FILTRATIONS ON PROJECTIVE MODULES FOR IWAHORI–HECKE ALGEBRAS

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ABSTRACT. We consider a generic Iwahori–Hecke algebra $H_{\mathcal{O}}$ associated with a finite Weyl group, defined over a suitable discrete valuation ring $\mathcal{O}$. We define filtrations on $H_{\mathcal{O}}$-modules in terms of Lusztig's a-function. For a projective module, we show that the quotients of this filtration are direct sums of irreducible lattices. As an application, we prove refinements of the results on decomposition numbers obtained by the first named author in [4].

1. INTRODUCTION

Let H be the generic Iwahori–Hecke algebra associated with a finite Weyl group W, defined over the ring $A = \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate (see, for example, [1, §68]). Let K be the field of fractions of A and $\mathcal{O} \subseteq K$ be a discrete valuation ring with residue field k such that $A \subseteq \mathcal{O}$ and the image of v in k has finite order. We have a corresponding decomposition map d_k^H between the Grothendieck groups of H_K and H_k (see (2.1) below). One of the main problems in the representation theory of Iwahori–Hecke algebras is the determination of such decomposition maps. For a survey of known results and their applications to the representation theory of finite reductive groups, see [3].

This paper is a continuation of the work begun in [4]. We assume that the characteristic of k is either 0 or a good prime for W. Using Lusztig's a-function, it is possible to attach a-values to the simple modules of H_K and H_k. Then it is shown in [4] that d_k^H is compatible with these a-values, in the following sense. Let $\mathrm{Irr}(H_K)$ and $\mathrm{Irr}(H_k)$ be complete sets of representatives of the isomorphism classes of simple modules for H_K and H_k, respectively. Then, in the Grothendieck group of finite-dimensional H_k-modules, we have equations

$$d_k^H([V]) = \sum_{M \in \mathrm{Irr}(H_k)} d_{V,M}\,[M] \qquad \text{where } V \in \mathrm{Irr}(H_K).$$

The compatibility result mentioned above can now be stated as follows, see [4, Theorem 3.3]. For any $V \in \mathrm{Irr}(H_K)$ and $M \in \mathrm{Irr}(H_k)$ we have

$$(1.1) \qquad d_{V,M} \neq 0 \quad \Longrightarrow \quad a_M \leq a_V.$$

Moreover, there exists a subset $B \subseteq \mathrm{Irr}(H_K)$ and a bijection $B \leftrightarrow \mathrm{Irr}(H_k)$, $V \leftrightarrow \bar{V}$, such that the following holds (see (2.5) below).

(1.2) $$d_{V,\bar{V}} = 1 \quad \text{and} \quad a_V = a_{\bar{V}} \quad \text{for all } V \in B.$$

The above two conditions together imply that the decomposition matrix of d_k^H has a lower triangular shape, with 1 along the diagonal. There is in fact an explicit construction of the subset $B \subseteq \mathrm{Irr}(H_K)$ which in turn is based on a construction due to Lusztig, in terms of the "asymptotic" algebra J.

The aim of this paper is to prove a number of refinements of the above results, which are announced in [4, Remark 3.7]. These refinements essentially assert that certain decomposition numbers $d_{V,M}$ must be 0. For example, we show in Corollary 4.3 that for any $V \in \mathrm{Irr}(H_K)$ and $M \in \mathrm{Irr}(H_k)$ we have the following implication:

(1.3) $$d_{V,M} \neq 0 \quad \text{and} \quad a_V = a_M \quad \Longrightarrow \quad V \in B \quad \text{and} \quad M = \bar{V}.$$

Combining this with (1.1) shows that there is a unique subset $B \subseteq \mathrm{Irr}(H_K)$ and a unique bijection $B \leftrightarrow \mathrm{Irr}(H_k)$ such that (1.2) holds.

These results follow from the study of certain filtrations of $H_{\mathcal{O}}$-modules defined in terms of the a-function; see Definition 3.2. For a projective module, we obtain strong assertions about the quotients of such a filtration in Theorems 3.4 and 4.2. This is partly based on the results in [4], which will be recalled in Section 2. The applications, especially to decomposition numbers, are discussed in Section 4.

