$ is cyclically ordered.

The next proposition follows immediately from Proposition 7.4.18 and Lemmas 7.4.19 and 7.4.20.

Proposition 7.4.33. *The functor F induces an isomorphism of differential pointed categories $\mathcal{H}_n \to \mathcal{S}^\bullet_{\mathbf{a}}(S^1)$. It restricts to isomorphisms of differential pointed categories*

$$\mathcal{H}_n^+ \to \mathcal{S}^\bullet_{\mathbf{a}}(\dot{S}^1),\ \mathcal{H}_n^{++} \to \mathcal{S}^\bullet_{\mathbf{a}}(\vec{S}^1),\ \mathcal{H}_n^f \to \mathcal{S}^\bullet_{\mathbf{a}}(I) \text{ and } \mathcal{H}_n^{f++} \to \mathcal{S}^\bullet_{\mathbf{a}}(\vec{I}).$$

Consider $Z = \mathbf{R}_{>0}$ as an unoriented curve. We denote by $\mathcal{S}^\bullet_\otimes(\mathbf{R}_{>0})$ the full subcategory of $\mathcal{S}^\bullet(\mathbf{R}_{>0})$ with objects the subsets of the form $\{1, \ldots, n\}$ for some $n \in \mathbf{Z}_{\geqslant 0}$. We define a monoidal structure on the differential pointed category $\mathcal{S}^\bullet_\otimes(\mathbf{R}_{>0})$ by $\{1, \ldots, n\} \otimes \{1, \ldots, m\} = \{1, \ldots, n+m\}$ and $\theta'' = \theta \otimes \theta'$ is defined by $\theta''_i = \theta_i$ if $i \leqslant n$ and $\theta''_i = \theta'_{i-n}$ otherwise.

The next theorem follows immediately from Proposition 7.4.33.

Theorem 7.4.34. *There is an isomorphism of differential pointed monoidal categories $\mathcal{U}^\bullet \xrightarrow{\sim} \mathcal{S}^\bullet_\otimes(\mathbf{R}_{>0})$ defined by $e \mapsto \{1\}$ and τ maps to the non-zero and non-identity element of* $\mathrm{End}_{\mathcal{S}^\bullet(\mathbf{R}_{>0})}(\{1,2\})$.

7.4.9. *Products and divisibility.*

Lemma 7.4.35. *Consider braids $\theta'' : I \to J$ and $\theta' : J \to K$ and assume $\theta = \theta' \cdot \theta''$ is non-zero. Let $\zeta \in D(\theta) \backslash (D(\theta) \cap D(\theta''))$. Assume ζ and ζ^{-1} are oriented.*

Define $\alpha'' : I \to J$ by

$$\alpha''_s = \begin{cases} \theta''_{\zeta(1)} \circ \zeta & \text{if } s = \zeta(0) \\ \theta''_{\zeta(0)} \circ \zeta^{-1} & \text{if } s = \zeta(1) \\ \theta''_s & \text{otherwise.} \end{cases}$$

Let $\zeta' = \theta''_{\zeta(1)} \circ \zeta \circ (\theta''_{\zeta(0)})^{-1}$ and $\alpha' = (\theta')^{\zeta'}$. Then α'' and α' are braids and $\theta = \alpha' \cdot \alpha''$.

Proof. Since ζ and ζ^{-1} are oriented, it follows that α''_s is oriented for all s. Also, it follows from Lemma 7.4.31 that α' is a braid.

Consider first the case where $Z = S^1$ unoriented. In that case, the lemma follows from Proposition 7.4.33 and Lemmas 7.4.20 and 6.2.9.

Assume now Z is smooth and connected. There is an injective morphism of curves $f : Z \to S^1$, where S^1 is unoriented. Since the lemma holds for S^1, we deduce that it holds for Z.

When Z is only assumed to be smooth, the lemma follows from the case of the connected component containing ζ.

Consider now the general case. Let $f : \tilde{Z} \to Z$ be a smooth cover. Let $\tilde{\theta}$ be a braid lifting θ. There are unique braids $\tilde{\theta}'$ and $\tilde{\theta}''$ in $\tilde{Z}$ with $\tilde{\theta} = \tilde{\theta}' \cdot \tilde{\theta}''$ and $f(\tilde{\theta}') = \theta'$, $f(\tilde{\theta}'') = \theta''$. There is a unique $\tilde{\zeta} \in D(\tilde{\theta})$ with $f(\tilde{\zeta}) = \zeta$ (Lemma 7.4.28). We have $\tilde{\zeta} \notin D(\tilde{\theta}'')$ (Lemma 7.4.28). Since the lemma holds for $\tilde{Z}$, we deduce it holds for Z. ∎

7.4.10. *Subcurves.* Let Z be a curve.

Let S and T be two finite subsets of Z. Let S_1 be a subset of S and $S_2 = S\backslash S_1$. Let T_1 be a subset of T and $T_2 = T\backslash T_1$. Let $\Phi_i \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S_i, T_i)$. We define $\Phi = \Phi_1 \boxtimes \Phi_2 \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S, T)$ by $\Phi_s = (\Phi_i)_s$ when $s \in S_i$. This gives an injective map of pointed sets

$$\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S_1, T_1) \wedge \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S_2, T_2) \hookrightarrow \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S, T).$$

Note that this is not compatible with composition in general. We obtain an isomorphism of pointed sets

$$\bigvee_{\substack{T_1' \subset T \\ |T_1'| = |S_1|}} \left(\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S_1, T_1') \wedge \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S_2, T\backslash T_1')\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S, T).$$

We have corresponding morphisms of $\mathbf{F}_2$-modules between Hom-spaces in $\mathcal{S}(Z)$. Note these are not compatible with the differential.

Assume $S_2 = T_2$. The map $\Phi_1 \mapsto \Phi_1 \boxtimes \mathrm{id}_{S_2}$ defines a canonical embedding of pointed sets (not compatible with the differential nor the multiplication in general)

$$\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S_1, T_1) \hookrightarrow \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S_1 \sqcup S_2, T_1 \sqcup S_2).$$

Given Z_1 and Z_2 two disjoint closed subcurves of Z, we obtain a faithful differential pointed functor

$$\mathcal{S}^\bullet(Z_1) \wedge \mathcal{S}^\bullet(Z_2) \to \mathcal{S}^\bullet(Z),\ (S_1, S_2) \mapsto S_1 \sqcup S_2.$$

Let $Z_1, \ldots, Z_r$ be the connected components of Z. The construction above induces an isomorphism of differential pointed categories (cf (7.4.2))

$$\mathcal{S}^\bullet(Z_1) \wedge \cdots \wedge \mathcal{S}^\bullet(Z_r) \xrightarrow{\sim} \mathcal{S}^\bullet(Z),\ (S_1, \ldots, S_r) \mapsto S_1 \sqcup \cdots \sqcup S_r. \tag{7.4.5}$$

Let us record a case where the tensor product construction $\boxtimes$ is compatible with composition and the differential in the following immediate lemma.

Lemma 7.4.36. *Let M be a subset of Z and let Z' be a subcurve of Z. Assume that given an admissible homotopy class of paths ζ in Z with endpoints in M, there is an admissible path γ in ζ contained in $Z - Z'$. There is a faithful functor of differential pointed categories*

$$\begin{aligned} \mathcal{S}^\bullet_M(Z) \wedge \mathcal{S}^\bullet(Z') &\to \mathcal{S}^\bullet_{M \cup Z'}(Z) \\ (S, T) &\mapsto S \sqcup T \\ (\alpha, \beta) &\mapsto \alpha \boxtimes \beta = (\alpha \boxtimes \mathrm{id}) \cdot (\mathrm{id} \boxtimes \beta) = (\mathrm{id} \boxtimes \beta) \cdot (\alpha \boxtimes \mathrm{id}). \end{aligned}$$

7.4.11. *Bordered Heegaard-Floer algebras.* We consider a chord diagram $(\mathcal{Z}, \mathbf{a}, \mu)$ as in §7.2.4. Let $Z_1, \ldots, Z_l$ be the connected components of $\mathcal{Z}$. Let $n_i = |\mathbf{a} \cap Z_i|$ and let $q : \tilde{Z} \to Z$ be the quotient map.

The isomorphism (7.4.5) associated with the decomposition $\tilde{Z} = \mathring{Z}_1 \coprod \cdots \coprod \mathring{Z}_l$ together with the strands algebra description of §6.3.2 and the isomorphism of Proposition 7.4.33 induce an isomorphism of differential algebras

$$\mathcal{A}(n_1) \otimes \cdots \otimes \mathcal{A}(n_l) \xrightarrow{\sim} \mathrm{End}_{\mathrm{add}(\mathcal{S}(\tilde{Z}))}\Big(\bigoplus_{I \subset \mathbf{a}} I\Big).$$

The differential algebra $\mathcal{A}(\mathcal{Z})$ associated to $\mathcal{Z}$ is a differential non-unital subalgebra of $\mathcal{A}(n_1) \otimes \cdots \otimes \mathcal{A}(n_l)$ (cf [Za, Definition 2.6] and [LiOzTh1, Definition 3.16] for the original setting where $l = 1$). There is a unique isomorphism of differential algebras

$$\mathcal{A}(\mathcal{Z}) \xrightarrow{\sim} \mathrm{End}_{\mathrm{add}(\mathcal{S}(Z))}\big(\bigoplus_{S \subset q(\mathbf{a})} S\big)$$

making the following diagram commutative

$$\begin{tikzcd}
\mathcal{A}(\mathcal{Z}) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{End}_{\mathrm{add}(\mathcal{S}(Z))}\big(\bigoplus_{S \subset q(\mathbf{a})} S\big) \arrow[d, hook, "q^{\#}"] \\
\mathcal{A}(n_1) \otimes \cdots \otimes \mathcal{A}(n_l) \arrow[r, "\sim"'] & \mathrm{End}_{\mathrm{add}(\mathcal{S}(\tilde{Z}))}\big(\bigoplus_{I \subset \mathbf{a}} I\big)
\end{tikzcd}$$

8. 2-representations on strand algebras

8.1. Action on ends of curves.

8.1.1. *Definition.* Let $\xi : \mathbf{R}_{>0} \to Z$ be an injective morphism of curves, where $\mathbf{R}_{>0}$ is viewed as an unoriented curve. Let M a subset of $Z \backslash \xi(\mathbf{R}_{\geqslant 1})$.

We say that ξ is *terminal* for (Z, M) if the following two conditions hold:

- given an admissible homotopy class of paths ζ in Z with endpoints in M, there is an admissible path γ in ζ contained in $Z \backslash \xi(\mathbf{R}_{\geqslant 1})$
- there is no admissible path in $Z \backslash \{\xi(1)\}$ from a point of M to $\xi(2)$.

Note that ξ is terminal for (Z, M) if and only if ξ is terminal for $(Z(\xi), Z(\xi) \cap M)$, where $Z(\xi)$ is the component of Z containing $\xi(\mathbf{R}_{>0})$.

We say that ξ is *outgoing* for Z if $\xi(\mathbf{R}_{\geqslant 1})$ is closed in Z. Note that if ξ is outgoing for Z then it is terminal for (Z, M) for any $M \subset Z \backslash \xi(\mathbf{R}_{\geqslant 1})$.

Remark 8.1.1. Assume ξ is not outgoing for Z and let $z_0 \in Z$ such that $\overline{\xi(\mathbf{R}_{\geqslant 1})} \backslash \xi(\mathbf{R}_{\geqslant 1}) = \{z_0\}$. Note that ξ is outgoing for $Z \backslash \{z_0\}$. The map ξ is terminal for (Z, M) if and only if $z_0 \notin M$ and the inclusion induces an isomorphism $\mathrm{Hom}_{\mathcal{A}^\bullet(Z \backslash \{z_0\}, 1)}(m, z) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}^\bullet(Z,1)}(m, z)$ for all $m \in M$ and $z \in M \cup \{\xi(1)\}$.

We assume now that ξ is terminal for (Z, M). Thanks to Lemma 7.4.36, we have a differential pointed functor

$$L^\bullet = L^\bullet_\xi : \mathcal{S}^\bullet_M(Z) \times \mathcal{S}^\bullet_M(Z)^{\mathrm{opp}} \times \mathcal{U}^\bullet \to \mathrm{diff}$$

$$L^\bullet(T, S, e^n) = \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S, T \sqcup \{\xi(1), \ldots, \xi(n)\})$$

$$L^\bullet(\beta, \alpha, \sigma)(f) = (\beta \boxtimes \xi(\sigma)) \cdot f \cdot \alpha \in L^\bullet(T', S', n)$$

for $\alpha \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S', S)$, $\beta \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(T, T')$, $\sigma \in \mathrm{End}_{\mathcal{U}^\bullet}(e^n)$, and $f \in L^\bullet(T, S, n)$. We have used the strands realization of $\mathcal{U}^\bullet$ given by Theorem 7.4.34.

We put $L^\bullet(T, S) = L^\bullet(T, S, e)$. As usual, we put $L_\xi = \mathbf{F}_2[L^\bullet_\xi]$.

The naturality in the next lemma is immediate as in Lemma 7.4.36.

Lemma 8.1.2. *Given $S \subset M$ and $n \geqslant 0$, there is an isomorphism of functors $\mathcal{S}^\bullet_M(Z) \to \mathrm{Sets}^\bullet$ (forgetting the differential)*

$$\bigvee_{\substack{S' \subset S \\ |S'|=n}} \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S', \{\xi(1), \ldots, \xi(n)\}) \wedge \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S \backslash S', -) \xrightarrow{\sim} L^\bullet(-, S, e^n)$$
$$(\alpha, \beta) \mapsto \alpha \boxtimes \beta.$$

Lemma 8.1.2 shows that there is an isomorphism of functors, functorial in S and T

$$L^\bullet(T, -, e^n) \wedge L^\bullet(-, S, e^m) \xrightarrow{\sim} L^\bullet(T, S, e^{n+m})$$
$$(\alpha, \beta) \mapsto (\alpha \boxtimes \xi([r \to n + r]_{1 \leqslant r \leqslant m})) \cdot \beta.$$

The functor $E = E_\xi = L^\bullet(-,-)$ gives a bimodule 2-representation on $\mathcal{S}^\bullet_M(Z)$. The endomorphism τ of $L^\bullet(-,-,e^2)$ is given by the non-identity non-zero braid $\{1,2\} \to \{1,2\}$.

We have obtained the following proposition.

Proposition 8.1.3. *The bimodule E and the endomorphism τ define a bimodule 2-representation on $\mathcal{S}^\bullet_M(Z)$ and on $\mathcal{S}_M(Z)$.*

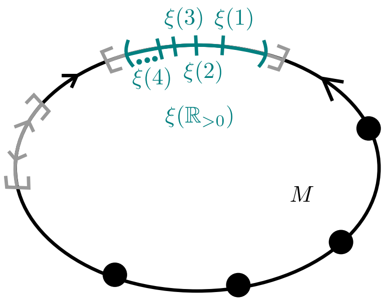
Lemma 8.1.2 shows that $L_\xi(-,-)$ is left finite.

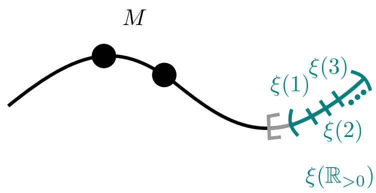
Remark 8.1.4. Proposition 8.1.3 generalizes and make more precise a result of Douglas and Manolescu [DouMa, §5.2].

Let $(\mathcal{Z}, \mathbf{a}, \mu)$ be a chord diagram where $\mathcal{Z} = [0,1]$. Let $\tilde{Z}' = (0, \infty)$, viewed as a curve with $\tilde{Z}'_o = \tilde{Z} = (0,1)$ (with its usual orientation). We extend the equivalence relation from $\tilde{Z}$ to $\tilde{Z}'$ by having all points of $[1,\infty)$ alone in their class. Let $Z' = \tilde{Z}'/\sim$. We have $Z'_o = Z_o$. Let $M = Z'_{exc}$ be the image of $\mathbf{a}$ in Z'. Let $\xi : \mathbf{R}_{>0} \to Z'$, $x \mapsto x+1$. Note that ξ is outgoing for Z'.

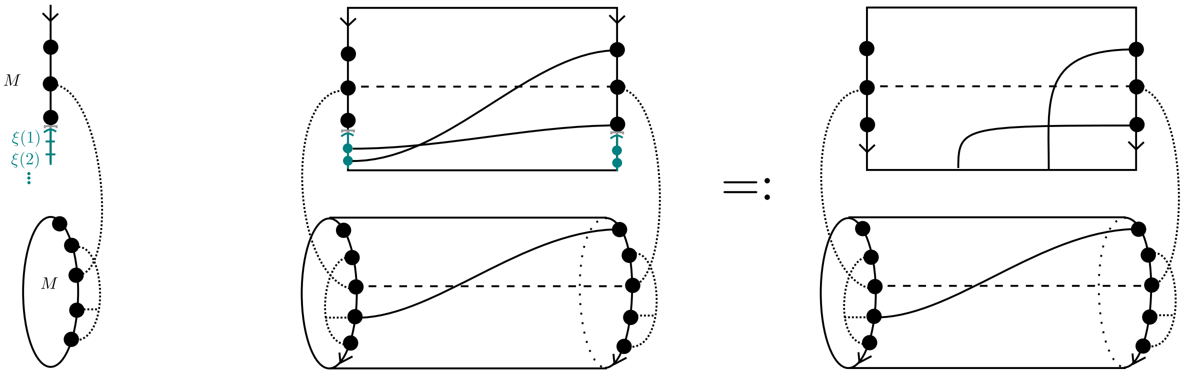
The lax 2-representation underlying the 2-representation on $\mathcal{S}_M(Z) = \mathcal{S}_M(Z')$ provided by Proposition 8.1.3 is the "bottom algebra module" constructed by Douglas and Manolescu, via the identification of §5.7.

Example 8.1.5. The left picture below gives an example where ξ is terminal for (Z, M) but not outgoing for Z. The right picture is an example where ξ is outgoing for Z.





The picture below consider the case of a curve quotient of the disjoint union of an interval and a circle, with an outgoing ξ at an end of the interval. The middle picture describes an element of $L^\bullet_\xi(-,-,e^2)$. The rightmost picture provides a different graphical representation of that element: the interval $\xi(\mathbf{R}_{\geqslant 1})$ has been moved to the bottom horizontal line.



The next remark discusses the dependence of $L^\bullet_\xi$ on ξ.

Remark 8.1.6. Assume ξ is terminal for (Z, M). Consider $f : Z \xrightarrow{\sim} Z$ an isomorphism of curves fixing M. Note that $f \circ \xi$ is terminal for (Z, M) and the map f induces an isomorphism $L^\bullet_\xi \xrightarrow{\sim} L^\bullet_{f\circ\xi}$.

Consider now another injective morphism of curves $\xi' : \mathbf{R}_{>0} \to Z$ such that $\overline{\xi'}$ is $\overline{\text{terminal}}$ for (Z, M). Assume there is a connected open subset U of Z_u containing $\overline{\xi(\mathbf{R}_{>0})}$ and $\overline{\xi'(\mathbf{R}_{>0})}$ and assume the canonical orientations on $\xi(\mathbf{R}_{>0})$ and $\xi'(\mathbf{R}_{>0})$ extend to an orientation of U. There is an isomorphism of curves $f : Z \xrightarrow{\sim} Z$ fixing $Z\backslash U$ such that $\xi' = f \circ \xi$. It induces an isomorphism $L^\bullet_\xi \xrightarrow{\sim} L^\bullet_{\xi'}$, and that isomorphism does not depend on the choice of f.

8.1.2. *Approximation.* Assume $\xi^{-1}(M)$ has no maximum. Fix an increasing sequence $m_0, m_1, \ldots$ of points of $(0, 1)$ with $\xi(m_i) \in M$ for all i and with $\lim_i m_i > t$ for all $t \in \xi^{-1}(M)$.

Fix $n \geqslant 0$ and define the braid $\beta_r : \{m_r, \ldots, m_{r+n-1}\} \to \{1, \ldots, n\}$ of $\mathbf{R}_{>0}$ by $(\beta_r)_{m_{r+i}} = [m_{r+i} \to i + 1]$.

Let S and T be two finite subsets of M. Consider r such that $m_r > \xi^{-1}(t)$ for all $t \in T \cap \xi(\mathbf{R}_{>0})$. There is an isomorphism

$$\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S, T \sqcup \xi(\{m_r, \ldots, m_{r+n-1}\})) \xrightarrow{\sim} L^\bullet(T, S, e^n),\ \alpha \mapsto (\mathrm{id}_T \boxtimes \xi(\beta_r)) \cdot \alpha.$$

It follows that there are isomorphisms functorial in S and T

$$\operatorname{colim}_{r\to\infty} \mathrm{Hom}_{\mathcal{S}^\bullet_M(Z)}(S, T \sqcup \xi(\{m_r, \ldots, m_{r+n-1}\})) \xrightarrow{\sim} L^\bullet(T, S, e^n). \tag{8.1.1}$$

Here, the colimit is taken over the invertible maps $\xi(\theta_r)$, where $\theta_r : \{m_r, \ldots, m_{r+n-1}\} \to \{m_{r+1}, \ldots, m_{r+n}\}$ is the braid in $\mathbf{R}_{>0}$ given by $(\theta_r)_{m_s} = [m_s \to m_{s+1}]$.

We deduce that $T \mapsto (S \mapsto L^\bullet(T, S, e^n))$ is isomorphic to the functor

$$\mathcal{S}^\bullet_M(Z) \to \mathcal{S}^\bullet_M(Z)\text{-diff},\ T \mapsto \operatorname{colim}_{r\to\infty} T \sqcup \xi(\{m_r, \ldots, m_{r+n-1}\}).$$

8.1.3. *2-representations and morphisms of curves.* Let $f : Z \to Z'$ be a morphism of curves. Assume ξ is terminal for (Z, M) and $f \circ \xi$ is terminal for $(Z', f(M))$.

Assume that $|f^{-1}(f(z))| = 1$ for all $z \in M$. Let M_f be the $(\mathcal{S}^\bullet_M(Z), \mathcal{S}^\bullet_{f(M)}(Z'))$-bimodule corresponding to f, *i.e.* given by $M_f(S, S') = \mathrm{Hom}_{\mathcal{S}^\bullet(Z')}(S', f(S))$. There is a morphism of functors

$E_\xi \wedge_{\mathcal{S}^\bullet_M(Z)} M_f \to M_f \wedge_{\mathcal{S}^\bullet_{f(M)}(Z')} E_{f\circ\xi}$ defined as making the following diagram commutative

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(-,T\sqcup\{\xi(1)\})\wedge\mathrm{Hom}_{\mathcal{S}^\bullet(Z')}(S',f(-)) \arrow[r, "\beta\wedge\alpha'\mapsto f(\beta)\cdot\alpha'"] \arrow[d] & \mathrm{Hom}_{\mathcal{S}^\bullet(Z')}(S',f(T)\sqcup\{f\circ\xi(1)\}) \\
\mathrm{Hom}_{\mathcal{S}^\bullet(Z')}(-,f(T))\wedge\mathrm{Hom}_{\mathcal{S}^\bullet(Z')}(S',-\sqcup\{f\circ\xi(1)\}) \arrow[ur, "\sim", "\beta'\wedge\alpha'\mapsto(\beta'\boxtimes\mathrm{id}_{f\circ\xi(1)})\cdot\alpha'"'] &
\end{tikzcd}$$

The following lemma is a consequence of (8.1.1).

Lemma 8.1.7. *If $\xi^{-1}(M)$ has no maximum, then the construction above gives an isomorphism*

$$E_\xi \wedge_{\mathcal{S}^\bullet_M(Z)} M_f \xrightarrow{\sim} M_f \wedge_{\mathcal{S}^\bullet_{f(M)}(Z')} E_{f\circ\xi},$$

and f provides a morphism of bimodule 2-representations $L^\bullet_{f\circ\xi} \to L^\bullet_\xi$.

We consider now an arbitrary M but we assume that f is strict. Let $M_{f^\#}$ be the $(\mathcal{S}_{f(M)}(Z'),\mathcal{S}_M(Z))$-bimodule corresponding to $f^\#$, *i.e.* given by $M_{f^\#}(S',S)=\bigoplus_{p:S'\to Z,\ f\circ p=\mathrm{id}_{S'}}\mathrm{Hom}_{\mathcal{S}_M(Z)}(S,p(S'))$.

There is a morphism of functors $E_{f\circ\xi}\otimes_{\mathcal{S}_{f(M)}(Z')} M_{f^\#} \to M_{f^\#}\otimes_{\mathcal{S}_M(Z)} E_\xi$ defined as making the following diagram commutative

$$\begin{tikzcd}
\bigoplus_{\substack{p:-\to Z\\ f\circ p=\mathrm{id}}}\mathrm{Hom}_{\mathcal{S}(Z')}(-,T'\sqcup\{f\circ\xi(1)\})\otimes\mathrm{Hom}_{\mathcal{S}(Z)}(S,p(-)) \arrow[r, "\beta'\wedge\alpha\mapsto f^\#(\beta')\cdot\alpha"] \arrow[d] & \bigoplus_{\substack{p:T'\to Z\\ f\circ p=\mathrm{id}_{T'}}}\mathrm{Hom}_{\mathcal{S}(Z)}(S,p(T')\sqcup\{\xi(1)\}) \\
\bigoplus_{\substack{p:T'\to Z\\ f\circ p=\mathrm{id}_{T'}}}\mathrm{Hom}_{\mathcal{S}(Z)}(-,p(T'))\otimes\mathrm{Hom}_{\mathcal{S}(Z)}(S,-\sqcup\{\xi(1)\}) \arrow[ur, "\sim", "\beta\wedge\alpha\mapsto(\beta\boxtimes\mathrm{id}_{\{\xi(1)\}})\cdot\alpha"'] &
\end{tikzcd}$$

The following lemma is a consequence of (8.1.1).

Lemma 8.1.8. *If $\xi^{-1}(M)$ has no maximum, then the construction above gives an isomorphism*

$$E_{f\circ\xi}\otimes_{\mathcal{S}_{f(M)}(Z')} M_{f^\#} \xrightarrow{\sim} M_{f^\#}\otimes_{\mathcal{S}_M(Z)} E_\xi,$$

and $f^\#$ provides a morphism of bimodule 2-representations $L_\xi \to L_{f\circ\xi}$.

8.1.4. *Twisted object description.* We explain how to obtain a version of Lemma 8.1.2 with a differential.

We say that a homotopy class of path in $Z(\xi)$ is positive if it has the same orientation as $\xi([1\to 2])$.

Fix a finite subset S of M and $n\geqslant 0$.

Let S'' be a subset of S with n elements, let $s'\in S\backslash S''$ and $s''\in S''$. Let $\zeta: s''\to s'$ be a positive smooth homotopy class of paths in Z. We put $S'=(S''\backslash\{s''\})\sqcup\{s'\}$.

We define a map

$g_{S'',\zeta}:\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S'',\{\xi(1),\dots,\xi(n)\})\to\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S',\{\xi(1),\dots\xi(n)\})\wedge\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S\backslash S',S\backslash S'')$.

We put

$$g_{S'',\zeta}(\alpha)=(\alpha_{|S'\backslash\{s'\}}\boxtimes(\alpha_{s''}\circ\zeta^{-1}))\wedge(\mathrm{id}_{S\backslash(S''\sqcup\{s'\})}\boxtimes\zeta)$$

if

- $\alpha_{s''} \circ \zeta^{-1}$ is smooth
- and given $s \in S''\backslash\{s''\}$ and $\zeta' : s \to s'$ and $\zeta'' : s'' \to s$ smooth positive with $\zeta = \zeta' \circ \zeta''$ and with $\alpha_{s''} \circ \zeta''^{-1}$ smooth, then $\alpha_{s''} \circ \zeta''^{-1} \circ \alpha_s^{-1}$ is negative.

We put $g_{S'',\zeta}(\alpha) = 0$ otherwise.

Remark 8.1.9. Note that if $\alpha_{s''} \circ \zeta^{-1}$ is smooth, then the support of ζ is contained in $Z(\xi)$. Given α non-zero, if ζ is positive, then both $\alpha_{s''} \circ \zeta^{-1}$ and ζ are oriented, since $\alpha_{s''}$ is oriented.

We obtain a map $f_{S'',\zeta} : \alpha \wedge \beta \mapsto (\mathrm{id} \wedge \beta) \circ g_{S'',\zeta}(\alpha)$
$\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S'', \{\xi(1), \ldots, \xi(n)\}) \wedge \mathrm{Hom}(S\backslash S'', -) \to \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S', \{\xi(1), \ldots \xi(n)\}) \wedge \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S\backslash S', -)$.

Let $r(S'')$ be the number of pairs $(s'', s) \in S'' \times (S\backslash S'')$ such that there exists a positive path $s'' \to s$.

We define now

$$V_r = \bigoplus_{\substack{S' \subset S,\ |S'|=n \\ r(S')=r}} \mathrm{Hom}(S', \{\xi(1), \ldots, \xi(n)\}) \otimes \mathrm{Hom}_{\mathcal{S}(Z)}(S\backslash S', -) \in \mathcal{S}_M(Z)\text{-diff}\,.$$

Given $r' < r''$, define $f_{r',r''} = \sum_{S'',\zeta} f_{S'',\zeta}$, where

- S'' is a subset of S with $|S''| = n$ and $r(S'') = r''$
- ζ is a positive admissible homotopy class of paths in Z with $\zeta(0) \in S''$ and $\zeta(1) \in S\backslash S''$

such that $\mathrm{supp}(\zeta) \cap S'' = \{s''\}$ and $r((S''\backslash\{\zeta(0)\}) \sqcup \{\zeta(1)\}) = r'$.

Let $V = V_n(S) = \bigoplus_r V_r$ and let $d_V = \sum_r d_{V_r} + \sum_{r',r''} f_{r',r''}$. We will show below (Proposition 8.1.10) that $d_V^2 = 0$, *i.e.* V is the object of $\mathcal{S}_M(Z)$-diff corresponding to the twisted object $[\bigoplus V_r, (f_{r',r''})]$.

Proposition 8.1.10. *Given $S \subset M$ and $n \geqslant 0$, then $d^2_{V_n(S)} = 0$ and the map of Lemma 8.1.2 defines an isomorphism of functors $\mathcal{S}_M(Z) \to k$-diff*

$$V_n(S) \xrightarrow{\sim} L(-, S, e^n).$$

Proof. By Remark 8.1.1, we can assume ξ is outgoing for Z. We will show that

(8.1.2) the isomorphism of Lemma 8.1.2 is compatible with the differentials.

The proposition will follow immediately from (8.1.2).

Let S'' be a subset of S with n elements, and let T be a finite subset of M. Let $a : \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S'', \{\xi(1), \ldots \xi(n)\}) \wedge \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S\backslash S'', T) \to L^\bullet(T, S, e^n)$ be the map of Lemma 8.1.2. Let $\alpha \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S'', \{\xi(1), \ldots \xi(n)\})$ and $\beta \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S\backslash S'', T)$. Let $\theta = \alpha \boxtimes \beta = a(\alpha \wedge \beta)$. The statement (8.1.2) will follow from the following property:

$$a(d(\alpha \otimes \beta)) = d(\theta). \tag{8.1.3}$$

We have

$$a(d(\alpha \otimes \beta)) = d(\alpha) \boxtimes \beta + \alpha \boxtimes d(\beta) + \sum_\zeta a((\mathrm{id} \otimes \beta) \cdot g_{S'',\zeta}(\alpha))$$

where ζ runs over positive admissible homotopy classes of paths starting in S'' and ending in $S\backslash S''$.

We have

$$D(\theta)/\mathrm{inv} = \big(D(\alpha)/\mathrm{inv}\big) \sqcup \big(D(\beta)/\mathrm{inv}\big) \sqcup \coprod_{(s_1,s_2)\in S''\times(S\backslash S'')} I(\alpha_{s_1},\beta_{s_2}) \cap D(\theta).$$

Fix $(s_1,s_2)\in S''\times(S\backslash S'')$. Let $S'=(S''\backslash\{s_1\})\sqcup\{s_2\}$. Let ζ be a smooth path $s_1\to s_2$. Let $u'=\mathrm{id}_{S\backslash(S'\sqcup\{s_1\})}\boxtimes\zeta$. Write $g_{S'',\zeta}(\alpha)=v\wedge u$ with $u: S\backslash S'\to S\backslash S''$ and $v: S'\to\{\xi(1),\ldots,\xi(n)\}$. We take $u=0$ and $v=0$ if $g_{S'',\zeta}(\alpha)=0$. If $g_{S'',\zeta}(\alpha)\neq 0$, then $u=u'$.

Assume $(\beta\cdot u)\neq 0$. Then $\alpha_{s_1}\circ\zeta^{-1}$ and $\beta_{s_2}\circ\zeta$ are smooth, and ζ and $\bar{\zeta}$ have opposite orientations, since $\bar{\zeta}$ is negative (it starts in $\xi(\mathbf{Z}_{\geqslant 1})$ and ends in M). It follows that $\zeta\in L(\theta)$.
Assume $\zeta\in L(\theta)$. Since $\bar{\zeta}$ is negative, it follows that ζ is positive, then $(\theta^\zeta)_s=\theta_s$ for $s\notin\{s_1,s_2\}$, while $(\theta^\zeta)_{s_1}=\beta_{s_2}\circ\zeta$ and $(\theta^\zeta)_{s_2}=\alpha_{s_1}\circ\zeta^{-1}$. We deduce that $(\beta\cdot u)\boxtimes v=\theta^\zeta$ if $\beta\cdot u\neq 0$. So, the assertion (8.1.3) is a consequence of the following:

(8.1.4) given $\zeta\in L(\theta)$ positive, we have $\beta\cdot u\neq 0$ if and only if $\zeta\in D(\theta)$.

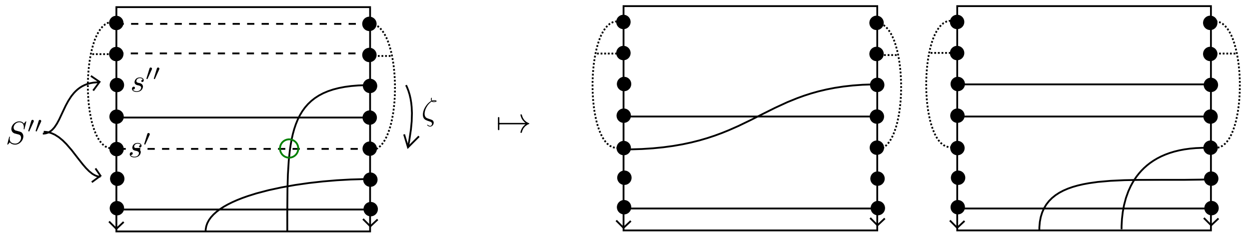
We will prove that statement by reduction to the non-singular case. Let $f:\hat{Z}\to Z$ be a non-singular cover. The morphism $\xi:\mathbf{R}_{>0}\to Z$ lifts uniquely to a morphism of curves $\hat{\xi}:\mathbf{R}_{>0}\to\hat{Z}$. Let $\hat{M}=f^{-1}(M)$ and let $\hat{\alpha}:\hat{S}''\to\{\hat{\xi}(1),\ldots\hat{\xi}(n)\}$ and $\hat{\zeta}$ be the unique lifts of α and ζ to $\hat{Z}$. There exists subsets $\hat{S},\hat{T}$ of $\hat{M}$ and a lift $\hat{\beta}:\hat{S}\backslash\hat{S}''\to\hat{T}$ of β such that $\hat{\zeta}\in L(\hat{\theta})$, where $\hat{\theta}=\hat{\alpha}\boxtimes\hat{\beta}$ (Lemma 7.4.28). We have $\zeta\in D(\theta)$ if and only if $\hat{\zeta}\in D(\hat{\theta})$ (Lemma 7.4.28).
Write $g_{\hat{S}'',\hat{\zeta}}(\hat{\alpha})=\hat{v}\wedge\hat{u}$ as above. We have $f(\hat{u})=u$ and $f(\hat{v})=v$. We have $g_{S'',\zeta}(\alpha)\neq 0$ if and only if $g_{\hat{S}'',\hat{\zeta}}(\hat{\alpha})\neq 0$. Finally, $\beta\cdot u\neq 0$ if and only if $\hat{\beta}\cdot\hat{u}\neq 0$. This completes the reduction of (8.1.4) to the case of $\hat{Z}$.

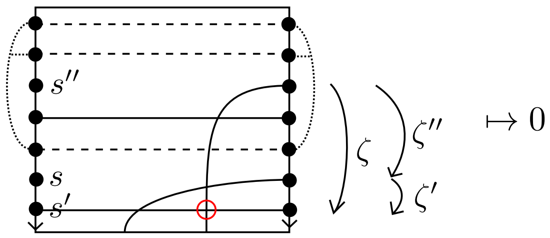
So, we now prove (8.1.4) assuming Z is smooth. Note that $Z(\xi)$ is isomorphic (as a 1-dimensional space) to an interval of $\mathbf{R}$. We consider $\zeta: s_1\to s_2$ in $L(\theta)$ positive with $s_1\in S''$ and $s_2\in S\backslash S''$.

Remark 7.4.11 shows that $\beta\cdot u'\neq 0$ if and only if $i(\beta_s,\beta_{s_2}\circ\zeta)=i(\beta_s,\beta_{s_2})+i(\mathrm{id}_s,\zeta)$ for all $s\in S\backslash(S'\sqcup\{s_1\})$. That equality is always satisfied unless there are $\zeta'': s_1\to s$ and $\zeta': s\to s_2$ positive. In that case, $\bar{\zeta}''$ is negative and the equality is satisfied if and only if $\bar{\zeta}'$ is positive.

We have $u\neq 0$ if and only if given $\zeta'': s_1\to s$ and $\zeta': s\to s_2$ positive with $s\in S'\backslash\{s_2\}$, then $\bar{\zeta}''=\alpha_{s_2}\circ\zeta''\circ\alpha_{s_1}^{-1}$ is positive.
We deduce that $\zeta\in D(\theta)$ if and only if $\beta\cdot u\neq 0$. The proposition follows. □

Example 8.1.11. The picture below gives two examples of description of the map $g_{S'',\zeta}$.





8.1.5. *Right action.* Consider now $\xi' : \mathbf{R}_{<0} \to Z$ an injective morphism of curves, where $\mathbf{R}_{<0}$ is unoriented. Identifying $(\mathbf{R}_{<0})^{\mathrm{opp}}$ with $\mathbf{R}_{>0}$ by $x \mapsto -x$, we obtain a morphism of curves $\xi : \mathbf{R}_{>0} \to Z^{\mathrm{opp}}$. Let M be a subset of $Z \backslash \xi'(\mathbf{R}_{\leqslant -1})$.

We say that ξ' is *initial* for (Z, M) if ξ is terminal for (Z^{opp}, M) and that ξ' is *incoming* for Z if $\xi'(\mathbf{R}_{\leqslant -1})$ is closed in Z.

Assume ξ' is initial for (Z, M). As in the left action case, we define a differential functor

$$\begin{aligned} R^\bullet = R^\bullet_{\xi'} : \mathcal{C} \times \mathcal{C}^{\mathrm{opp}} \times \mathcal{U} &\to k\text{-diff} \\ R^\bullet(S, T, e^n) &= \mathrm{Hom}(T \sqcup \{\xi'(-1), \ldots, \xi'(-n)\}, S) \\ R^\bullet(\beta, \alpha, \sigma)(f) &= \beta \cdot f \cdot (\alpha \boxtimes \xi'(\sigma^{\mathrm{revopp}})) \in R^\bullet(S', T', n) \end{aligned}$$

for $\alpha \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(T', T)$, $\beta \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S, S')$ and $\sigma \in \mathrm{End}_{\mathcal{U}^\bullet}(e^n)$, and $f \in R^\bullet(S, T, n)$.

We put $R^\bullet_\xi(S, T) = R^\bullet_\xi(S, T, e)$ and $R_\xi = \mathbf{F}_2[R^\bullet_\xi]$.

Recall that the isomorphism (7.4.4) of differential categories $\mathcal{S}^\bullet_M(Z) \xrightarrow{\sim} \mathcal{S}^\bullet_M(Z^{\mathrm{opp}})^{\mathrm{opp}}$. This isomorphism provides an isomorphism $R^\bullet_{\xi'}(S, T, e^n) \xrightarrow{\sim} L^\bullet_\xi(T, S, e^n)$ functorial in S, T and e^n.