2. Decomposition maps and the asymptotic algebra

Throughout this paper, W is a finite Weyl group with set of simple reflections S. Let $A = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v over $\mathbb{Z}$. Let H be the corresponding generic Iwahori–Hecke algebra over A with parameter v^2. Then H has a free A-basis $\{T_w\}_{w \in W}$ with the well-known multiplication rules, i.e., we have $(T_s - v^2)(T_s + 1) = 0$ for $s \in S$ and $T_w = T_{s_1} \cdots T_{s_m}$ whenever $w = s_1 \cdots s_m$ with $s_i \in S$ is a reduced expression.

If $A \to R$ is any homomorphism into a commutative ring R and Y is any A-module, we regard R as an A-module and set $Y_R := R \otimes_A Y$. This applies in particular to H itself. If R is a field we denote by $R_0(H_R)$ the Grothendieck group of finite-dimensional H_R-modules. (All modules are left modules, unless explicitly stated otherwise.)

(**2.1**) Decomposition maps are defined in the following setting (see [5] and [3, §2]). Let K be the field of fractions of A. By extension of scalars, we obtain a K-algebra H_K. Let $\mathfrak{p} \subset A$ be a non-zero prime ideal. We consider the corresponding localized ring $A_{\mathfrak{p}}$, and denote its residue field by $k_{\mathfrak{p}}$. We assume that the image of v in $k_{\mathfrak{p}}$ has finite order. Then, by [4, Corollary 3.6], the algebra $H_{k_{\mathfrak{p}}}$ is split.

There exists a discrete valuation ring $\mathcal{O} \subseteq K$ with maximal ideal $J(\mathcal{O})$ such that $A \subseteq \mathcal{O}$ and $J(\mathcal{O}) \cap A = \mathfrak{p}$. (This follows from the fact that A is a regular ring, cf. [3, §2]). Let k be the residue field of $\mathcal{O}$; we may regard k as an extension of $k_{\mathfrak{p}}$. Since $H_{k_{\mathfrak{p}}}$ is split, the scalar extension from $k_{\mathfrak{p}}$ to k defines an isomorphism $R_0(H_{k_{\mathfrak{p}}}) \cong R_0(H_k)$ which preserves the classes of simple modules. We will henceforth identify

$$R_0(H_{k_{\mathfrak{p}}}) = R_0(H_k).$$

Then we have a corresponding decomposition map

$$d_k^H \colon R_0(H_K) \to R_0(H_k),$$

defined in the usual way by choosing $\mathcal{O}$-forms for the simple H_K-modules and reducing them modulo the maximal ideal of $\mathcal{O}$ (see [2, Section I.17]). Note that d_k^H only depends on $\mathfrak{p}$ but not on the choice of $\mathcal{O}$, since A is integrally closed in K and $H_{k_{\mathfrak{p}}}$ is split (see [3, Proposition 2.3]).

(**2.2**) It is shown in [4] that the decomposition map d_k^H has an alternative interpretation, as follows. Let J be Lusztig's "asymptotic" algebra and $\phi\colon H \to J_A$ be the corresponding A-algebra homomorphism; see [7]. (By definition, J is a free abelian group with a basis labelled by the elements of W, and we set $J_A = A \otimes_{\mathbb{Z}} J$.) If $A \to R$ is any homomorphism into a commutative ring R, we obtain a corresponding homomorphism of R-algebras $\phi_R \colon H_R \to J_R$.

Consider the k-algebra homomorphism $\phi_k \colon H_k \to J_k$, which in turn determines a group homomorphism

$$(\phi_k)_* \colon R_0(J_k) \to R_0(H_k).$$

Assume from now on that the characteristic of k is either 0 or a good prime for W. Then it is shown in [4, Remark 2.5] that $R_0(J_k)$ and $R_0(H_K)$ can be naturally identified and that, under this identification, $(\phi_k)_*$ coincides with d_k^H. The identification of $R_0(J_k)$ and $R_0(H_K)$ is based on the following two facts.

(1) The K-algebra homomorphism $\phi_K \colon H_K \to J_K$ is an isomorphism; see [7, Theorem 2.8].
(2) The decomposition map $d_k^J \colon R_0(J_K) \to R_0(J_k)$ (defined in a similar way as d_k^H above) is an isomorphism preserving the classes of simple modules; see [4, Remark 2.5].

Thus, the map $(\phi_K)_* \circ (d_k^J)^{-1} \colon R_0(J_k) \to R_0(H_K)$ is also an isomorphism which preserves the classes of simple modules. Given a simple J_k-module E, we denote by E_* a simple H_K-module corresponding to E under that map. With this notation, we have the following identity in $R_0(H_k)$, see [4, Theorem 3.3]:

$$d_k^H([E_*]) = (\phi_k)_*([E]) \quad \text{for all simple } J_k\text{-modules } E.$$

This means that the problem of determining d_k^H is equivalent to that of determining $(\phi_k)_*$.

(**2.3**) It is possible to attach non-negative integers ("a-values") to the simple modules of H_k, J_k, and H_K, as follows.

First, let $\{C_w\}_{w\in W}$ be the Kazhdan–Lusztig basis of H and $a\colon W \to \mathbb{N}_0$ be Lusztig's a-function. Then, following [8, Lemma 1.9], to any simple H_k-module M we attach a non-negative integer a_M by the requirement that

$$C_w M = 0 \quad \text{for all } w \in W \text{ with } a(w) > a_M,$$
$$C_w M \neq 0 \quad \text{for some } w \in W \text{ with } a(w) = a_M.$$

Next, let $\{t_w\}_{w\in W}$ be the natural basis of J. To any simple J_k-module, we can also attach a non-negative integer a_E by the requirement that $t_w E \neq 0$ for some $w \in W$ with $a(w) = a_E$. This is well-defined since $J_k = \bigoplus_i J_k^i$ is a decomposition into 2-sided ideals, where J_k^i is the subspace generated by all t_w with $a(w) = i$ (see [8, (1.3)(d)]).

Finally, we can also attach a-values to the simple modules of H_K, in terms of the generic degrees associated to these modules (see [9, (3.4)]). With the notation in (2.2), we have

$$a_{E_*} = a_E \quad \text{for every simple } J_k\text{-module } E.$$

The basic relation between the a-values for the modules of H_k and J_k is as follows.

Theorem 2.4 ([8, Lemma 1.9]). *For any simple H_k-module M, there exists a J_k-module $\tilde{M}_J$ such that, in $R_0(H_k)$, we have $(\phi_k)_*([\tilde{M}_J)] = [M]$+sum of terms $[M']$ where M' are simple H_k-modules with $a_{M'} < a_M$. Moreover, for all composition factors E of $\tilde{M}_J$, we have $a_E = a_M$.*

(Lusztig gives an explicit construction for $\tilde{M}_J$; we will come back to this point in Proposition 4.5 below.)

(**2.5**) We can now describe the subset $B \subseteq \operatorname{Irr}(H_K)$ and the bijection $B \leftrightarrow \operatorname{Irr}(H_k)$, $V \leftrightarrow \bar{V}$, satisfying condition (1.2). Let $M \in \operatorname{Irr}(H_k)$ and $\tilde{M}_J$ be as in Theorem 2.4. Then $\tilde{M}_J$ has a unique composition factor $E(M)$ such that $[M]$ occurs in the decomposition of $(\phi_k)_*([E(M)])$. In $R_0(H_k)$, we have $(\phi_k)_*([E(M)] = [M]$+sum of terms $[M']$ where M' are simple H_k-modules with $a_{M'} < a_M$. It is easily seen that non-isomorphic modules M, M' yield non-isomorphic modules $E(M), E(M')$. So we can set

$$B = \{E(M)_* \mid M \in \operatorname{Irr}(H_k)\}$$

and define a bijection $B \leftrightarrow \operatorname{Irr}(H_k)$ by $E(M)_* \leftrightarrow M$. Note that we have

$$a_{E(M)_*} = a_{E(M)} = a_M.$$

For details of the proof, see [4, Section 3]. In Proposition 4.5 below, we will show that $\tilde{M}_J$ is simple and so $\tilde{M}_J = E(M)$.