In particular, $R^\bullet$ provides a "right" 2-representation on $\mathcal{S}^\bullet_M(Z)$ and all results of §8.1.1–8.1.4 have counterparts for $R^\bullet$.

Given $S \subset M$ and $n \geqslant 0$, there is an isomorphism of functors

$$\bigvee_{\substack{S' \subset S \\ |S'| = n}} \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(\{\xi'(-1), \ldots, \xi'(-n)\}, S') \wedge \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(-, S \backslash S') \xrightarrow{\sim} R^\bullet(S, -, e^n).$$

There is an isomorphism of functors, functorial in S and T

$$\begin{aligned} R^\bullet(T, -, e^n) \wedge R^\bullet(-, S, e^m) &\xrightarrow{\sim} R^\bullet(T, S, e^{n+m}) \\ (\alpha, \beta) &\mapsto \alpha \cdot (\beta \boxtimes \xi'([-m - r \to -r]_{1 \leqslant r \leqslant n})). \end{aligned}$$

Assume there is a decreasing sequence $m_0, m_{-1}, \ldots$ of points of $\xi'^{-1}(M)$ with $\lim_i m_i < t$ for all $t \in \xi'^{-1}(M)$.

We obtain as in (8.1.1) isomorphisms functorial in S and T

$$\operatorname{colim}_{r\to\infty} \operatorname{Hom}_{\mathcal{S}^\bullet_M(Z)}(T \sqcup \xi'(\{m_{-r}, \ldots, m_{-r-n+1}\}), S) \xrightarrow{\sim} R^\bullet(S, T, e^n). \tag{8.1.5}$$

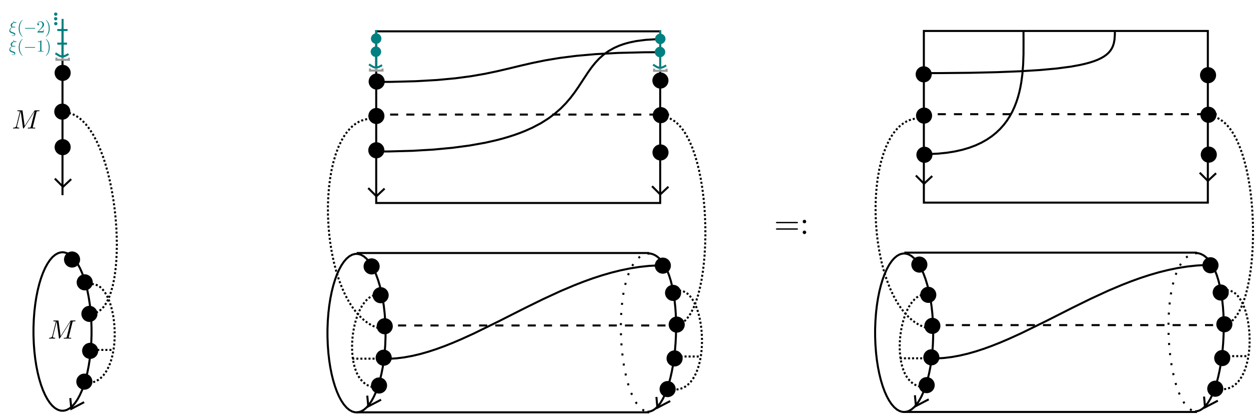
Let us finally consider functoriality as in §8.1.3. Let $f : Z \to Z'$ a morphism of curves and assume $f \circ \xi'$ is initial for $(Z', f(M))$.

The functor $f : \mathcal{S}^\bullet_{f,M}(Z) \to \mathcal{S}^\bullet_{f(M)}(Z')$ induces a morphism of bimodule 2-representations $R^\bullet_{f\circ\xi'} \to R^\bullet_{\xi'}$, when $|f^{-1}(f(z))| = 1$ for all $z \in M$.

If f is strict, then the functor $f^\# : \operatorname{add}(\mathcal{S}_{f(M)}(Z')) \to \operatorname{add}(\mathcal{S}_M(Z))$ induces a morphism of bimodule 2-representations $R_{\xi'} \to R_{f\circ\xi'}$.

Remark 8.1.12. As in Remark 8.1.4, we recover the construction of "top algebra module" of Douglas and Manolescu by taking the underlying lax 2-representation of $R_{\xi'}$.

Example 8.1.13. As in Example 8.1.5, we use an alternative graphical description for $R^\bullet_{\xi'}$. This is illustrated in the example of $R^\bullet_{\xi'}(-,-,e^2)$ below.



8.1.6. *Duality.* Let $Z' = \mathbf{R}$ be the smooth curve with $Z'_o = (-\frac{1}{2}, \frac{1}{2})$, with its standard orientation. Consider a morphism of curves $\tilde{\xi} : Z' \to Z$ such that $\tilde{\xi}(Z')$ is a component of Z.

Fix an increasing homeomorphism $\alpha : \mathbf{R}_{>0} \xrightarrow{\sim} \mathbf{R}_{>\frac{1}{2}}$ fixing the positive integers and define $\alpha' : \mathbf{R}_{<0} \xrightarrow{\sim} \mathbf{R}_{<-\frac{1}{2}}$ by $\alpha'(t) = -\alpha(-t)$. Let $\xi^+ = \tilde{\xi} \circ \alpha : \mathbf{R}_{>0} \to Z$ and $\xi^- = \tilde{\xi} \circ \alpha' : \mathbf{R}_{<0} \to Z$. These are injective morphisms of curves, ξ^+ is outgoing for Z and ξ^- is incoming for Z.

Given $n \geqslant 0$, we denote by $\theta(n) \in \operatorname{Hom}_{\mathcal{S}^\bullet(Z')}(\{-n, \ldots, -1\}, \{1, \ldots, n\})$ the braid given by $\theta(n)_{-i} = [-i \to i]$.

Let T and T' two finite subsets of Z and $I \subset \mathbf{Z}_{\geqslant 1}$ finite. Assume that $\tilde{\xi}(-I) \subset T$ and that given $x \in \mathbf{R}$ with $x < i$ for all $i \in -I$, we have $\tilde{\xi}(x) \notin T$. Assume also that $\tilde{\xi}(I) \subset T'$ and that given $x \in \mathbf{R}$ with $x > i$ for all $i \in I$, we have $\tilde{\xi}(x) \notin T'$.

We consider the pointed map

$$\kappa_I : \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(T,T') \to \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(T\backslash(T\cap\tilde{\xi}(-I)), T'\backslash(T'\cap\tilde{\xi}(I)))$$
$$\theta \mapsto \begin{cases}(\theta_t)_{t\in T\backslash\tilde{\xi}(-I)} & \text{if } \chi(\theta)(\tilde{\xi}(-i)) = \tilde{\xi}(i) \text{ for } i\in I\\ 0 & \text{otherwise.}\end{cases}$$

We put $\kappa_n = \kappa_{\{1,\ldots,n\}}$. Note that $\kappa_n = \kappa_{\{n\}}\circ\cdots\circ\kappa_{\{2\}}\circ\kappa_{\{1\}}$.

Let $f : Z \to \bar{Z}$ be a morphism of curves such that $f\circ\tilde{\xi}$ is a homeomorphism from Z' to a component of $\bar{Z}$. Put $\bar{\tilde{\xi}} = f\circ\tilde{\xi}$. Denote by $\bar{\kappa}_n$ the map defined as above for Z replaced by $\bar{Z}$.

Let T and T' be two finite subsets of Z such that $|f(T)| = |T|$ and $|f(T')| = |T'|$. Put $\mathring{T} = T\backslash(T\cap\tilde{\xi}(\{-n,\ldots,-1\}))$ and $\mathring{T}' = T'\backslash(T'\cap\tilde{\xi}(\{-n,\ldots,-1\}))$. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(T,T') \arrow[r, "\kappa_n"] \arrow[d, "f"'] & \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(\mathring{T},\mathring{T}') \arrow[d, "f"] \\
\mathrm{Hom}_{\mathcal{S}^\bullet(\bar{Z})}(f(T),f(T')) \arrow[r, "\bar{\kappa}_n"'] & \mathrm{Hom}_{\mathcal{S}^\bullet(\bar{Z})}(f(\mathring{T}),f(\mathring{T}'))
\end{tikzcd} \tag{8.1.6}$$

Similarly, if f is strict and U and U' are two finite subsets of $\bar{Z}$, there is a commutative diagram

(8.1.7)
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{S}(\bar{Z})}(U,U') \arrow[r, "\bar{\kappa}_n"] \arrow[d, "f^\#"'] & \mathrm{Hom}_{\mathcal{S}(\bar{Z})}(U\backslash(U\cap\bar{\tilde{\xi}}(\{-n,\ldots,-1\})), U'\backslash(U'\cap\bar{\tilde{\xi}}(\{1,\ldots,n\}))) \arrow[d, "f^\#"] \\
\bigoplus_{T,T'}\mathrm{Hom}_{\mathcal{S}(Z)}(T,T') \arrow[r, "\kappa_n"'] & \bigoplus_{T,T'}\mathrm{Hom}_{\mathcal{S}(Z)}(\mathring{T},\mathring{T}')
\end{tikzcd}$$

where T (resp. T') runs over finite subsets of Z such that $f(T) = U$ (resp. $f(U') = T'$).

Lemma 8.1.14. *The map κ_n commutes with differentials.*

Proof. Assume first $\tilde{\xi}$ is a homeomorphism and $Z_o = \varnothing$. Let T and T' be two finite subsets of $\mathbf{R}$ with same cardinality m. Let $a : \{1,\ldots,m\} \xrightarrow{\sim} T$ and $a' : \{1,\ldots,m\} \xrightarrow{\sim} T'$ be the increasing bijections. There is an isomorphism of differential modules (Proposition 7.4.33) $\phi : \mathrm{Hom}_{\mathcal{S}(Z)}(T,T') \xrightarrow{\sim} H_m$: given $\theta\in\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(T,T')$ non-zero and given $i\in\{1,\ldots,m\}$, we put $\phi(\theta)(i) = a'^{-1}(\theta_{a(i)}(1))$.

Assume in addition that $\{-n,\ldots,-1\}\subset T$ and $T\backslash\{-n,\ldots,-1\}\subset(-1,\infty)$ and $\{1,\ldots,n\}\subset T'$ and $T'\backslash\{1,\ldots,n\}\subset(-\infty,1)$. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{S}(Z)}(T,T') \arrow[r, "\kappa_n"] \arrow[d, "\phi"', "\sim"] & \mathrm{Hom}_{\mathcal{S}(Z)}(T\backslash\{-n,\ldots,-1\}, T'\backslash\{1,\ldots,n\}) \arrow[d, "\phi", "\sim"'] \\
H_m \arrow[r, "t^-_{m,m-n}"'] & H_{m-n}
\end{tikzcd}$$

The lemma follows now from §6.1.1.

Assume now Z is smooth. If $Z(\xi^+)$ is unoriented, then the lemma holds by the discussion above, using §7.4.10. In general, we consider the morphism of curves $f : Z \to \bar{Z}$ that is an isomorphism outside $Z(\xi^+)$ and the identity on $Z(\xi^+)$, with $f(Z(\xi^+))_o = \varnothing$. The vertical maps of the commutative diagram (8.1.6) are injective, hence the lemma holds for Z since it holds for $\bar{Z}$.

Consider now a general Z. Let $f : \hat{Z} \to Z$ be a non-singular cover. The vertical maps of the commutative diagram (8.1.7) are injective, hence the lemma holds for Z since it holds for $\hat{Z}$. $\square$

Let M be a subset of $Z \backslash \tilde{\xi}\big((-\infty,-1] \cup [1,\infty)\big)$.

Given S a finite subset of M, the pointed map

$$L^\bullet_{\xi^+}(T,S,e^n) \wedge R^\bullet_{\xi^-}(S,T',e^n) \to \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(T',T),\ (\theta',\theta) \mapsto \kappa_n(\theta'\cdot\theta)$$

induces an $\mathbf{F}_2$-linear map

$$\begin{aligned}\hat{\kappa}(T,S) : L_{\xi^+}(T,S,e^n) &\to \mathrm{Hom}_{\mathcal{S}(Z)^{\mathrm{opp}}\text{-diff}}(R_{\xi^-}(S,-,e^n),\mathrm{Hom}(-,T))\\ \theta' &\mapsto \big((\theta \in R^\bullet_{\xi^-}(S,T',e^n)) \mapsto \kappa_n(\theta'\cdot\theta)\big).\end{aligned}$$

Proposition 8.1.15. *The map $\hat{\kappa}$ induces an isomorphism of differential pointed bimodules $L_{\xi^+}(-_2,-_1,e^n) \xrightarrow{\sim} R_{\xi^-}(-_1,-_2,e^n)^\vee$.*

Proof. Lemma 8.1.14 shows that $\hat{\kappa}$ commutes with differentials.

Let S be a finite subset of M of cardinality n.

Assume $\tilde{\xi}$ is a homeomorphism and $Z_o = \varnothing$. There is a commutative diagram (see the proof of Lemma 8.1.14 with $(T,T') = (S,\xi^+(\{1,\ldots,n\})$ and $(T,T') = (\xi^-(\{-n,\ldots,-1\}),S))$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{S}(Z)}(S,\xi^+(\{1,\ldots,n\}) \arrow[r, "\hat{\kappa}"] \arrow[d, "\phi"', "\sim"] & \mathrm{Hom}_{\mathcal{S}(Z)}(\xi^-(\{-n,\ldots,-1\},S)^* \arrow[d, "\sim"', "(\phi^*)^{-1}"] \\
H_n \arrow[r, "\hat{t}^-_{S,\varnothing}"'] & H_n^*
\end{tikzcd}$$

The bottom horizontal map is bijective by Corollary 3.1.2, hence $\hat{\kappa}(\varnothing,S)$ is bijective.

Assume now $Z(\xi^+)$ is smooth unoriented. The map $\hat{\kappa}(\varnothing,S)$ is the same for Z and for $Z(\xi^+)$, so $\hat{\kappa}(\varnothing,S)$ is still bijective.

Assume $Z(\xi^+)$ is smooth. There is a morphism of curves $f : Z \to \bar{Z}$ that is an isomorphism outside $Z(\xi^+)$ and the identity on $Z(\xi^+)$ with $f(Z(\xi^+))_o = \varnothing$. The map $\hat{\kappa}(\varnothing,S)$ is the same for Z and for $\bar{Z}$, so $\hat{\kappa}(\varnothing,S)$ is still bijective.

Consider now a general Z and let $f : \hat{Z} \to Z$ be a non-singular cover. Let $\tilde{\hat{\xi}} : Z' \to \hat{Z}$ be the morphism of curves such that $\tilde{\xi} = f \circ \tilde{\hat{\xi}}$. The functors f and $f^\#$ are inverse bijections between $\mathrm{Hom}_{\mathcal{S}(Z)}(S,\xi^+(\{1,\ldots,n\})$ and $\bigoplus_{S'} \mathrm{Hom}_{\mathcal{S}(\hat{Z})}(S',\tilde{\hat{\xi}}(\{1,\ldots,n\})$ (resp. $\mathrm{Hom}_{\mathcal{S}(Z)}(\xi^-(\{-n,\ldots,-1\},S)$ and $\bigoplus_{S'} \mathrm{Hom}_{\mathcal{S}(\hat{Z})}(\tilde{\hat{\xi}}(\{-n,\ldots,-1\},S'))$, where S' runs over n-elements subsets of $\hat{Z}$ such that $f(S') = S$. Furthermore, $\hat{\kappa}(\varnothing,S)$ is compatible with these bijections (see the proof of Lemma 8.1.14). It follows that $\hat{\kappa}(\varnothing,S)$ is bijective.

We consider now two arbitrary subsets S and T of M. The canonical isomorphisms of Lemma 8.1.2 and of §8.1.5 fit in a commutative diagram of $\mathbf{F}_2$-modules

$$\begin{tikzcd}
\bigoplus_{S'} L_{\xi^+}(\varnothing, S', e^n) \otimes \mathrm{Hom}_{\mathcal{S}(Z)}(S\backslash S', T) \arrow[r, "\sim"] \arrow[d, "\sum_{S'} \hat{\kappa}(\varnothing, S') \otimes \mathrm{id}"'] & L_{\xi^+}(T, S, e^n) \arrow[d, "\hat{\kappa}(T,S)"] \\
\bigoplus_{S'} R_{\xi^-}(S', \varnothing, e^n)^* \otimes \mathrm{Hom}_{\mathcal{S}(Z)}(S\backslash S', T) \arrow[r, "\sim"'] & \mathrm{Hom}(R_{\xi^-}(S, -, e^n), \mathrm{Hom}(-, T))
\end{tikzcd}$$

where S' runs over n elements subsets of S. The discussion above shows that the left vertical arrow is an isomorphism, hence $\hat{\kappa}(T, S)$ is an isomorphism. $\square$

Given $x_1, x_2 \in [-1, 1]$, the homotopy class $\tilde{\xi}([x_1 \to x_2])$ is admissible if $x_1 \leqslant x_2$ or $x_1 \leqslant -\frac{1}{2}$ or $x_2 \geqslant \frac{1}{2}$. Given $x \in [-1, 1]$ and ζ an admissible class of paths in Z with $\zeta(1) = \tilde{\xi}(x)$ and $\tilde{\xi}([x \to 1]) \cdot \zeta \neq 0$, there is a unique $y \in [-1, 1]$ such that $\zeta = \tilde{\xi}([y \to x])$.

Let us describe now the unit of the adjunction when $n = 1$.

Lemma 8.1.16. *The unit of the adjunction* $(L_{\xi^+}(-,-) \otimes -, R_{\xi^-}(-,-) \otimes -)$ *is given by the morphism of bimodules whose evaluation at* (T, S) *is*

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{S}(Z)}(S, T) \to R_{\xi^-}(T, -) \otimes L_{\xi^+}(-, S) \\
&\qquad \gamma \mapsto \sum_{x \in \tilde{\xi}^{-1}(S)} (\gamma_{|S \backslash \{\tilde{\xi}(x)\}} \boxtimes \tilde{\xi}([-1 \to x])) \otimes (\mathrm{id}_{S \backslash \{\tilde{\xi}(x)\}} \boxtimes \tilde{\xi}([x \to 1])).
\end{aligned}$$

Proof. The counit of the adjunction is $\varepsilon = \kappa_1 \circ \mathrm{mult}$. Let $\gamma \in R^\bullet_{\xi^-}(T, S)$. Let η be the map defined in the lemma. We have

$$\eta(\mathrm{id}_T) = \sum_{x \in \tilde{\xi}^{-1}(T)} (\tilde{\xi}([-1 \to x]) \boxtimes \mathrm{id}_{T \backslash \{\tilde{\xi}(x)\}}) \otimes (\tilde{\xi}([x \to 1]) \boxtimes \mathrm{id}_{T \backslash \{\tilde{\xi}(x)\}}),$$

hence

$$(\mathrm{id} \otimes \varepsilon) \circ (\eta \otimes \mathrm{id})(\gamma) = \sum_{x \in \tilde{\xi}^{-1}(T)} (\tilde{\xi}([-1 \to x]) \boxtimes \mathrm{id}_{T \backslash \{\tilde{\xi}(x)\}}) \cdot \kappa_1\big((\tilde{\xi}([x \to 1]) \boxtimes \mathrm{id}_{T \backslash \{\tilde{\xi}(x)\}}) \cdot \gamma\big)$$

Let x be the unique element of $\tilde{\xi}^{-1}(\chi(\gamma)(\xi^-(-1)))$. We have $\gamma_{\xi^-(-1)} = \tilde{\xi}([-1 \to x])$ and $\kappa_1\big((\tilde{\xi}([x \to 1]) \boxtimes \mathrm{id}_{T \backslash \{\tilde{\xi}(x)\}}) \cdot \gamma\big) = \gamma_{|S}$, hence

$$(\mathrm{id} \otimes \varepsilon) \circ (\eta \otimes \mathrm{id})(\gamma) = (\gamma_{\xi^-(-1)} \boxtimes \mathrm{id}_{T \backslash \{\chi(\gamma)(\xi^-(-1))\}}) \otimes \gamma_{|S}$$

We deduce that

$$\mathrm{mult} \circ (\mathrm{id} \otimes \varepsilon) \circ (\eta \otimes \mathrm{id})(\gamma) = \gamma$$

and the lemma follows. $\square$

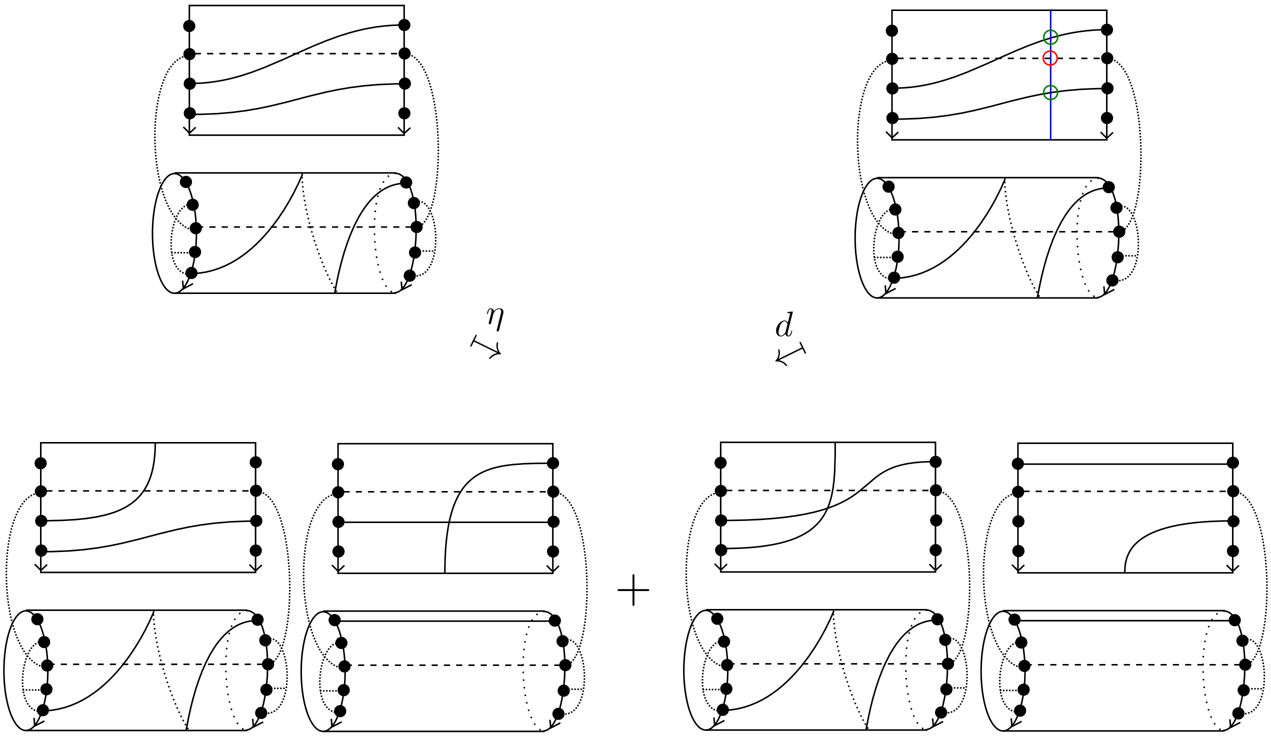
Remark 8.1.17. There is a bifunctorial injective map

$$R^\bullet_{\xi^-}(T, -) \wedge L^\bullet_{\xi^+}(-, S) \to \mathrm{Hom}(S \sqcup \{\xi^-(-1)\}, T \sqcup \{\xi^+(1)\}),\ \beta \wedge \alpha \mapsto (\beta \boxtimes \mathrm{id}_{\xi^+(1)}) \cdot (\alpha \boxtimes \mathrm{id}_{\xi^-(-1)}).$$

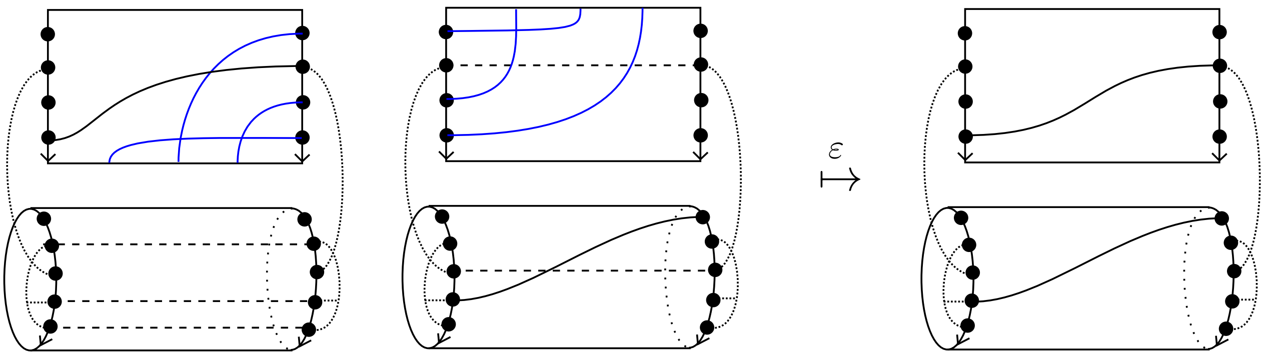
The composition of the unit given by Lemma 8.1.16 with this map is the following map

$$\mathrm{Hom}(S, T) \to \mathrm{Hom}(S \sqcup \{\xi^-(-1)\}, T \sqcup \{\xi^+(1)\}),\ \gamma \mapsto (\gamma \otimes \mathrm{id}_{\xi^+(1)}) \cdot d(\tilde{\xi}([-1 \to 1]) \boxtimes \mathrm{id}_S).$$

Example 8.1.18. The first picture below provides an example of description of the unit of adjunction as in Lemma 8.1.16.



The second picture describes a calculation of an image by the counit.



θ' θ $\kappa_3(\theta' \cdot \theta)$

8.1.7. *Actions for the line.* We consider the unoriented curve $\mathbf{R}$. Let $M = \{\pm(1-\frac{1}{n})\}_{n\in\mathbf{Z}_{>0}}$.

Consider S, T two finite subsets of $\mathbf{R}$ with $|S| = |T| = n$. Let $f_S : S \xrightarrow{\sim} \{1,\ldots,n\}$ and $f_T : T \xrightarrow{\sim} \{1,\ldots,n\}$ be the unique increasing bijections. We define

$$\phi(S,T) : \mathrm{Hom}_{\mathcal{S}^\bullet(\mathbf{R})}(S,T) \xrightarrow{\sim} H_n^\bullet = \mathrm{End}_{\mathcal{U}^\bullet}(e^n),\ \theta \mapsto T_{f_T\circ\chi(\theta)\circ f_S^{-1}}.$$

We define a functor $\Phi : \mathcal{S}^\bullet_M(\mathbf{R}) \to \mathcal{U}^\bullet$. We put $\Phi(S) = e^{|S|}$ and $\Phi(f) = \phi(S,T)(f)$ for $f \in \mathrm{Hom}_{\mathcal{S}^\bullet(\mathbf{R})}(S,T)$.

The next proposition follows from Proposition 7.4.33.

Proposition 8.1.19. *The functor* $\Phi : \mathcal{S}^\bullet_M(\mathbf{R}) \to \mathcal{U}^\bullet$ *is an equivalence of differential pointed categories.*

Consider $\xi_+ : \mathbf{R}_{>0} \to \mathbf{R}$ and $\xi_- : \mathbf{R}_{<0} \to \mathbf{R}$ the inclusion maps.

We define $\varphi_\pm : L_{\xi_\pm}(-,-,e^n) \xrightarrow{\sim} L^\pm(-,-,e^n) \circ (\Phi \wedge \Phi)$ by

$$\varphi_\pm(T,S) = \phi(S, T \sqcup \xi_\pm(\{\pm 1,\ldots,\pm n\})) : \mathrm{Hom}_{\mathcal{S}^\bullet(\mathbf{R})}(S, T \sqcup \xi_\pm(\{\pm 1,\ldots,\pm n\})) \xrightarrow{\sim} \delta_{|S|=n+|T|} L^\pm(|T|, n).$$

Similarly, we define $\varphi'_\pm : R_{\xi_\pm}(-,-,e^n) \xrightarrow{\sim} R^\pm(-,-,e^n) \circ (\Phi \wedge \Phi)$ by

$$\varphi'_\pm(T,S) = \phi(T \sqcup \xi_\pm(\{\pm 1,\ldots,\pm n\}), S) : \mathrm{Hom}_{\mathcal{S}^\bullet(\mathbf{R})}(T \sqcup \xi_\pm(\{\pm 1,\ldots,\pm n\}), S) \xrightarrow{\sim} \delta_{|S|=n+|T|} R^\pm(|T|, n).$$

Proposition 8.1.20. *Together with* $\varphi_\pm$ *(resp.* $\varphi'_\pm$*), the functor* Φ *induces equivalences of bimodule* 2*-representations between* $L_{\xi_\pm}$ *and* $L^\pm$ *(resp.* $R_{\xi_\pm}$ *and* $R^\pm$*).*

8.2. **Gluing.**

8.2.1. *Construction.* Consider two injective morphisms of curves $\xi_1^+ : \mathbf{R}_{>0} \to Z$ and $\xi_2^- : \mathbf{R}_{<0} \to Z$ where $\mathbf{R}_{<0}$ and $\mathbf{R}_{>0}$ are unoriented. We assume that ξ_1^+ is outgoing for Z, that ξ_2^- is incoming for Z and that $\xi_1^+(\mathbf{R}_{>0}) \cap \xi_2^-(\mathbf{R}_{<0}) = \varnothing$. We write r instead of $\xi_1^+(r)$ and $-r$ instead of $\xi_2^-(-r)$, for $r \in \mathbf{Z}_{>0}$.

Let M be a subset of $Z \backslash (\xi_1^+(\mathbf{R}_{\geqslant 1}) \sqcup \xi_2^-(\mathbf{R}_{\leqslant -1}))$.

Fix an oriented diffeomorphism $\mathbf{R}_{>0} \xrightarrow{\sim} \mathbf{R}_{<-1}$ and let $i_+ : \mathbf{R}_{>0} \to \mathbf{R}$ be its composition with the inclusion map. Similarly, fix an oriented diffeomorphism $\mathbf{R}_{<0} \xrightarrow{\sim} \mathbf{R}_{>1}$ and let $i_- : \mathbf{R}_{<0} \to \mathbf{R}$ be its composition with the inclusion map.

Consider $m,n \geqslant 0$. Let $E_{m,n}$ be the $(\mathcal{S}^\bullet_M(Z), \mathcal{S}^\bullet_M(Z))$-bimodule given by

$$E_{m,n}(T,S) = \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S \sqcup (-n,-1), T \sqcup (1,m)).$$

Note that $E_{0,1} = R^\bullet_{\xi_2^-}$ and $E_{1,0} = L^\bullet_{\xi_1^+}$, but $E_{m,n}$ is not isomorphic to $(R^\bullet_{\xi_2^-})^n (L^\bullet_{\xi_1^+})^m$ in general.

There is an action of $H^\bullet_m \wedge H^\bullet_n$ on $E_{m,n}$ given by

$$(T_a \wedge T_b) \cdot \sigma = (\mathrm{id}_T \boxtimes ([i \mapsto a(i)]_{1\leqslant i\leqslant m}) \cdot \sigma \cdot (\mathrm{id}_S \boxtimes (-i \mapsto b^{-1}(n+1-i)-n-1)_{1\leqslant i\leqslant n})$$

for $\sigma \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S \sqcup (-n,-1), T \sqcup (1,m))$, $a \in \mathfrak{S}_m$ and $b \in \mathfrak{S}_n$.

There is a map $* : E_{m,n}E_{m',n'} \to E_{m+m',n+n'}$ given by

$$\alpha \wedge \beta \mapsto \alpha * \beta = (\alpha \boxtimes ([i \to i+m])_{1\leqslant i\leqslant m'}) \cdot (\beta \boxtimes ([-n'-i \to -i])_{1\leqslant i\leqslant n}).$$

This map is compatible with the action of $(H^\bullet_m \wedge H^\bullet_n) \wedge (H^\bullet_{m'} \wedge H^\bullet_{n'})$ via the canonical embeddings $H^\bullet_m H^\bullet_{m'} \to H^\bullet_{m+m'}$ and $H^\bullet_n H^\bullet_{n'} \to H^\bullet_{n+n'}$. We have $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.

So, we have defined a bimodule lax bi-2-representation on $\mathcal{S}^\bullet_M(Z)$.

Let $Z_\xi = Z \sqcup_{\mathbf{R}_{>0} \sqcup \mathbf{R}_{<0}} \mathbf{R}$, where the gluing is done along the maps $\xi_1^+ \sqcup \xi_2^- : \mathbf{R}_{>0} \sqcup \mathbf{R}_{<0} \to Z$ and $i_+ \sqcup i_- : \mathbf{R}_{>0} \sqcup \mathbf{R}_{<0} \to \mathbf{R}$. Note that Z_ξ is a 1-dimensional space and it comes with an injective open morphism of 1-dimensional spaces $\xi : \mathbf{R} \to Z_\xi$. We endow $\mathbf{R}$ with a curve structure by setting $\mathbf{R}_u = \mathbf{R}_{\leqslant -1} \sqcup \mathbf{R}_{\geqslant 1}$ and by endowing $(-1,1)$ with its usual orientation. We extend the curve structure on Z by endowing $\xi(\mathbf{R})$ with the curve structure of $\mathbf{R}$. Note that $(Z_\xi)_u = \overline{Z_u}$.

Given $\varepsilon, \varepsilon' \in \{+,-\}$ and $a \in \mathbf{R}_\varepsilon$, $b \in \mathbf{R}_{\varepsilon'}$, we put $[a \to b] = \xi([i_\varepsilon(a), i_{\varepsilon'}(b)])$.

We consider the differential pointed category $T_{\mathcal{S}_M^\bullet(Z)}(R^\bullet_{\xi_2^-}L^\bullet_{\xi_1^+})$ with objects those of $\mathcal{S}_M^\bullet(Z)$ and with

$$\mathrm{Hom}_{\tilde{\mathcal{S}}_M^\bullet(Z)}(S,T)=\bigvee_{i\geqslant 0} R^\bullet_{\xi_2^-}(T,-_i)\wedge L^\bullet_{\xi_1^+}(-_i,-_{i-1})\wedge\cdots\wedge R^\bullet_{\xi_2^-}(-_2,-_1)\wedge L^\bullet_{\xi_1^+}(-_1,S).$$

We define a differential pointed functor $\tilde{\Xi}:T_{\mathcal{S}_M^\bullet(Z)}(R^\bullet_{\xi_2^-}L^\bullet_{\xi_1^+})\to\mathcal{S}_M^\bullet(Z_\xi)$. It is the identity on objects and defined on maps by

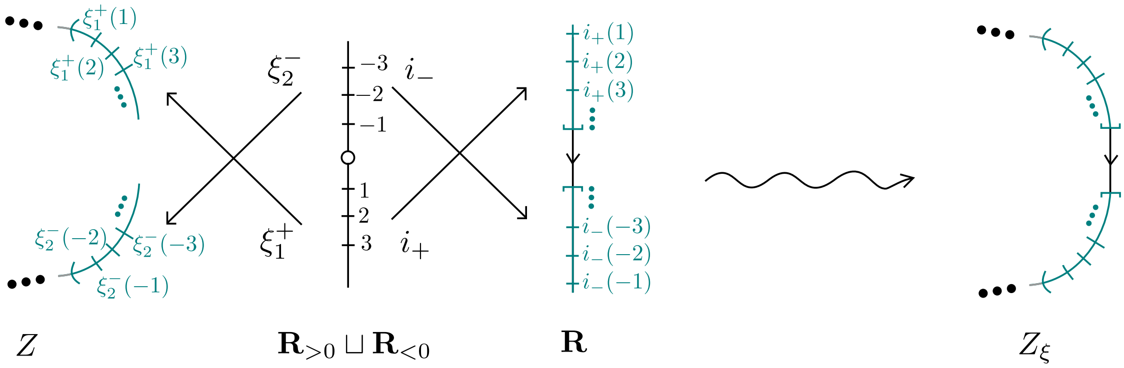
$$\beta_i\wedge\alpha_i\wedge\cdots\wedge\beta_1\wedge\alpha_1\mapsto(\beta_i\cdot(\mathrm{id}\boxtimes[1\to -1])\cdot\alpha_i)\cdot\cdots\cdot(\beta_1\cdot(\mathrm{id}\boxtimes[1\to -1])\cdot\alpha_1):$$

$$S\xrightarrow{\alpha_1}U_1\sqcup\{\xi_1^+(1)\}\xrightarrow{\mathrm{id}_{U_1}\boxtimes[1\to -1]}U_1\sqcup\{\xi_2^-(-1)\}\xrightarrow{\beta_1}V_1\xrightarrow{\alpha_2}\cdots\to T.$$

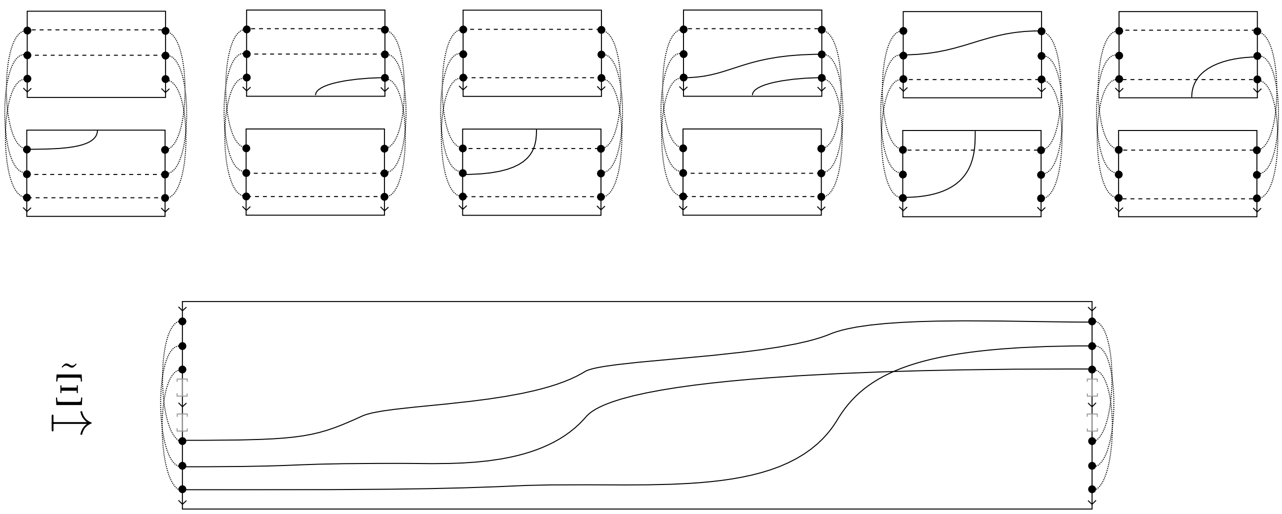
Theorem 8.2.1. *The functor $\tilde{\Xi}$ factors through $\Delta_E\mathcal{S}_M^\bullet(Z)$ and induces an isomorphism of differential pointed categories $\Xi:\Delta_E\mathcal{S}_M^\bullet(Z)\xrightarrow{\sim}\mathcal{S}_M^\bullet(Z_\xi)$.*

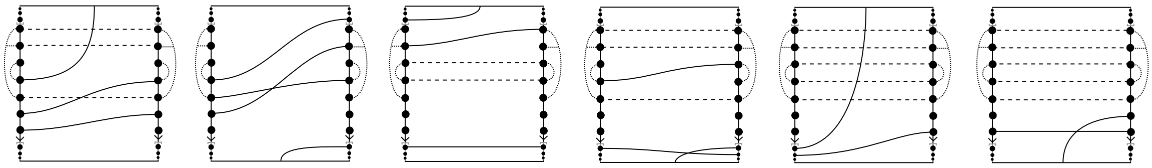
The sections §8.2.2-8.2.4 below are devoted to the proof of Theorem 8.2.1.

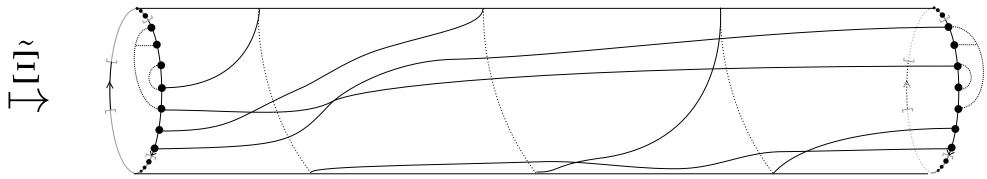
Example 8.2.2. We give below an illustration of the gluing data.