3. Filtrations on modules

Let $\mathcal{O} \subseteq K$ be a discrete valuation ring as in (2.1). Then $(K, \mathcal{O}, k)$ is a "modular system" for the $\mathcal{O}$-algebra $H_\mathcal{O}$, as in the usual setting for modular representation theory (e.g., in [2, Section I.17]). We will only be dealing with $H_\mathcal{O}$-modules which are finitely generated and free over $\mathcal{O}$ (such modules will be called *lattices*). Recall that all modules are left modules, unless explicitly stated otherwise.

Note that, since H_K is split semisimple (see [6]), a result of Heller shows that the Krull–Schmidt–Azuyama Theorem holds for $H_{\mathcal{O}}$-lattices (see [1, Theorem 30.18]). Moreover, idempotents can be lifted from H_k to $H_{\mathcal{O}}$ (see [1, Ex. 6.16]). Thus, the results in [2, Section I.17] are valid without passing to the completion of $\mathcal{O}$.

For any lattice V, we shall write $V_K = K \otimes_{\mathcal{O}} V$ and $V_k = k \otimes_{\mathcal{O}} V$.

(3.1) For the following constructions, see [8, Section 1]. For any $i \geq 0$, let

$$H^{\geqslant i} = A\text{-submodule of } H \text{ generated by all } C_w \text{ with } a(w) \geq i.$$

Note that $H^{\geqslant 0} = H$ and $H^{\geqslant N} = \{0\}$ where N is the length of the longest element of W (see [7, Proposition 1.2]). We obtain a filtration

$$\{0\} = H^{\geqslant N} \subseteq H^{\geqslant N-1} \subseteq \cdots \subseteq H^{\geqslant 1} \subseteq H^{\geqslant 0} = H, \tag{1}$$

where each term $H^{\geqslant i}$ is a 2-sided ideal of H which is finitely generated and free over A. Let $H^i := H^{\geqslant i}/H^{\geqslant i+1}$; this is an (H,H)-bimodule which is finitely generated and free over A. (A basis is given by the images of the C_w with $a(w) = i$.) There is also a natural (J_A, J_A)-bimodule structure on H^a which is compatible with the action of H itself; denoting the action of J_A by $\circ$, we have for $f \in H^i$, $h \in H$ and $j \in J_A$:

$$hf = \phi(h) \circ f, \quad j \circ (fh) = (j \circ f)h, \quad (hf) \circ j = h(f \circ j), \tag{2}$$

where $\phi \colon H \to J_A$ is Lusztig's homomorphism.

If $A \to R$ is any homomorphism into a commutative ring R, we can apply extension of scalars and obtain 2-sided ideals $H_R^{\geqslant i}$ in H_R with similar properties as above. In particular, each quotient H_R^i admits left and right actions by elements of H_R and of J_R, and these satisfy similar compatibility properties as above, i.e., we have for $f \in H_R^i$, $h \in H_R$ and $j \in J_R$:

$$hf = \phi_R(h) \circ f, \quad j \circ (fh) = (j \circ f)h, \quad (hf) \circ j = h(f \circ j). \tag{2'}$$

In particular, this means that H_R^i is a (J_R, H_R)-bimodule.

Definition 3.2. Consider the filtration of $H_{\mathcal{O}}$ obtained by extending scalars from A to $\mathcal{O}$ in (3.1)(1):

$$\{0\} = H_{\mathcal{O}}^{\geqslant N} \subseteq H_{\mathcal{O}}^{\geqslant N-1} \subseteq \cdots \subseteq H_{\mathcal{O}}^{\geqslant 1} \subseteq H_{\mathcal{O}}^{\geqslant 0} = H_{\mathcal{O}}.$$

Recall that each term $H_{\mathcal{O}}^{\geqslant i}$ is a 2-sided ideal in $H_{\mathcal{O}}$; moreover, $H_{\mathcal{O}}^{\geqslant i}$ and $H_{\mathcal{O}}^i$ are finitely generated and free over $\mathcal{O}$. Let V be any $H_{\mathcal{O}}$-module. Then we have a filtration

$$\{0\} = V^{\geqslant N} \subseteq V^{\geqslant N-1} \subseteq \cdots \subseteq V^{\geqslant 1} \subseteq V^{\geqslant 0} = V, \quad \text{where} \quad V^{\geqslant i} := H_{\mathcal{O}}^{\geqslant i} V.$$

We also set $V^i := V^{\geqslant i}/V^{\geqslant i+1}$ for any i.