Example 8.2.3. The pictures below give two examples of description of $\tilde{\Xi}$. The first picture corresponds to the gluing of two intervals to form an interval. The second picture corresponds to the self-gluing of an interval to form a circle.







8.2.2. *Bimodules.* If $\mathrm{Hom}_{\mathcal{S}(Z_\xi)^\bullet}(\{-1\},\{1\}) \neq 0$, then there is $\kappa' \in \mathrm{Hom}_{\mathcal{S}(Z_\xi)^\bullet}(\{-1\},\{1\})$ such that $\mathrm{Hom}_{\mathcal{S}(Z_\xi)^\bullet}(\{-1\},\{1\}) = \{\kappa^n \cdot \kappa'\}_{n\geqslant 0}$, where $\kappa = \kappa' \cdot [1 \to -1]$.

When $\mathrm{Hom}_{\mathcal{S}(Z_\xi)^\bullet}(\{-1\},\{1\}) = 0$, we put $\kappa = \mathrm{id}_1$.

We define a partial order on the component Z' of Z containing 1. We define $s < s'$ if there exists an admissible path $\zeta : s' \to 1$ in Z' whose support does not contain s.

We consider the map μ of §7.4.6 for the curve Z_ξ and its point $z_0 = 0$.

Given $n \geqslant 0$, we put $G_n = E_{n,n}$.

Lemma 8.2.4. *Let $\alpha \in G_n - \{0\}$.*

Given $i \in (1, n-1)$, the following assertions are equivalent

(1) $\alpha(i-n-1) > \alpha(i-n)$
(2) $L(\alpha_{|\{i-n-1,i-n\}}) \neq \varnothing$
(3) $[i-n-1 \to i-n] \in D(\alpha)$
(4) $\alpha \in G_n T_i$
(5) $\alpha T_i = 0$.

There exists $i \in (1, n-1)$ such that $\alpha \in G_n T_i$ if and only if $L(\alpha_{|(-n,-1)}) \neq \varnothing$.

Proof. The equivalence between (1) and (2) follows from Lemma 7.4.20.

Assume (2). We deduce that $[i-n-1 \to i-n] \in L(\alpha)$, hence $[i-n-1 \to i-n] \in D(\alpha)$. So (3) holds.

Assume (3). Writing $\alpha = \alpha \cdot 1$, we deduce from Lemma 7.4.35 that (4) holds.

The implication (4)⇒(5) is immediate.

Asssume (5). We have $\alpha_{|\{i-n-1,i-n\}} \cdot ([i-n-1 \to i-n] \boxtimes [i-n \to i-n-1]) = 0$ by Remark 7.4.11. Lemma 7.4.9 shows that $i(\alpha_{|\{i-n-1,i-n\}}) \neq 0$, hence (2) holds.

Assume now $L(\alpha_{|(-n,-1)}) \neq \varnothing$. It follows from Lemma 7.4.20 that there is $i \in (1, n-1)$ with $\alpha(i-n-1) > \alpha(i-n)$, hence $\alpha \in G_n T_i$. This shows the last statement of the lemma. ☐

There is a map $\nu_n : R^\bullet_{\xi_2^-}(-,-,e^n) L^\bullet_{\xi_1^+}(-,-,e^n) \to G_n$ given by

$$\mathrm{Hom}(- \sqcup (-n,-1), T) \wedge \mathrm{Hom}(S, - \sqcup (1,n)) \to \mathrm{Hom}(S \sqcup (-n,-1), T \sqcup (1,n))$$

$$\beta \wedge \alpha \mapsto (\beta \boxtimes \mathrm{id}_{(1,n)}) \cdot (\alpha \boxtimes \mathrm{id}_{(-n,-1)})$$

We have

$$\nu_n\big((\beta \cdot T_b) \wedge (T_a \cdot \alpha)\big) = T_a \cdot \nu_n(\beta \wedge \alpha) \cdot \iota_n(T_b)$$

for $a, b \in \mathfrak{S}_n$.

The multiplication map on E defines a map $\mu_n : (R^\bullet_{\xi_2^-} L^\bullet_{\xi_1^+})^n = (E_{0,1}E_{1,0})^n \to E_{n,n} = G_n$, hence gives a morphism $T^*(R^\bullet_{\xi_2^-} L^\bullet_{\xi_1^+}) \to G = \bigvee_{n\geqslant 0} G_n$ compatible with multiplication.

We define $(\mathcal{S}^\bullet_M(Z), \mathcal{S}^\bullet_M(Z))$-subbimodules A_n, B_n, C_n, D_n, E_n and F_n of G_n. Let $\sigma \in G_n(T, S)$.

We have

- $\sigma \in A_n$ if $\sigma(-i) \in T \sqcup (1, n-i)$ for $1 \leqslant i \leqslant n$
- $\sigma \in B_n$ if there exists $1 \leqslant j \leqslant i \leqslant n$ with $\sigma(-i) = n - j + 1$
- $\sigma \in C_n$ if it is in the image of μ_n
- $\sigma \in D_n$ if $\sigma(-i) \in T$ for $1 \leqslant i \leqslant n$
- $\sigma \in E_n$ if $\sigma \in A_n$ and $L(\sigma_{|(-n,-1)}) = \varnothing$.
- $\sigma \in F_n$ if $\sigma \in A_n$ and $L(\sigma_{|\sigma^{-1}(1,n)}) = \varnothing$.

We put $A = \bigvee_{n\geqslant 0} A_n$, $B = \bigvee_{n\geqslant 0} B_n$, etc.
Note that $G_n = A_n \vee B_n$.
We have $C_n, D_n, E_n, F_n \subset A_n$.

Lemma 8.2.5. *We have an isomorphism* $\nu_n : R^\bullet_{\xi_2^-}(-, -, e^n) L^\bullet_{\xi_1^+}(-, -, e^n) \xrightarrow{\sim} D_n$.

In particular, we have an isomorphism $\mu_1 : R^\bullet_{\xi_2^-} \xrightarrow{\sim} D_1 = A_1 = C_1$ *and* $C_n = C_1^{*n} = A_1^{*n}$.

Proof. Let $\beta \wedge \alpha \in \mathrm{Hom}(- \sqcup (-n,-1), T) \wedge \mathrm{Hom}(S, - \sqcup (1,n))$. We have $\beta \wedge \alpha = \beta' \wedge \alpha'$ where $\beta' = \beta_{|(-n,-1)} \boxtimes \mathrm{id}$ and $\alpha' = (\beta_{|-} \boxtimes \mathrm{id}_{(1,n)}) \cdot \alpha$. If $\nu_n(\beta' \wedge \alpha') = 0$, then $\beta' = \alpha' = 0$ (cf beginning of §7.4.10). Now ν_n has an inverse given by $\sigma \mapsto (\mathrm{id} \boxtimes \sigma_{|(-n,-1)}) \wedge \sigma_{|S}$. ☐

Remark 8.2.6. Consider $\bar{\xi}_1^+ : \mathbf{R}_{>0} \to Z^{\mathrm{opp}},\ x \mapsto \xi_2^-(-x)$ and $\bar{\xi}_2^- : \mathbf{R}_{<0} \to Z^{\mathrm{opp}},\ x \mapsto \xi_1^+(-x)$. There is an isomorphism $(Z^{\mathrm{opp}})_{\bar{\xi}} \xrightarrow{\sim} (Z_\xi)^{\mathrm{opp}}$ that is the identity on Z and $x \mapsto -x$ on $\mathbf{R}$. This provides an isomorphism $(\mathcal{S}^\bullet(Z_\xi))^{\mathrm{opp}} \xrightarrow{\sim} \mathcal{S}^\bullet(Z^{\mathrm{opp}}_{\bar{\xi}})$. It induces isomorphisms

$$\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S \sqcup (-n,-1), T \sqcup (1,n)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{S}^\bullet(Z^{\mathrm{opp}})}(T \sqcup (-n,-1), S \sqcup (1,n)).$$

This restricts to isomorphisms between A_n (resp. B_n, D_n, E_n, F_n) for Z and A_n (resp. B_n, D_n, F_n, E_n) for Z^{opp}.

Lemma 8.2.7.
- *B_n and D_n are stable under the action of $H_n^\bullet \wedge (H_n^\bullet)^{\mathrm{opp}}$.*
- *E_n is stable under the action of $H_n^\bullet$ and F_n is stable under the action of $(H_n^\bullet)^{\mathrm{opp}}$.*
- *A and C are stable under multiplication*
- *Given $\alpha \in B$ and $\beta \in G$, we have $\alpha * \beta \in B$ and $\beta * \alpha \in B$.*

Proof. Let $\sigma \in B_n$ and $r \in \{1, \ldots, n-1\}$. Assume $\sigma T_r \neq 0$.

If there is $1 \leqslant j \leqslant i \leqslant n$ with $\sigma(-i) = n - j + 1$ and $i \neq n + 1 - r$, then $\sigma T_r \in B_n$.

Assume now $\sigma(-i) \in T \sqcup (1, n-i)$ for all $i \neq n+1-r$. We deduce that $L(\sigma_{|\{-(n+1-r),-(n-r)\}}) \neq \varnothing$, hence $\sigma T_r = 0$ (cf Lemma 8.2.4), a contradiction.

Using Remark 8.2.6, we deduce that $T_r\sigma \in B_n$.

The other assertions of the lemma are immediate. ☐

8.2.3. *Gluing map.* We define a morphism of $(\mathcal{S}^\bullet_M(Z),\mathcal{S}^\bullet_M(Z))$-bimodules $q : G \to \mathrm{Id}_{\mathcal{S}^\bullet_M(Z_\xi)}$:

$$\mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S \sqcup (-n,-1), T \sqcup (1,n)) \to \mathrm{Hom}_{\mathcal{S}^\bullet(Z_\xi)}(S,T).$$

Let $\alpha \in A_n$. We put $T_1 = \alpha(S) \cap T$ and $I_1 = \alpha(S) \cap (1,n)$. We define inductively $T_m \subset T$ and $I_m \subset (1, n-m+1)$ for $1 < m \leqslant n+1$ by $T_m = T_{m-1} \sqcup (\alpha(-n+I_{m-1}-1) \cap T)$ and $I_m = \alpha(-n+I_{m-1}-1) \cap (1,n)$.

Note that $-n+I_{m-1}-1 \subset (-n,-m+1)$, hence $I_m \subset (1,n-m+1)$ since $\alpha \in A_n$.

Note that $T_{n+1} = T$ and $I_{n+1} = \varnothing$.

Define

$$\beta^m = \mathrm{id}_{T_m} \boxtimes \Big(\boxed{\times}_{r\in I_m} (\alpha_{-n+r-1} \cdot [r \to -n+r-1]) \Big) : T_m \sqcup I_m \to T_{m+1} \sqcup I_{m+1}$$

for $1 \leqslant m \leqslant n$. We define $q(\alpha) = \beta^n \cdot \beta^{n-1} \cdots \beta^1 \cdot \alpha_{|S}$

$$q(\alpha) : S \xrightarrow{\alpha_{|S}} T_1 \sqcup I_1 \xrightarrow{\beta^1} T_2 \sqcup I_2 \to \cdots \to T_n \sqcup I_n \xrightarrow{\beta^n} T.$$

We put $q(\alpha) = 0$ if $\alpha \in B_n$.

Assume now $q(\alpha) \neq 0$, hence $\alpha \in A_n$. Let $S' = S \cap \alpha^{-1}(T)$ and $T' = T \cap \alpha(S)$. Let $S'' = S - S'$ and $T'' = T - T'$.

Given $s \in S'$, we have $q(\alpha)_s = \alpha_s$.

Note in particular that $S' = \{s \in S \mid \mu(q(\alpha)_s) = 0\}$.

Let $s \in S''$, $t = q(\alpha)(s)$ and $i = \alpha^{-1}(t)$. Put $d_s = \mu(q(\alpha)_s) - 1 \geqslant 0$. We have

$$q(\alpha)_s = \alpha_i \cdot [1 \to i] \cdot \kappa^{d_s} \cdot [\alpha(s) \to 1] \cdot \alpha_s.$$

Given a decomposition $q(\alpha)_s = \xi \cdot [1 \to -1] \cdot \kappa^{d_s} \cdot \xi'$ with $\xi' \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(\{s\},\{1\})$ and $\xi \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(\{-1\},\{t\})$, we have $\alpha_i = \xi \cdot [i \to -1]$ and $\alpha_s = [1 \to \alpha(s)] \cdot \xi'$.

The next lemma is immediate.

Lemma 8.2.8. *The map q defines a morphism of $(\mathcal{S}^\bullet_M(Z),\mathcal{S}^\bullet_M(Z))$-bimodules $G \to \mathrm{Id}_{\mathcal{S}^\bullet_M(Z_\xi)}$ and $q(\alpha * \alpha') = q(\alpha) \cdot q(\alpha')$.*

Given $h \in H^\bullet_n$ and $\alpha \in G_n$, we have $q(h\alpha) = q(\alpha h)$.

Lemma 8.2.9. *The restrictions of q to E and to F are injective.*

Proof. Let $\alpha : S \sqcup (-n,-1) \to T \sqcup (1,n)$ be a non-zero element of F_n. Let $s \in S''$. Given $1 \leqslant m \leqslant n$, we put $i_m(s) = \beta^{m-1} \circ \cdots \circ \beta^1 \circ \alpha(s)$. We put $d_s = \min\{m | i_{m+1}(s) \in T_{m+1}\}$.

Let s, s' be two distinct elements of S and let $\theta = \beta^n_{|\beta^{n-1}\circ\cdots\circ\beta^1\circ\alpha(\{s,s'\})} \cdots \beta^1_{|\alpha(\{s,s'\})} \cdot \alpha_{|\{s,s'\}}$.

- If $s, s' \in S'$, then $\theta = \alpha_{|\{s,s'\}} \neq 0$.
- Assume $s \in S'$ and $s' \in S''$. We have $\theta = \theta^1 \cdot \alpha_{|\{s,s'\}}$ where

$$\theta^1 = (\mathrm{id}_{\alpha(s)} \boxtimes (\alpha_{-n+i_{d_{s'}}(s')-1} \cdot [1 \to -n + i_{d_{s'}}(s') - 1] \cdot \kappa^{d_{s'}-1} \cdot [i_1(s') \to 1])).$$

We have

$$i(\theta^1 \circ \alpha_{|\{s,s'\}}) = i(\alpha_s, \alpha_{s'}) + i(\alpha_s, \alpha_{-n+i_{d_{s'}}(s')-1}) + d_{s'} - 1 = i(\alpha_{|\{s,s'\}}) + i(\theta^1).$$

Ir follows that $\theta \neq 0$.

- Assume finally $s, s' \in S''$ and $d_{s'} \geqslant d_s$. We have $\theta = \theta^1 \cdot \theta^2 \cdot \theta^3 \cdot \alpha_{|\{s,s'\}}$ where

$$\theta^1 = (\alpha_{-n+i_{d_{s'}}(s')-1} \cdot [1 \to -n + i_{d_{s'}}(s') - 1] \cdot \kappa^{d_{s'}-d_s} \cdot [i_{d_s+1}(s') \to 1]) \boxtimes \mathrm{id}_{in(s)}$$

$$\theta^2 = [-n + i_{d_s}(s') + 1 \to i_{d_s+1}(s')] \boxtimes (\alpha_{n-i_{d_s}(s)+1} \cdot [-n \to n - i_{d_s}(s) + 1])$$
$$\theta^3 = (([1 \to -n + i_{d_s}(s') + 1] \cdot \kappa^{d_s-1} \cdot [i_1(s') \to 1]) \boxtimes ([1 \to -n] \cdot \kappa^{d_s-1} \cdot [i_1(s) \to 1])).$$

We have

$$i(\theta^1 \circ \theta^2 \circ \theta^3 \circ \alpha_{|\{s,s'\}}) = i(\alpha_{i_{d_s}(s)}, \alpha_{i_{d_{s'}}(s')}) + d_{s'} - d_s + i(\alpha_s, \alpha_{s'}) = i(\theta^1) + i(\theta^2) + i(\theta^3) + i(\alpha_{|\{s,s'\}}).$$

It follows that $\theta \neq 0$.

It follows from Remark 7.4.11 that $q(\alpha) \neq 0$.

Define S' and S'' as above. Let $r = |S''|$. We have $\alpha(S'') = (n-r+1, n)$ and $\alpha^{-1}(i) < \alpha^{-1}(i')$ for $i < i'$ in $(n-r+1, n)$.

Given $i < i'$ in $(-n, -1)$ with $\alpha(i), \alpha(i') \in (1, n)$, we have $\alpha(i) < \alpha(i')$.

Consider now $\tilde{\alpha} : S \sqcup (-n, -1) \to T \sqcup (1, n)$ another non-zero element of F_n and assume $q(\alpha) = q(\tilde{\alpha}) \neq 0$. We have $S \cap \tilde{\alpha}^{-1}(T) = S'$ and $T \cap \tilde{\alpha}(S) = T'$. The discussion above shows that $\alpha(s) = \tilde{\alpha}(s)$ for $s \in S''$. Note also that $\alpha_s = \tilde{\alpha}_s$ for $s \in S'$. As a consequence, $\alpha = \tilde{\alpha}$ if $\mu(q(\alpha)) = 0$.

Let $s \in S''$, $t = q(\alpha)(s)$, $\tilde{t} = q(\tilde{\alpha})(s)$, $i = \alpha^{-1}(t)$ and $\tilde{i} = \tilde{\alpha}^{-1}(t)$. Since $q(\alpha)_s = q(\tilde{\alpha})_s$, it follows that $\tilde{t} = t$, $\mu(q(\alpha)_s) = \mu(q(\tilde{\alpha})_s)$, $[\alpha(s) \to 1] \cdot \alpha_s = [\tilde{\alpha}(s) \to 1] \cdot \tilde{\alpha}_s$, and $\alpha_i \cdot [-1 \to i] = \tilde{\alpha}_{\tilde{i}} \cdot [-1 \to \tilde{i}]$. We deduce that $\tilde{\alpha}_s = \alpha_s$ for $s \in S''$.

We proceed now by induction on $\mu(q(\alpha))$ to show that $q(\alpha)$ determines α, for $\alpha \in F$.

Assume there is $s \in S''$ such that $\mu(q(\alpha)_s) = 1$. Let $j = \alpha(s) \in (1, n)$ and $i = -n + j - 1$. We have $t = \alpha(i) = q(\alpha)(s) \in T$. Define $\alpha' : S\backslash\{s\} \sqcup (-n+1, -1) \to T\backslash\{t\} \sqcup (1, n-1)$ an element of F_{n-1} as follows. Given $s' \in S\backslash\{s\}$, we put $\alpha'_{s'} = \alpha_{s'}$ if $\alpha(s') < j$, $\alpha'_{s'} = [\alpha(s') \to \alpha(s') - 1] \cdot \alpha_{s'}$ if $\alpha(s') > j$. Given $i' \in (-i+1, -1)$, we put $\alpha'_{i'} = \alpha_{i'}$. Given $i' \in (-n+1, -i)$, we put $\alpha'_{i'} = \alpha_{i'-1}$. This defines an element of F_{n-1}. Furthermore, $q(\alpha') = q(\alpha)_{|S\backslash\{s\}}$.

We define similarly $\tilde{i}$, $\tilde{j}$, $\tilde{t}$ and $\tilde{\alpha}'$ starting with $\tilde{\alpha}$ and s. We have $\tilde{j} = j$ and $\tilde{t} = t$, hence also $\tilde{i} = i$. We have $q(\alpha') = q(\tilde{\alpha}')$, hence $\alpha' = \tilde{\alpha}'$ by induction. Since $\alpha_s = \tilde{\alpha}_s$ and $\alpha_i = \tilde{\alpha}_i$, it follows that $\alpha = \tilde{\alpha}$.

Assume $\mu(q(\alpha)_s) \geqslant 2$ for all $s \in S''$. We have $\alpha^{-1}((1, n-r)) = \{i_1 < \cdots < i_{n-r}\} \subset (-n, -1)$. Note that $\alpha_{-i_d} = [-i_d \to d]$ for $1 \leqslant d \leqslant n-r$. Let $\varphi : (-r, -1) \to (-n, -1)\backslash\alpha^{-1}((1, n-r))$ be the unique increasing bijection. We define $\alpha' : S \sqcup (-r, -1) \to T \sqcup (1, r)$ and element of F_r as follows. We put $\alpha'_s = \alpha_s$ for $s \in S'$, $\alpha'_s = [\alpha(s) \to \alpha(s) - n + r] \cdot \alpha_s$ for $s \in S''$ and $\alpha_i = \alpha_{\varphi(i)} \cdot [i \to \varphi(i)]$ for $i \in (-r, -1)$.

Let $s \in S''$, $t = q(\alpha)(s)$ and $i = \alpha^{-1}(t)$. We have

$$q(\alpha')_s = \alpha_i \cdot [1 \to i] \cdot [\alpha(s) \to 1] \cdot \alpha_s.$$

Define $\tilde{\alpha}'$ similarly, starting with $\tilde{\alpha}$ instead of α. We have $q(\alpha') = q(\tilde{\alpha}')$. By induction, we deduce that $\alpha' = \tilde{\alpha}'$, hence $\alpha = \tilde{\alpha}$.

This completes the proof that the restriction of q to F is injective.

We deduce that the restriction of q to E is injective using Remark 8.2.6 ☐

Lemma 8.2.10. *The restrictions of q to $E \cap C$ and to $F \cap C$ are surjective.*

Proof. Let $\theta \in \mathrm{Hom}_{\mathcal{S}^\bullet_M(Z_\xi)}(I, J)$. Let $n = \mu(\theta)$. We show by induction on n that there exists $\alpha \in F_n \cap C_n$ such that $q(\alpha) = \theta$.

Assume $n = 1$. Let $s \in I$ such that $\mu(\theta_s) = 1$. There is a decomposition $\theta_s = \theta_s^{r-} \cdot \theta_s^r$ as in §7.4.6. We define $\alpha \in \mathrm{Hom}_{\mathcal{S}_M^\bullet(Z)}(I \sqcup \{-1\}, J \sqcup \{1\})$ by $\alpha_{s'} = \theta_{s'}$ for $s' \neq s$, $\alpha_s = [0 \to 1] \cdot \theta_s^r$ and $\alpha_{-1} = \theta_s^{r-} \cdot [-1 \to 0]$. We have $\alpha \in A_1 = F_1 \cap C_1$ and $q(\alpha) = \theta$.

Assume now $n > 1$. Consider a decomposition $\theta = r'(\theta) \cdot r(\theta)$ as in Lemma 7.4.27. There exists $\alpha \in A_1$ and $\beta \in F_{n-1} \cap C_{n-1}$ such that $q(\alpha) = r(\theta)$ and $q(\beta) = r'(\theta)$. Let $\gamma = \beta * \alpha \in C_n$. We have $q(\gamma) = \theta$.

Let $s = \gamma^{-1}(n) = \alpha^{-1}(1)$. We have $\mu(r(\theta)_s) = 1$. Let $i \in (1, n-1)$ and $s' = \gamma^{-1}(i)$. If $s' \in (-n, -1)$, then $I(\gamma_{|\{s',s\}}) = \varnothing$. Assume $s' \notin (-n, -1)$. We have $\theta_{s'}^r = [i \to 0] \cdot \gamma_{s'}$. Since $\mathrm{supp}(\theta_s^r) \subset \mathrm{supp}(\theta_{s'}^r)$, it follows that $I(\gamma_{|\{s',s\}}) = \varnothing$. Since $\beta \in F_{n-1}$, we deduce that $\gamma \in F_n$.

The case of $E \cap C$ follows from that of $F \cap C$ applied to Z^{opp}, cf Remark 8.2.6. □

8.2.4. *Equivalence relation.* We define an equivalence relation $\sim$ on G as the transitive, symmetric and reflexive closure of the relation $T_i\sigma \sim \sigma T_i$ for $\sigma \in G_n$ and $1 \leqslant i < n$ and $\sigma \sim 0$ if $\sigma \in B_n$.

Lemma 8.2.11. *Let $\alpha \in G_n$. There exists $\sigma \in E_n$ and $\sigma' \in F_n$ such that $\alpha \sim \sigma \sim \sigma'$.*

Proof. If $\alpha \in B_n$, then $\alpha \sim 0$ and we are done. Assume now $\alpha \in A_n$. We proceed by induction on $M(\alpha) = \frac{1}{2}|L(\alpha_{|(-n,-1)})|$ and then on $N(\alpha) = n - \max\{i \mid [-n+i-1 \to -n+i] \in L(\alpha)\}$ if $M(\alpha) \neq 0$ to show that there exists $\sigma \in E_n$ with $\alpha \sim \sigma$.

If $M(\alpha) = 0$, then $\alpha \in E_n$ and we are done. Assume now $M(\alpha) > 0$. By Lemma 8.2.4, there are $i \in (1, n-1)$ and $\beta \in G_n$ such that $\alpha = \beta T_i$, and we choose i maximal with this property, so that $N(\alpha) = n - i$. We have $\alpha \sim T_i\beta$. If $T_i\beta \in B_n$ then we are done. We assume now $T_i\beta \notin B_n$. We have $L(\beta_{|(-n,-1)}) = L(\alpha_{|(-n,-1)}) \backslash \{[-n+i-1 \to -n+i], [-n+i \to -n+i-1]\}$.

If $\beta^{-1}(\{i, i+1\}) \not\subset (-n, -1)$, then $L(T_i\beta_{|(-n,-1)}) = L(\beta_{|(-n,-1)})$, hence $M(T_i\beta) < M(\alpha)$. By induction, there is $\sigma \in E_n$ with $T_i\beta \sim \sigma$, hence $\alpha \sim \sigma$.

Assume now there are $j, k \in (1, n)$ with $\beta(-n+j-1) = i$ and $\beta(-n+k-1) = i+1$. Since $T_i\beta \neq 0$, we have $j < k$. Since $\beta \in A_n$, we have $j > i$ and $k > i+1$. We have $M(T_i\beta) \leqslant M(\beta) + 1 = M(\alpha)$. On the other hand, $[j \to k] \in L(T_i\beta)$ (cf Lemma 7.4.20), hence $N(T_i\beta) < N(\alpha)$. We conclude by induction.

The case of F_n follows by applying Remark 8.2.6. □

Lemma 8.2.12. *Let $\alpha, \beta \in G_n$. We have $q(\alpha) = q(\beta)$ if and only if $\alpha \sim \beta$.*

Proof. Lemma 8.2.8 shows that if $\alpha \sim \beta$, then $q(\alpha) = q(\beta)$. Assume now $q(\alpha) = q(\beta)$. There are $\alpha', \beta' \in E_n$ with $\alpha' \sim \alpha$ and $\beta' \sim \beta$ (Lemma 8.2.11) and we have $q(\alpha') = q(\alpha) = q(\beta) = q(\beta')$. It follows now from Lemma 8.2.9 that $\alpha' = \beta'$, hence $\alpha \sim \beta$. □

Proof of Theorem 8.2.1. The canonical surjective map $T^*(R^\bullet_{\xi_2^-} L^\bullet_{\xi_1^+}) \to \mathrm{Id}_{\Delta_E(\mathcal{S}_M^\bullet(Z))}$ factors through a surjective map $C \to \mathrm{Id}_{\Delta_E(\mathcal{S}_M^\bullet(Z))}$. Its equalizer is given by the equivalence relation $\sim$, so it induces an isomorphism $C/\sim \xrightarrow{\sim} \mathrm{Id}_{\Delta_E(\mathcal{S}_M^\bullet(Z))}$. On the other hand, the canonical map $T^*(R^\bullet_{\xi_2^-} L^\bullet_{\xi_1^+}) \to \mathrm{Id}_{\mathcal{S}_M^\bullet(Z_\xi)}$ factors through the map q, and the restriction of q to C is surjective (Lemma 8.2.10). Lemma 8.2.12 completes the proof of the theorem. □

8.2.5. *Complement.* We provide here a more direct description of the equivalence relation $\sim$ on C.

Corollary 8.2.13. *We have $E \subset C$ and $F \subset C$.*

We define an equivalence relation $\sim'$ on C as the relation generated by $\alpha' * (T_1\alpha) * \alpha'' \sim' \alpha' * (\alpha T_1) * \alpha''$ for $\alpha', \alpha'' \in C$ and $\alpha \in D_2$.

Lemma 8.2.14. *Let $\sigma \in G_n$ and $i \in \{1, \ldots, n-1\}$*
If $\sigma T_i \in C_n \backslash \{0\}$, then $T_i\sigma \in C_n$ and $\sigma T_i \sim' T_i\sigma$.
If $T_i\sigma \in C_n \backslash \{0\}$, then $\sigma T_i \in C_n$ and $\sigma T_i \sim' T_i\sigma$.

Proof. Put $\sigma' = \sigma T_i$ and assume $\sigma' \in C_n \backslash \{0\}$. There are $\gamma \in C_{n-i-1}$, $\beta \in C_2$ and $\alpha \in C_{i-1}$ such that $\sigma' = \alpha * \beta * \gamma$.

Lemma 8.2.4 shows that $[-n+i-1 \to -n+i] \in D(\sigma')$. We have $\sigma'_{|\{-n+i-1,-n+i\}} = (\alpha * \beta)_{|(-i-1,-i)} \circ ([-n+i-1 \to -i-1] \boxtimes [-n+i \to -i])$. It follows from Lemma 8.2.4 that $[-i-1 \to -i] \in D(\alpha * \beta)$. Since $[-i-1 \to -i] \in L((\alpha * \beta)_{|(-i-1,-i)})$, it follows that $\beta(-1) \neq 1$, hence $\beta \in D_2$.

• Assume $[-1 \to -2] \in D(\beta)$. We have $\beta = \beta' T_1$ for some $\beta' \in G_2$ by Lemma 8.2.4. Since $\beta \in D_2$, we have $\beta' \in D_2 \subset A_2$. We deduce that $\beta' \in E_2$, hence $T_1\beta' \in E_2 \subset C_2$ (Corollary 8.2.13). So, $\sigma T_i = \alpha * (\beta' T_1) * \gamma \sim' \alpha * (T_1\beta') * \gamma = T_i\sigma$.

• Assume now $[-1 \to -2] \notin D(\beta)$, i.e., $\beta \in E_2$. We have $T_1\beta, \beta T_1 \in D_2 \subset A_2$ and $T_1\beta \subset E_2 \subset C_2$ (Corollary 8.2.13).

⋄ Assume $T_1\beta = 0$. There is $\beta'' \in G_2$ such that $\beta = T_1\beta''$ (Lemma 8.2.4). Since $\beta \in E_2 \cap D_2$, we have $\beta'' \in E_2 \cap D_2 \subset C_2$, hence also $\beta'' \in F_2$. As a consequence, $\beta'' T_1 \in F_2 \subset C_2$. We deduce that $\alpha * \beta \sim' \alpha * (\beta'' T_1)$. We have $L(\alpha_{|\beta((-2,-1))}) \neq \varnothing$ and $L((\beta'' T_1)_{|(-2,-1)}) \neq \varnothing$, hence $(\alpha * (\beta'' T_1))_{|(-2,-1)} = 0$ and $\alpha * (\beta'' T_1) = 0$. We have $\sigma T_i = \alpha * (T_1\beta'') * \gamma \sim' \alpha * (\beta'' T_1) * \gamma = 0$. Since $T_i\sigma T_i = 0$ and $\sigma T_i \neq 0$, it follows that $L((\sigma T_i)_{|(\sigma T_i)^{-1}(\{i,i+1\})}) \neq \varnothing$, by applying Lemma 8.2.4 to Z^{opp}. Since $\sigma T_i \in A_n$, we deduce that $L(\sigma_{|\sigma^{-1}(\{i,i+1\})}) \neq \varnothing$, hence $T_i\sigma = 0 \sim' \sigma T_i$ (using Lemma 8.2.4 for Z^{opp} again).

⋄ Assume now $T_1\beta \neq 0$. It follows that $\beta \in F_2$, hence $\beta T_1 \in F_2 \subset C_2$.

There are $\alpha^1, \ldots, \alpha^{i-1} \in C_1$ with $\alpha = \alpha^{i-1} * \cdots * \alpha^1$. Let $s_i = \beta(-i)$ for $i \in \{1, 2\}$. Consider $j \geqslant 1$ minimum such that $L((\alpha^j * \cdots * \alpha^1)_{|\{s_1,s_2\}}) \neq \varnothing$.

Define $u' = \alpha^j \boxtimes ([l \to l+1])_{1 \leqslant l \leqslant j+1}$ and $u'' = (\alpha^{j-1} * \cdots \alpha^1 * \beta) \boxtimes [-j-2 \to -1]$.

Let $\zeta = u''_{-2} \circ [-1 \to -2] \circ (u''_{-1})^{-1}$. Define I and J to be the domain and codomain of u'', intersected with M. Note that $\zeta(0), \zeta(1) \in M$. Let $v' = (u')^\zeta = (\alpha^j)^\zeta \boxtimes ([l \to l+1])_{1 \leqslant l \leqslant j+1}$ and define $v'' : I \sqcup (-j-2,-1) \to J \sqcup (1,2) \sqcup \{-1\} \sqcup (1, j+1)$ by

$$v''_s = \begin{cases} u''_{-2} \circ [-1 \to -2] & \text{if } s = -1 \\ u''_{-1} \circ [-2 \to -1] & \text{if } s = -2 \\ u''_s & \text{otherwise.} \end{cases}$$

Lemma 7.4.35 shows that v' and v'' are braids and $\alpha^j * \cdots * \alpha^1 * \beta = u' \cdot u'' = v' \cdot v''$. We have $v'' = (\alpha^{j-1} * \cdots \alpha^1 * (\beta T_1)) \boxtimes [-j-2 \mapsto -1]$ and we deduce that $\alpha * \beta = \alpha' * (\beta T_1)$, where $\alpha' = \alpha^{i-1} * \cdots * \alpha^{j+1} * (\alpha^j)^\zeta * \alpha^{j-1} \cdots * \alpha^1 \in C_{i-1}$. We have $\sigma T_i = \alpha' * (\beta T_1) * \gamma \sim' \alpha' * (T_1\beta) * \gamma = T_i\sigma$. This completes the proof of the first statement of the lemma.

The second statement of the lemma follows from the first one applied to Z^{opp} thanks to Remark 8.2.6. □

Proposition 8.2.15. *Let $\alpha, \beta \in C_n$. We have $\alpha \sim' \beta$ if and only if $\alpha \sim \beta$.*

Proof. It is clear that $\alpha \sim' \beta$ implies $\alpha \sim \beta$. The converse follows from Lemma 8.2.14. □

Corollary 8.2.16. *We have* $C/\sim' = G/\sim$.

Proof. The surjectivity of $C/\sim' \to G/\sim$ is given by Lemma 8.2.10. The injectivity follows from Lemmas 8.2.12 and 8.2.15. ☐

8.3. **Diagonal Action.**

8.3.1. *Isomorphism Theorem.* Let $Z' = \mathbf{R}$ be the smooth curve with $Z'_o = (-\frac{1}{2}, \frac{1}{2})$ with its standard orientation. Fix an increasing homeomorphism $\alpha : \mathbf{R}_{>0} \xrightarrow{\sim} \mathbf{R}_{>\frac{1}{2}}$ fixing the positive integers and define $\alpha' : \mathbf{R}_{<0} \xrightarrow{\sim} \mathbf{R}_{<-\frac{1}{2}}$ by $\alpha'(t) = -\alpha(-t)$.

Assume $Z(\xi_1^+) \neq Z(\xi_2^-)$ and assume there is a morphism $\tilde{\xi}_1 : Z' \to Z$ with image $Z(\xi_1^+)$ and such that $\xi_1^+ = \tilde{\xi}_1 \circ \alpha$. Put $\xi_1^- = \tilde{\xi}_1 \circ \alpha' : \mathbf{R}_{<0} \to Z$ and denote by ξ^- the composition $\mathbf{R}_{<0} \xrightarrow{\xi_1^-} Z \hookrightarrow Z_{\xi_1}$.

Proposition 8.1.15 gives an isomorphism of differential pointed bimodules $\hat{\kappa}_1 : L_{\xi_1^+}(-_2, -_1) \xrightarrow{\sim} R_{\xi_1^-}(-_1, -_2)^\vee$.

Since there is no admissible path from $\xi_2^-(-1)$ to $\xi_1^+(1)$ in Z, we have $A_n = D_n = G_n$ (with the notations of §8.2.2), hence we have an isomorphism (Lemma 8.2.5)

$$\nu_n : R^\bullet_{\xi_2^-}(T, -, e^n) \wedge L^\bullet_{\xi_1^+}(-, S, e^n) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{S}^\bullet(Z)}(S \sqcup \{\xi_2^-(-n), \ldots, \xi_2^-(-1)\}, T \sqcup \{\xi_1^+(1), \ldots, \xi_1^+(n)\}).$$

Consider

$$\begin{aligned} \lambda : L^\bullet_{\xi_1^+}(T, -) R^\bullet_{\xi_2^-}(-, S) &\to R^\bullet_{\xi_2^-}(T, -) L^\bullet_{\xi_1^+}(-, S) \\ \alpha \wedge \beta &\mapsto \nu_1^{-1}(\alpha \cdot \beta) = ((\alpha \cdot \beta)_{\xi_2^-(-1)} \boxtimes \mathrm{id}_{T \setminus \{\chi(\alpha \circ \beta)(\xi_2^-(-1))\}}) \wedge (\alpha \cdot \beta)_{|S}. \end{aligned}$$

There is an isomorphism of differential pointed categories (cf Remark 5.4.1)

$$\Delta_E \mathcal{S}^\bullet_M(Z) \xrightarrow{\sim} \Delta_\lambda \mathcal{S}^\bullet_M(Z).$$

Composing its inverse with Ξ, we deduce from Theorem 8.2.1 an isomorphism of differential pointed categories

$$\Xi' : \Delta_\lambda \mathcal{S}^\bullet_M(Z) \xrightarrow{\sim} \mathcal{S}^\bullet_M(Z_\xi).$$

Theorem 8.3.1. *The isomorphism* Ξ' *provides an isomorphism of* 2*-representations, where* $\Delta_\lambda \mathcal{S}^\bullet_M(Z)$ *is equiped with the diagonal action and* $\mathcal{S}^\bullet_M(Z_\xi)$ *with the action of* R_{ξ^-}.

The remainder of §8.3 is devoted to the proof of Theorem 8.3.1.

8.3.2. *Setting.* Let $\sigma : R_{\xi_2^-}(T, -) \otimes R_{\xi_1^-}(-, S) \to R_{\xi_1^-}(T, -) \otimes R_{\xi_2^-}(-, S)$ be defined as in (4.4.1).