Note that, a priori, each V^i is an $H_{\mathcal{O}}$-module but it is not clear that it is free over $\mathcal{O}$, even if V is an $H_{\mathcal{O}}$-lattice.

(**3.3**) Let P be a projective $H_{\mathcal{O}}$-module which is of the form $P = H_{\mathcal{O}}e$ where $e \in H_{\mathcal{O}}$ is an idempotent. We consider the filtration of P as given by Definition 3.2. Let $i \geq 0$. Then we have

$$P^{\geqslant i} = H_{\mathcal{O}}^{\geqslant i} P = H_{\mathcal{O}}^{\geqslant i} e \quad \text{and} \quad P^{\geqslant i+1} = H_{\mathcal{O}}^{\geqslant i+1} P = H_{\mathcal{O}}^{\geqslant i+1} e.$$

Now consider the exact sequence

$$\{0\} \to H_{\mathcal{O}}^{\geqslant i+1} \to H_{\mathcal{O}}^{\geqslant i} \to H_{\mathcal{O}}^{i} \to \{0\}.$$

Multiplying with e yields a sequence

$$\{0\} \to H_{\mathcal{O}}^{\geqslant i+1} e \to H_{\mathcal{O}}^{\geqslant i} e \to H_{\mathcal{O}}^{i} e \to \{0\},$$

which is also exact. (To see this, also work with the idempotent $1-e$.) It follows that we have a canonical isomorphism of $H_{\mathcal{O}}$-modules

$$P^i \cong H_{\mathcal{O}}^i e.$$

Hence, P^i is isomorphic to a direct summand of $H_{\mathcal{O}}^i$. This shows, in particular, that P^i is an $H_{\mathcal{O}}$-lattice.

Theorem 3.4. *Recall that the characteristic of the residue field of $\mathcal{O}$ is either* 0 *or a prime which is good for W. Let P be a projective $H_{\mathcal{O}}$-module as in (3.3) and $i \geq 0$ be such that $P^i \neq \{0\}$. We consider a decomposition $P^i = \bigoplus_\lambda V_\lambda$, where each term V_λ is an indecomposable $H_{\mathcal{O}}$-lattice. Then $(V_\lambda)_K$ is simple for all λ and the a-value of $(V_\lambda)_K$ is $\geq i$.*

Proof. By (3.3), we can assume that $P^i = H_{\mathcal{O}}^i e$. The crux of the proof consists of using the compatible actions of$J_{\mathcal{O}}$ and of $H_{\mathcal{O}}$ on $H_{\mathcal{O}}^i$, see (3.1). Since P^i is a submodule of $H_{\mathcal{O}}^i$ obtained by right multiplication with some element of $H_{\mathcal{O}}$, it is clear that P^i is also a $J_{\mathcal{O}}$-submodule of $H_{\mathcal{O}}^i$. Let us now regard P^i as a $J_{\mathcal{O}}$-module and write $P^i = \bigoplus_\lambda E_\lambda$ where each E_λ is an indecomposable $J_{\mathcal{O}}$-lattice. By Lemma 3.6 below, $(E_\lambda)_K$ is a simple J_K-module for all λ.

Now we return to the $H_{\mathcal{O}}$-module structure on P^i. Formula (2') in (3.1) shows that the natural left action of $H_{\mathcal{O}}$ on $H_{\mathcal{O}}^i$ actually factors through the action of $J_{\mathcal{O}}$, via Lusztig's homomorphism $\phi_{\mathcal{O}}$. It follows that $P^i = \bigoplus_\lambda E_\lambda$ is also a decomposition of P^i into $H_{\mathcal{O}}$-lattices. Extending scalars from $\mathcal{O}$ to K yields a decomposition $P_K^i = \bigoplus_\lambda (E_\lambda)_K$, where the H_K-action on each $(E_\lambda)_K$ is pulled back from the J_K-action via Lusztig's homomorphism ϕ_K. But ϕ_K is an isomorphism; see (2.2)(1). Hence, each $(E_\lambda)_K$ is also a simple H_K-module.