Lemma 8.3.2. *The morphism* σ *is invertible. Given* $\alpha \in R^\bullet_{\xi_2^-}(T, U)$ *and* $\beta \in R^\bullet_{\xi_1^-}(U, S)$*, we have*

$$\sigma(\alpha \otimes \beta) = \delta_{\alpha_{|U} \cdot \beta \neq 0} \big(\mathrm{id} \boxtimes (\alpha_{\chi(\beta)(\xi_1^-(-1))} \cdot \beta_{\xi_1^-(-1)})\big) \otimes \big(\alpha_{\xi_2^-(-1)} \boxtimes (\alpha_{|U \setminus \{\chi(\beta)(\xi_1^-(-1))\}} \cdot \beta_{|S})\big).$$

Given $\alpha' \in R^\bullet_{\xi_1^-}(T, U')$ *and* $\beta' \in R^\bullet_{\xi_2^-}(U', S)$*, we have*

$$\sigma^{-1}(\alpha' \otimes \beta') = \delta_{\alpha'_{|U'} \cdot \beta' \neq 0} \big(\mathrm{id} \boxtimes (\alpha'_{\chi(\beta')(\xi_2^-(-1))} \cdot \beta'_{\xi_2^-(-1)})\big) \otimes \big(\alpha'_{\xi_1^-(-1)} \boxtimes (\alpha'_{|U' \setminus \chi(\beta')(\xi_2^-(-1))} \circ \beta'_{|S})\big).$$

Proof. We have

$$\sigma = (R_{\xi_1^-} \circ \mathrm{mult}) \circ (R_{\xi_1^-} \otimes R_{\xi_2^-} \otimes \varepsilon_{L_{\xi_1^+}, R_{\xi_1^-}}) \circ (R_{\xi_1^-} \otimes \lambda \otimes R_{\xi_1^-}) \circ (\eta_{L_{\xi_1^+}, R_{\xi_1^-}} \otimes \mathrm{id}).$$

We have $\alpha = (\mathrm{id} \boxtimes \alpha_{\xi_2^-(-1)}) \cdot (\alpha_{|U} \boxtimes \mathrm{id})$, hence $\alpha \otimes \beta = (\mathrm{id} \boxtimes \alpha_{\xi_2^-(-1)}) \otimes (\alpha_{|U} \cdot \beta)$. As a consequence, it is enough to prove the first statement of the lemma assuming that $\alpha_{|U} = \mathrm{id}_U$. In that case, the composition above is given by

$$\begin{aligned}
\alpha \otimes \beta &\mapsto \sum_{x \in \tilde{\xi}_1^{-1}(T)} (\mathrm{id}_{T \setminus \{\tilde{\xi}_1(x)\}} \boxtimes \tilde{\xi}_1([-1 \to x])) \otimes (\mathrm{id}_{T \setminus \{\tilde{\xi}_1(x)\}} \boxtimes \tilde{\xi}_1([x \to 1])) \otimes \alpha \otimes \beta \\
&\mapsto \sum_{x \in \tilde{\xi}_1^{-1}(T)} (\mathrm{id}_{T \setminus \{\tilde{\xi}_1(x)\}} \boxtimes \tilde{\xi}_1([-1 \to x])) \otimes (\alpha_{\xi_2^-(-1)} \boxtimes \mathrm{id}) \otimes (\mathrm{id} \boxtimes \tilde{\xi}_1([x \to 1])) \otimes \beta \\
&\mapsto (\mathrm{id} \boxtimes \beta_{\xi_1^-(-1)}) \otimes (\alpha_{\xi_2^-(-1)} \boxtimes \mathrm{id}) \otimes \beta_{|S} \\
&\mapsto (\mathrm{id} \boxtimes \beta_{\xi_1^-(-1)}) \otimes (\alpha_{\xi_2^-(-1)} \boxtimes \beta_{|S}).
\end{aligned}$$

It is immediate to check that the formula for σ^{-1} does produce an inverse. $\square$

Consider the map $\rho : L_{\xi_1^+}(T, -) \otimes R_{\xi_1^-}(-, S) \to R_{\xi_1^-}(T, -) \otimes L_{\xi_1}^-(-, S)$ defined in §4.4.2.

Lemma 8.3.3. *Given $\alpha \in L^\bullet_{\xi_1^+}(T, U)$ and $\beta \in R^\bullet_{\xi_1^-}(U, S)$, we have*

$$\rho(\alpha \otimes \beta) = \delta_1 \big(\alpha_{|U \setminus \chi(\alpha)^{-1}(\xi_1^+(1))} \cdot (\beta_{\xi_1^-(-1)} \boxtimes \mathrm{id})\big) \otimes \big((\alpha_{\chi(\alpha)^{-1}(\xi_1^+(1))} \boxtimes \mathrm{id}) \cdot \beta_{|S}\big)$$

where $\delta_1 = 1$ if $\chi(\alpha \circ \beta)(\xi_1^-(-1)) \neq \xi_1^+(1)$ and $(\mathrm{id}_{\chi(\beta)(\xi_1^-(-1))} \boxtimes \alpha_{\chi(\alpha)^{-1}(\xi_1^+(1))}) \cdot (\beta_{\xi_1^-(-1)} \boxtimes \mathrm{id}_{\chi(\alpha)^{-1}(\xi_1^+(1))}) \neq 0$ and $\delta_1 = 0$ otherwise.

Proof. Assume first $\alpha_{|U \setminus \{\chi(\alpha)^{-1}(\xi_1^+(1))\}} = \mathrm{id}$ and $\beta_{|S} = \mathrm{id}$. We have

$$\begin{aligned}
\rho(\alpha \otimes \beta) &= \varepsilon_1 R_{\xi_1^-} L_{\xi_1^+} \circ L_{\xi_1^+} \tau L_{\xi_1^+} (\alpha \otimes \beta \otimes \eta_1(\mathrm{id}_S)) \\
&= \varepsilon_1 R_{\xi_1^-} L_{\xi_1^+} \Bigg(\sum_{x \in \tilde{\xi}_1^{-1}(S)} \alpha \otimes \tau \Big((\beta_{\xi_1^-(-1)} \boxtimes \mathrm{id}) \otimes \big(\mathrm{id} \boxtimes \tilde{\xi}_1([-1 \to x])\big) \Big) \otimes \big(\tilde{\xi}_1([x \to 1]) \boxtimes \mathrm{id}\big) \Bigg) \\
&= \varepsilon_1 R_{\xi_1^-} L_{\xi_1^+} \Big(\sum_{x \in I} \alpha \otimes \big(\mathrm{id} \boxtimes \tilde{\xi}_1([-1 \to x])\big) \otimes (\beta_{\xi_1^-(-1)} \boxtimes \mathrm{id}) \otimes \big(\tilde{\xi}_1([x \to 1]) \boxtimes \mathrm{id}\big) \Big) \\
&= \delta_1 (\beta_{\xi_1^-(-1)} \boxtimes \mathrm{id}) \otimes (\alpha_{\chi(\alpha)^{-1}(\xi_1^+(1))} \boxtimes \mathrm{id})
\end{aligned}$$

where $I = \{x \in \tilde{\xi}_1^{-1}(S) \mid (\tilde{\xi}_1([-1 \to x]) \boxtimes (\beta_{\xi_1^-(-1)} \circ [\xi_1^-(-2) \to \xi_1^-(-1)]) \cdot \tau \neq 0\}$.

Since ρ is a morphism of $(\mathcal{S}_M(Z), \mathcal{S}_M(Z))$-bimodules, the general result follows using the decompositions $\alpha = (\alpha_{|U \setminus \{\chi(\alpha)^{-1}(\xi_1^+(1))\}} \boxtimes \mathrm{id}_{\xi_1^+(1)}) \cdot (\mathrm{id} \boxtimes \alpha_{\chi(\alpha)^{-1}(\xi_1^+(1))})$ and $\beta = (\mathrm{id} \boxtimes \beta_{\xi_1^-(-1)}) \cdot (\beta_{|S} \boxtimes \mathrm{id}_{\xi_1^-(-1)})$. $\square$

8.3.3. *Diagonal bimodule.* Recall that we have a $(\Delta_\lambda \mathcal{S}_M(Z), \Delta_\lambda \mathcal{S}_M(Z))$-bimodule E. Its restriction to a $(\mathcal{S}_M(Z), \Delta_\lambda \mathcal{S}_M(Z))$-bimodule is the cone of $\pi : R_{\xi_2^-} \otimes_{\mathcal{S}_M(Z)} \mathrm{Id}_{\Delta_\lambda \mathcal{S}_M(Z)} \to R_{\xi_1^-} \otimes_{\mathcal{S}_M(Z)} \mathrm{Id}_{\Delta_\lambda \mathcal{S}_M(Z)}$.

The $(\mathcal{S}_M(Z), \mathcal{S}_M(Z_\xi))$-bimodule $E' = E \circ (1 \otimes \Xi'^{-1})$ is the cone of the map u defined as follows.

Given $\alpha \in R^\bullet_{\xi_2^-}(T,U)$ with $\alpha_{|U} = \mathrm{id}$ and given $\beta \in \mathrm{Hom}_{\mathcal{S}^\bullet(Z_\xi)}(S,U)$, we have

$$u(\alpha\otimes\beta) = \sum_{x\in\tilde{\xi}_1^{-1}(T)} \big(\mathrm{id}\boxtimes\tilde{\xi}_1([-1\to x])\big)\otimes\Big(\big(\alpha_{\xi_2^-(-1)}\cdot[\xi_1^+(1)\to\xi_2^-(-1)]\cdot\tilde{\xi}_1([x\to 1])\big)\boxtimes\mathrm{id}\Big)\cdot\beta.$$

We construct now an isomorphism between E' and the restriction of R_{ξ^-} to a $(\mathcal{S}_M(Z),\mathcal{S}_M(Z_\xi))$-bimodule.

We define two morphisms of pointed sets

$$\begin{gathered}
f_1 : R^\bullet_{\xi_1^-}(T,-)\wedge\mathrm{Hom}_{\mathcal{S}^\bullet(Z_\xi)}(S,-)\to R^\bullet_{\xi^-}(T,S)\\
(\alpha : U\sqcup\{\xi_1^-(-1)\}\to T)\wedge(\beta : S\to U)\mapsto \alpha\cdot(\beta\boxtimes\mathrm{id}_{\xi_1^-(-1)}) = (\alpha_{|U}\cdot\beta)\boxtimes\alpha_{\xi_1^-(-1)}
\end{gathered}$$

and

$$\begin{aligned}
&f_2 : R^\bullet_{\xi_2^-}(T,-)\wedge\mathrm{Hom}_{\mathcal{S}^\bullet(Z_\xi)}(S,-)\to R^\bullet_{\xi^-}(T,S)\\
&(\alpha : U\sqcup\{\xi_2^-(-1)\}\to T)\wedge(\beta : S\to U)\mapsto \alpha\cdot(\beta\boxtimes[\xi_1^-(-1)\to\xi_2^-(-1)])\\
&\qquad\qquad = (\alpha_{|U}\cdot\beta)\boxtimes(\alpha_{\xi_2^-(-1)}\cdot[\xi_1^-(-1)\to\xi_2^-(-1)]).
\end{aligned}$$

Note that we have an isomorphism of pointed sets

$$f_2\vee f_1 : \big(R^\bullet_{\xi_2^-}(T,-)\wedge\mathrm{Hom}_{\mathcal{S}^\bullet(Z_\xi)}(S,-)\big)\vee\big((R^\bullet_{\xi_1^-}(T,-)\wedge\mathrm{Hom}_{\mathcal{S}^\bullet(Z_\xi)}(S,-)\big)\xrightarrow{\sim} R^\bullet_{\xi^-}(T,S).$$

Lemma 8.3.4. *We have $d(f_1)=0$ and $d(f_2)=f_1\circ u$. There is an isomorphism of differential modules*

$$(f_2,f_1) : E'(T,S)\xrightarrow{\sim} R_{\xi^-}(T,S)$$

functorial in $T\in\mathcal{S}_M(Z)$ and $S\in\mathcal{S}_M(Z_\xi)$.

Proof. It is immediate that $d(f_1)=0$. For the second equality, consider $\alpha\in R^\bullet_{\xi_2^-}(T,U)$ and $\beta\in\mathrm{Hom}_{\mathcal{S}^\bullet(Z_\xi)}(S,U)$. Since $\alpha\otimes\beta = (\mathrm{id}\boxtimes\alpha_{\xi_2^-(-1)})\otimes(\alpha_{|U}\cdot\beta)$, we can assume that $\alpha_{|U}=\mathrm{id}$. We have

$$d(f_2)(\alpha\otimes\beta) = \alpha\cdot\big(d(\beta\boxtimes[\xi_1^-(-1)\to\xi_2^-(-1)]) + d(\beta)\boxtimes[\xi_1^-(-1)\to\xi_2^-(-1)]\big).$$

We have

$$\begin{aligned}
d(\beta\boxtimes[\xi_1^-(-1)\to\xi_2^-(-1)]) &= d\big((\mathrm{id}\boxtimes[\xi_1^-(-1)\to\xi_2^-(-1)])\cdot(\beta\boxtimes\mathrm{id}_{\xi_1^-(-1)})\big)\\
&= (\mathrm{id}\boxtimes[\xi_1^+(1)\to\xi_2^-(-1)])\cdot d(\mathrm{id}_U\boxtimes[\xi_1^-(-1)\to\xi_1^+(1)])\cdot(\beta\boxtimes\mathrm{id}_{\xi_1^-(-1)})\\
&\quad + (\mathrm{id}\boxtimes[\xi_1^-(-1)\to\xi_2^-(-1)])\cdot(d(\beta)\boxtimes\mathrm{id}_{\xi_1^-(-1)})\\
&= (\mathrm{id}_U\boxtimes[\xi_1^+(1)\to\xi_2^-(-1)])\sum_{x\in\tilde{\xi}_1^{-1}(U)}\big(\mathrm{id}\boxtimes\tilde{\xi}_1([x\to 1])\boxtimes\tilde{\xi}_1([-1\to x])\big)\\
&\quad\cdot(\beta\boxtimes\mathrm{id}_{\xi_1^-(-1)}) + d(\beta)\boxtimes[\xi_1^-(-1)\to\xi_2^-(-1)]
\end{aligned}$$

hence

$$d(f_2)(\alpha\otimes\beta) = \alpha\cdot\sum_{x\in\tilde{\xi}_1^{-1}(U)}\Big(\mathrm{id}\boxtimes\big([\xi_1^+(1)\to\xi_2^-(-1)]\cdot\tilde{\xi}_1([x\to 1])\big)\boxtimes\tilde{\xi}_1([-1\to x])\Big)\cdot(\beta\boxtimes\mathrm{id}_{\xi_1^-(-1)})$$

$$= \sum_{x\in\tilde{\xi}_1^{-1}(U)} \Big(\mathrm{id}\boxtimes\tilde{\xi}_1([-1\to x])\boxtimes\big(\alpha_{\xi_2^-(-1)}\cdot[\xi_1^+(1)\to\xi_2^-(-1)]\cdot\tilde{\xi}_1([x\to 1])\big)\Big)\cdot(\beta\boxtimes\mathrm{id}_{\xi_1^-(-1)})$$
$$= f_1\circ u(\alpha\otimes\beta).$$

The lemma follows. $\square$

8.3.4. *Matching of extended action.* Recall that E' is the restriction of the $(\mathcal{S}_M(Z_\xi),\mathcal{S}_M(Z_\xi))$-bimodule $E\circ(\Xi'^{-1}\otimes\Xi'^{-1})$.

We show here that the previous isomorphism is functorial in $T\in\mathcal{S}_M(Z_\xi)$. Consider the diagram

$$\begin{tikzcd} R_{\xi_2^-}(T,-)\otimes L_{\xi_1^+}(-,U)\otimes E(U,S) \arrow[r, "w"] \arrow[d, "\Xi\otimes(f_2,f_1)"'] & E(T,S) \arrow[d, "(f_2,f_1)"] \\ \mathrm{Hom}_{\mathcal{S}(Z_\xi)}(T,U)\otimes R_{\xi^-}(U,S) \arrow[r, "\text{action}"'] & R_{\xi^-}(T,S) \end{tikzcd} \tag{8.3.1}$$

where $w=\begin{pmatrix} w_{11} & w_{12}\\ 0 & w_{22}\end{pmatrix}$ (cf §5.4.2) with

$$w_{11}=(R_{\xi_2^-}(\mathrm{mult}\circ\Xi\,\mathrm{Hom}))\circ(\tau L_{\xi_1^+}\,\mathrm{Hom})\circ(R_{\xi_2^-}\lambda\,\mathrm{Hom})$$
$$w_{12}=R_{\xi_2^-}\varepsilon\,\mathrm{Hom}$$
$$w_{22}=(R_{\xi_1^-}(\mathrm{mult}\circ\Xi\,\mathrm{Hom}))\circ(\sigma L_{\xi_1^+}\,\mathrm{Hom})\circ(R_{\xi_2^-}\rho\,\mathrm{Hom}).$$

Lemma 8.3.5. *The diagram (8.3.1) is commutative.*

Proof. Note first that all the maps of the diagram are functorial with respect to $S\in\mathcal{S}_M(Z_\xi)$.

- Let $\gamma\in R_{\xi_2^-}(U,S)$, $\beta\in L_{\xi_1^+}(V,U)$ and $\alpha\in R_{\xi_2^-}(T,V)$. We will show that

$$\text{action}\circ(\Xi\otimes f_2)(\alpha\otimes\beta\otimes\gamma)=f_2\circ R_{\xi_2^-}(\mathrm{mult}\circ\Xi)\circ\tau L_{\xi_1^+}\circ R_{\xi_2^-}\lambda(\alpha\otimes\beta\otimes\gamma). \tag{8.3.2}$$

Since $\gamma=(\mathrm{id}\boxtimes\gamma_{\xi_2^-(-1)})\cdot\gamma_{|S}$ and since $\text{action}\circ(\Xi\otimes f_2)$ and $f_2\circ R_{\xi_2^-}(\mathrm{mult}\circ\Xi)\circ\tau L_{\xi_1^+}\circ R_{\xi_2^-}\lambda$ are morphisms of $\mathcal{S}_M(Z)^{\mathrm{opp}}$-modules, we can assume $\gamma_{|S}=\mathrm{id}$. We have $\alpha\otimes\beta=(\mathrm{id}\boxtimes\alpha_{\xi_2^-(-1)})\otimes(\alpha_{|V}\boxtimes\mathrm{id}_{\xi_1^+(1)}\cdot\beta)$, hence we can assume $\alpha_{|V}=\mathrm{id}$. We can also assume that $\beta\otimes\gamma\neq 0$.

We have

$$\begin{aligned}\text{action}\circ(\Xi\otimes f_2)(\alpha\otimes\beta\otimes\gamma)&=\big(\mathrm{id}_V\boxtimes(\alpha_{\xi_2^-(-1)}\cdot[\xi_1^+(1)\to\xi_2^-(-1)])\big)\cdot\beta\\ &\quad\cdot\big(\mathrm{id}_S\boxtimes(\gamma_{\xi_2^-(-1)}\cdot[\xi_1^-(-1)\to\xi_2^-(-1)])\big)\\ &=\delta_1\beta_{|S\setminus\chi(\beta)^{-1}(\xi_1^+(1))}\boxtimes\big(\alpha_{\xi_2^-(-1)}\cdot[\xi_1^+(1)\to\xi_2^-(-1)]\cdot\beta_{\chi(\beta)^{-1}(\xi_1^+(1))}\big)\\ &\quad\boxtimes\big((\beta\circ\gamma)_{\xi_2^-(-1)}\cdot[\xi_1^-(-1)\to\xi_2^-(-1)]\big)\end{aligned}$$

where $\delta_1=\delta_{i(\alpha_{\xi_2^-(-1)},(\beta\circ\gamma)_{\xi_2^-(-1)})=0}$. On the other hand, we have

$$f_2\circ R_{\xi_2^-}(\mathrm{mult}\circ\Xi)\circ\tau L_{\xi_1^+}\circ R_{\xi_2^-}\lambda(\alpha\otimes\beta\otimes\gamma)=$$

$= f_2 \circ R_{\xi_2^-}(\mathrm{mult} \circ \Xi) \circ \tau L_{\xi_1^+}\Big(\alpha \otimes \big((\beta_{\chi(\gamma)(\xi_2^-(-1))} \cdot \gamma_{\xi_2^-(-1)}) \boxtimes \mathrm{id}\big) \otimes \beta_{|S}\Big)$
$= \delta_1 f_2 \circ R_{\xi_2^-}(\mathrm{mult} \circ \Xi)\Big(\big(\mathrm{id} \boxtimes (\beta_{\chi(\gamma)(\xi_2^-(-1))} \cdot \gamma_{\xi_2^-(-1)})\big) \otimes (\mathrm{id} \boxtimes \alpha_{\xi_2^-(-1)}) \otimes \beta_{|S}\Big)$
$= \delta_1 f_2\Big(\big(\mathrm{id} \boxtimes (\beta_{\chi(\gamma)(\xi_2^-(-1))} \cdot \gamma_{\xi_2^-(-1)})\big) \otimes \Big(\big(\mathrm{id} \boxtimes (\alpha_{\xi_2^-(-1)} \cdot [\xi_1^+(1) \to \xi_2^-(-1)])\big) \cdot \beta_{|S}\Big)\Big)$
$= \delta_1\big((\beta_{\chi(\gamma)(\xi_2^-(-1))} \cdot \gamma_{\xi_2^-(-1)}) \cdot [\xi_1^-(-1) \to \xi_2^-(-1)]\big) \boxtimes \Big(\big(\mathrm{id} \boxtimes (\alpha_{\xi_2^-(-1)} \cdot [\xi_1^+(1) \to \xi_2^-(-1)])\big) \cdot \beta_{|S}\Big)$
$= \mathrm{action} \circ (\Xi \otimes f_2)(\alpha \otimes \beta \otimes \gamma).$

We deduce that (8.3.2) holds.

- Let $\gamma \in R_{\xi_1^-}(U, S)$, $\beta \in L_{\xi_1^+}(V, U)$ and $\alpha \in R_{\xi_2^-}(T, V)$. We will show that

(8.3.3) $\mathrm{action} \circ (\Xi \otimes f_1)(\alpha \otimes \beta \otimes \gamma) = \big(f_1 \circ R_{\xi_1^-}(\mathrm{mult} \circ \Xi) \circ \sigma L_{\xi_1^+} \circ R_{\xi_2^-}\rho + f_2 \circ R_{\xi_2^-}\varepsilon\big)(\alpha \otimes \beta \otimes \gamma).$

As before, we can assume $\gamma_{|S} = \mathrm{id}$, $\alpha_{|V} = \mathrm{id}$ and $\beta \otimes \gamma \neq 0$. We put $u_1 = \chi(\gamma)(\xi_1^-(-1))$ and $u_2 = \chi(\beta)^{-1}(\xi_1^+(1))$.

We have

$$\mathrm{action} \circ (\Xi \otimes f_1)(\alpha \otimes \beta \otimes \gamma) = \big(\mathrm{id}_V \boxtimes (\alpha_{\xi_2^-(-1)} \cdot [\xi_1^+(1) \to \xi_2^-(-1)])\big) \cdot \beta \cdot \big(\mathrm{id}_S \boxtimes \gamma_{\xi_1^-(-1)}\big)$$
$$= \begin{cases} \delta_2(\alpha_{\xi_2^-(-1)} \cdot [\xi_1^-(1) \to \xi_2^-(-1)]) \boxtimes \beta_{|S} & \text{if } u_1 = u_2 \\ \delta_3(\alpha_{\xi_2^-(-1)} \cdot [\xi_1^+(1) \to \xi_2^-(-1)] \cdot \beta_{u_2}) \boxtimes (\beta_{u_1} \circ \gamma_{\xi_1^-(-1)}) \boxtimes \beta_{|S \setminus \{u_2\}} & \text{otherwise} \end{cases}$$

where

- $\delta_2 = 1$ if $\gamma_{\xi_1^-(-1)}(1-) = \iota(\beta_{u_2}(0+))$ and $\delta_2 = 0$ otherwise
- $\delta_3 = 1$ if $\beta_{|U} \cdot (\mathrm{id}_S \boxtimes \gamma_{\xi_1^-(-1)}) \neq 0$ and $\delta_3 = 0$ otherwise.

We have

$$f_2 \circ R_{\xi_2^-}\varepsilon(\alpha \otimes \beta \otimes \gamma) = \delta_2 f_2(\alpha \otimes \beta_{|S}) = \delta_2(\alpha_{\xi_2^-(-1)} \cdot [\xi_1^-(1) \to \xi_2^-(-1)]) \boxtimes \beta_{|S}.$$

We have

$$f_1 \circ R_{\xi_1^-}(\mathrm{mult} \circ \Xi) \circ \sigma L_{\xi_1^+} \circ R_{\xi_2^-}\rho(\alpha \otimes \beta \otimes \gamma) =$$
$$= \delta_3' f_1 \circ R_{\xi_1^-}(\mathrm{mult} \circ \Xi) \circ \sigma L_{\xi_1^+}\Big(\alpha \otimes \big(\beta_{|U \setminus \{u_2\}} \cdot (\gamma_{\xi_1^-(-1)} \boxtimes \mathrm{id})\big) \otimes (\beta_{u_2} \boxtimes \mathrm{id})\Big)$$
$$= \delta_3' \delta_3'' f_1 \circ R_{\xi_1^-}(\mathrm{mult} \circ \Xi) \circ \sigma L_{\xi_1^+}\Big(\alpha \otimes \big(\beta_{|S \setminus \{u_2\}} \boxtimes (\beta_{u_1} \circ \gamma_{\xi_1^-(-1)})\big) \otimes (\beta_{u_2} \boxtimes \mathrm{id})\Big)$$
$$= \delta_3' \delta_3'' f_1 \circ R_{\xi_1^-}(\mathrm{mult} \circ \Xi)\Big(\big((\beta_{u_1} \circ \gamma_{\xi_1^-(-1)}) \boxtimes \mathrm{id}\big) \otimes (\alpha_{\xi_2^-(-1)} \boxtimes \beta_{|S \setminus \{u_2\}}) \otimes (\beta_{u_2} \boxtimes \mathrm{id})\Big)$$
$$= \delta_3' \delta_3'' (\beta_{u_1} \circ \gamma_{\xi_1^-(-1)}) \boxtimes (\alpha_{\xi_2^-(-1)} \cdot [\xi_1^+(1) \to \xi_2^-(-1)] \cdot \beta_{u_2}) \boxtimes \beta_{|S \setminus \{u_2\}}$$

where

- $\delta_3' = 1$ if $u_1 \neq u_2$ and $(\mathrm{id}_{u_1} \boxtimes \beta_{u_2}) \cdot (\gamma_{\xi_1^-(-1)} \boxtimes \mathrm{id}_{u_2}) \neq 0$ and $\delta_3' = 0$ otherwise
- $\delta_3'' = 1$ if $\beta_{|U \setminus \{u_2\}} \cdot (\mathrm{id}_{S \setminus \{u_2\}} \boxtimes \gamma_{\xi_1^-(-1)}) \neq 0$ and $\delta_3'' = 0$ otherwise.

Since $\delta_3 = \delta_3' \delta_3''$, we deduce that (8.3.3) holds and the lemma follows. ☐

8.3.5. *Action of τ.* The action of τ on $E(T,-)\otimes E(-,S)$ corresponds to an endomorphism of $R^2_{\xi_2^-}\oplus R_{\xi_2^-}R_{\xi_1^-}\oplus R_{\xi_1^-}R_{\xi_2^-}\oplus R^2_{\xi_1^-}$ given in (5.3.4).

Lemma 8.3.6. *We have $\tau\circ((f_2,f_1)\otimes(f_2,f_1))=((f_2,f_1)\otimes(f_2,f_1))\circ\tau$.*

Proof. Consider $\alpha_i\in R^\bullet_{\xi_i^-}(T,U)$ and $\beta_i\in R^\bullet_{\xi_i^-}(U,S)$. In order to prove that the equality of the lemma holds when applied to $((\alpha_2,\alpha_1)\otimes(\beta_2,\beta_1))$, we can assume that $(\alpha_i)_{|T}=\mathrm{id}$ and $(\beta_i)_{|S}=\mathrm{id}$, since the morphisms involved in the equality commute with the right action of $\mathcal{S}_M(Z)$.

We have

$$\tau\circ(f_i\otimes f_j)(\alpha_i\otimes\beta_j)=$$

$$=\tau\Big(\big((\alpha_i)_{\xi_i^-(-1)}\cdot[\xi_1^-(-1)\to\xi_i^-(-1)]\boxtimes\mathrm{id}\big)\otimes\big(\mathrm{id}\boxtimes(\beta_j)_{\xi_j^-(-1)}\cdot[\xi_1^-(-1)\to\xi_j^-(-1)]\big)\Big)$$
$$=d_{i,j}\big((\beta_j)_{\xi_j^-(-1)}\cdot[\xi_1^-(-1)\to\xi_j^-(-1)]\boxtimes\mathrm{id}\big)\otimes\big(\mathrm{id}\boxtimes(\alpha_i)_{\xi_i^-(-1)}\cdot[\xi_1^-(-1)\to\xi_i^-(-1)]\big)$$

where $d_{2,1}=0$, $d_{1,2}=1$ and $d_{i,i}=1$ if $\big((\alpha_i)_{\xi_i^-(-1)}\cdot[\xi_i^-(-2)\to\xi_i^-(-1)]\boxtimes(\beta_i)_{\xi_i^-(-1)}\big)\cdot\tau\neq 0$ and $d_{i,i}=0$ otherwise. We deduce that the lemma holds when applied to $((\alpha_2,\alpha_1)\otimes(\beta_2,\beta_1))$, hence it holds in general. □

Index of notations

$\langle\alpha,\beta\rangle$, **68**, **91**
$\sigma' < \sigma$, **69**
$g_1 < g_2$, **92**
$w < w'$, **13**
$[x_0 \to x_1]$, **77**
$[\![\theta]\!]$, **101**
$[\![f]\!]$, **79**
$\bigotimes$, **51**

$\mathcal{A}(n)$, **72**
$\mathcal{A}^\bullet(Z)$, **109**
$\mathrm{add}(\mathcal{V})$, **5**

Bimod , **8**
$\mathrm{Bimod}^\bullet$, **11**

$\mathcal{C}_+$, **10**
$\chi(\theta)$, **99**
$C(x)$, **73**
$C(z)^+$, **92**

$\underline{D(\theta)}$, **105**
$\overline{\deg}(\theta)$, **101**
$\overline{\deg}_D(\theta)$, **101**
$\deg'(\theta)$, **101**
$\deg(\theta)$, **101**
$\deg_D(\theta)$, **101**
$d(\theta)$, **114**
$\mathrm{dm}(\sigma)$, **68**
$\deg(\sigma)$, **69**
$\deg_D(\sigma)$, **69**
$\Delta_E(B)$, **53**
$\Delta_E(\mathcal{C})$, **59**
$\Delta_E(\mathcal{V})$, **60**
$\Delta_E\mathcal{W}$, **30**
$\Delta_\lambda(\mathcal{C})$, **59**
$\Delta_\lambda(\mathcal{V})$, **60**
$\Delta_\sigma\mathcal{W}$, **31**
$\Delta_\lambda(B)$, **57**
$\Delta'_\lambda(\mathcal{C})$, **59**
$\Delta'_\lambda(\mathcal{V})$, **60**
$\Delta'_\lambda(B)$, **53**
$\Delta_\sigma(B)$, **54**
A-diff , **6**

diff , **10**
$\mathcal{V}$-diff , **5**
$D(\sigma)$, **69**
$d(\sigma)$, **70**
$d(T_w)$, **16**

E^+ , **92**
e_c , **91**
e_Ω , **91**
$\varepsilon_{L,R}$, **5**
$\eta_{L,R}$, **5**

$f^\#$, **93**, **100**
f_n , **28**
F_α , **71**
F_I , **66**

$\Gamma(Z)$, **91**
$\Gamma_D(Z)$, **92**
Γ_n , **69**
$\bar{\Gamma}_D(Z)$, **92**
Γ_D , **69**

$\mathcal{H}^+$, **66**
H_n , **17**
$H_n^\bullet$, **27**
$\hat{H}_n$, **22**
$\mathcal{H}_n$, **67**
$\mathcal{H}_n^+$, $\mathcal{H}_n^{++}$, **72**
$\mathcal{H}_n^f$, **72**
$\mathcal{H}_n^{f++}$, **72**
$\mathcal{H}om$, **52**

$i(\theta)$, **101**
$\mathcal{V}^i$, **5**
$I_c^\pm(\gamma)$, **79**
inv , **98**
ι_n , **28**
$I(\zeta_1,\zeta_2)$, **98**
$i(\zeta,\zeta')$, **96**

$L(\theta)$, **104**
L_n , **68**
$L(\sigma)$, **20**, **67**

$\ell(\sigma)$, **67**
$\ell(w)$, **13**
$L(Z)$, **91**

$m(\theta)$, **101**
m_c , **79**
$m_c^{\pm}$, **79**
$M\langle g\rangle, \langle g\rangle M$, **7**
$m(\sigma)$, **68**
$M^{\vee}$, **7**

n_x , **73**

opp , **27, 28**

$\mathcal{P}(Z)$, **99**
$\mathcal{P}^{\bullet}(Z)$, **99**
$\mathcal{P}^{\bullet}_M(Z)$, **99**
$\mathcal{P}^{\bullet}_f(Z)$, **100**
$\Pi(X)$, **77**
pt , **73**

$R(X)$, **80**
rev , **27**
R_n , **68**

$\mathcal{S}(Z)$, **109**
$\mathcal{S}^{\bullet}(Z)$, **109**
$\mathcal{S}^{\bullet}(Z,n)$, **90**, **109**
$\mathcal{S}^{\bullet}_M(Z)$, **109**
$\mathcal{S}^{\bullet}_f(Z)$, **109**
$\mathcal{S}^{\bullet}_{M,f}(Z)$, **109**
Sets$^{\bullet}$, **9**
σ^{i_1,i_2} , **69**
s_{ij} , **18**, **66**
$\hat{\mathfrak{S}}_n$, **18**
$\hat{\mathfrak{S}}_n^{+,\mathrm{nil}}$, **27**
$\hat{\mathfrak{S}}_n^{\mathrm{nil}}$, **27**
$\mathcal{S}_n$, **66**
$\hat{\mathfrak{S}}_n^{++}$, **72**
$\mathcal{S}_n^{+}$, $\mathcal{S}_n^{++}$, **72**
$\mathcal{S}_n^{f}$, **72**
$\mathcal{S}_n^{f++}$, **72**
St , **73**, **81**
supp(ζ) , **78**

θ^{ζ} , **104**
T_a , **23**
$T_{\mathcal{C}}(M)$, **8**
Θ , **66**
$T_{\mathcal{V}}(M)$, **11**
T_w , **13**
$T(X)$, **73**

$\mathcal{U}$, **27**
$\mathcal{U}^{\bullet}$, **29**

$\bar{\mathcal{V}}$, **6**

W_n , **18**
W^{nil} , **17**

X_{exc} , **73**
X_f , **73**
Ξ , **130**

Z^{opp} , **81**
Z_o , **81**
Z_u , **81**
$\hat{Z}$, **84**
$\zeta(0+)$, **79**
$\zeta(1-)$, **79**
$\bar{\zeta}$, **98**
$Z(\xi)$, **117**
Z_{ξ} , **129**

Terms

$(\mathcal{C}, \mathcal{C}')$-bimodule, 8
1-dimensional space, 73
1-dimensional subspace, 74
2-representation on a differential algebra, 52
2-representation on a differential category, 28
2-representation on a differential pointed category, 29
Z-monoid, 7
$\mathcal{V}$-modules, 5

admissible path, 89

bimodule 2-representation, 58, 60
bimodule lax bi-2-representation, 59, 60
bounded map, 9
braid, 99

chord diagram, 85
closed element, 5
components of a curve, 82
cone, 6
curve, 81

differential G-graded pointed strctures, 11
differential G-graded structure, 7
differential category, 5
differential module, 5
differential pointed category, 11
differential pointed set, 10

finite relation, 76, 83

idempotent complete category, 5
idempotent completion, 5
incoming morphism, 123
initial morphism, 123
internal Hom 2-representation, 52

lax bi-2-representation, 29, 53
left dual 2-representation, 28, 53
left dual bimodule, 7, 8
left finite bimodule, 7, 8

minimal path, 77
morphism of 1-dimensional spaces, 74
morphism of 2-representations, 28, 52
morphism of curves, 81

nil Hecke algebra, 16
non-singular 1-dimensional space, 73
non-singular cover, 84

opposite 2-representation, 28, 52
opposite curve, 81
opposite orientation, 90
oriented path, 88
outgoing morphism, 117

parametrized braid, 99

right dual 2-representation, 28
right finite bimodule, 7, 8

same orientation, 90
small open neighbourhood, 73
smooth path, 89
strict morphism of curves, 82
strictly perfect module, 6
strongly pretriangulated category, 6
subcurve, 82
support of a path, 78
sutured surface, 85

tensor product 2-representation, 51
terminal morphism, 117
twisted object, 6

References

[AnChePeReiSu] J. E. Andersen, L.O. Chekhov, R. C. Penner, C. M. Reidys and P. Sulkowski, *Topological recursion for chord diagrams, RNA complexes, and cells in moduli spaces*, Nuclear Phys. B **866** (2013), 414–443.

[Au1] D. Auroux, *Fukaya categories of symmetric products and bordered HeegaardFloer homology*, J. Gökova Geom. Topol. **4** (2010), 1–54.

[Au2] D. Auroux, *Fukaya categories and bordered Heegaard-Floer homology*, in "Proc. International Congress of Mathematicians (Hyderabad, 2010)", Vol. II, Hindustan Book Agency, 917–941, 2010.

[BjBr] A. Björner and F. Brenti, "Combinatorics of Coxeter groups", Springer Verlag, 2005.

[CrFr] L. Crane and I. Frenkel, *Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases*, J. Math. Phys. **35**, (1994), 5136–5154.

[DouLiMa] C. Douglas, R. Lipshitz and C. Manolescu, *Cornered Heegaard Floer homology*, Mem. Amer. Math. Soc. vol. 262, no. 1266 (2019).

[DouMa] C. Douglas and C. Manolescu, *On the algebra of cornered Floer homology*, J. of Topology **7** (2014), 1–68.

[GePf] M. Geck and G. Pfeiffer, "Characters of finite Coxeter groups and Iwahori-Hecke algebras", Oxford Univ. Press, 2000.

[GuPuVa] S. Gukov, P. Putrov and C. Vafa, *Fivebranes and 3-manifold homology*, J. High Energy Phys. 2017, **7**, 071, front matter+80 pp.

[Hu] J.E. Humphreys, "Reflection Groups and Coxeter groups", Cambridge Univ. Press, 1990.

[KaSal] L. H. Kauffman and H. Saleur, *Free fermions and the Alexander-Conway polynomial*, Comm. Math. Phys. **141** (1991), 293–327.

[Kh] M. Khovanov, *How to categorify one-half of quantum* $\mathfrak{gl}(1|2)$, in "Knots in Poland III", Part III, 211-232, Banach Center Publ., **103**, 2014 (Preprint arXiv:1007.3517).

[LePo] Y. Lekili and A. Polishchuk, *Homological mirror symmetry for higher dimensional pairs of pants*, Compos. Math. **156** (2020), 1310–1347.

[LiOzTh1] R. Lipshitz, P. Ozsváth and D. Thurston, "Bordered Heegaard Floer homology", Memoirs of the Amer. Math. Soc. **1216**, 2018.

[LiOzTh2] R. Lipshitz, P. Ozsváth and D. Thurston, work in preparation.

[Lus] G. Lusztig, *Some examples of square integrable representations of semisimple p-adic groups*, Trans. Amer. Math. Soc. **277** (1983), 623–653.

[Man] A. Manion, *Toward the Heegaard Floer homology of a point*, in preparation.

[ManMarWi] A. Manion, M. Marengon and M. Willis, *Strands algebras and Ozsváth-Szabó's Kauffman–states functor*, preprint arXiv:1903.05655.

[OsSz1] P. Ozsváth and Z.Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. **159** (2004), 1027–1158.

[OsSz2] P. Ozsváth and Z.Szabó, *Kauffman states, bordered algebras, and a bigraded knot invariant*, Adv. Math. **328** (2018), 1088–1198.

[OsSz3] P. Ozsváth and Z.Szabó, *Bordered knot algebras with matchings*, Quantum Topol. **10** (2019), 481–592.

[OsSz4] P. Ozsváth and Z.Szabó, *Algebras with matchings and knot Floer homology*, preprint arXiv:1912.01657.

[Rou1] R. Rouquier, 2-*Kac-Moody algebras*, preprint arXiv:0812.5023.

[Rou2] R. Rouquier, *Quiver Hecke algebras and 2-Lie algebras*, Alg. Colloquium **19** (2012), 359–410.

[Rou3] R. Rouquier, *Hopf categories*, in preparation.

[Sar] A. Sartori, *The Alexander polynomial as quantum invariant of links*, Ark. Math. **53** (2015), 177–202.

[Sh] J. Shi, "The Kazhdan-Lusztig cells in certain affine Weyl groups", Lecture Notes in Math. vol. **1179**, Springer Verlag, 1986.

[Ti] Y. Tian, *A categorification of* $U_T(\mathfrak{sl}(1|1))$ *and its tensor product representations*, Geom. Topol. **18** (2014), 1635–1717.

[Za] R. Zarev, *Bordered Floer homology for sutured manifolds*, preprint, arXiv:0908.1106.

A.M.: Department of Mathematics, University of Southern California, 3620 S. Vermont Ave, KAP 104, Los Angeles, CA 90089-2532, USA.
E-mail address: amanion@usc.edu

R.R.: UCLA Mathematics Department, Los Angeles, CA 90095-1555, USA.
E-mail address: rouquier@math.ucla.edu