Thus, we have shown that P^i is direct sum of $H_{\mathcal{O}}$-lattices which are simple after tensoring with K. Then a similar statement holds for any decomposition of P^i as a direct sum of indecomposable $H_{\mathcal{O}}$-lattices, since the Krull–Schmidt–Azumaya Theorem holds (see the remarks at the beginning of this section).

It remains to prove the statement about the a-values. Since P^i is a direct summand of $H_{\mathcal{O}}^i$ and V_λ is a direct summand of P^i, we have a surjective $H_{\mathcal{O}}$-module homomorphism $H_{\mathcal{O}}^{\geqslant i} \to V_\lambda$. Then we also have a surjective homomorphism of H_k-modules $H_k^{\geqslant i} \to (V_\lambda)_k$. It follows that there exists some $w \in W$ with $a(w) = i$ such that C_w acts non-trivially on some composition factor M of $(V_\lambda)_k$. By definition and property (1.1), this implies that the a-value of $(V_\lambda)_K$ is $\geq a_M \geq i$. □

Remark 3.5. The above proof shows that each $H_{\mathcal{O}}$-lattice V_λ occurring in the decomposition of P^i also has a structure as $J_{\mathcal{O}}$-module, in such a way that the original $H_{\mathcal{O}}$-action is pulled back from that of $J_{\mathcal{O}}$ via Lusztig's homomorphism $\phi_{\mathcal{O}} \colon H_{\mathcal{O}} \to J_{\mathcal{O}}$.

The following result, which is used in the proof of Theorem 3.4, shows that $J_{\mathcal{O}}$ is a hereditary order in J_K (see [1, §26B]).

Lemma 3.6. *Recall the assumptions on the residue field of $\mathcal{O}$ as in Theorem 3.4. Let V be a $J_{\mathcal{O}}$-lattice. Then V is projective. If V is indecomposable, then V_K is a simple J_K-module.*

Proof. This easily follows from the fact that J_k is semisimple (see [4, Remark 2.5]). For the convenience of the reader, we give the details here. Since J_k is semisimple, the module V_k is semisimple and, hence, projective. By [1, Theorem 30.11], this implies that V itself is projective. Now assume that V is such that V_K is not simple. We must show that then V is not indecomposable. By [1, Proposition 23.7], there exists a $J_{\mathcal{O}}$-sublattice $V' \subseteq V$ such that $\{0\} \neq V' \neq V$ and V/V' is also a $J_{\mathcal{O}}$-lattice. Then, as we have already seen, V/V' is projective and so the sequence $\{0\} \to V' \to V \to V/V' \to \{0\}$ splits. Thus, we have shown that V is not indecomposable, as desired. □

4. Applications

We now discuss some applications of Theorem 3.4. We keep the set-up and the notation of Section 3.

(4.1) Let $\mathrm{Irr}(H_K)$ and $\mathrm{Irr}(H_k)$ be as in Section 1. Then, in $R_0(H_k)$, we have equations

$$(1) \qquad d_k^H([V]) = \sum_{M \in \mathrm{Irr}(H_k)} d_{V,M}\,[M] \qquad \text{for } V \in \mathrm{Irr}(H_K),$$

where $d_{V,M}$ are the decomposition numbers. By property (1.1), we know that the sum need only be extended over those M for which $a_M \leq a_V$.

The numbers $d_{V,M}$ also have an alternative interpretation in terms of projective modules, as follows. It is well-known (see [2, Chapter 1]) that for each simple H_k-module M there exists a projective indecomposable $H_{\mathcal{O}}$-module $P(M)$, unique up to isomorphism, such that $P(M)_k$ has M as a simple quotient. Then, by Brauer reciprocity (see [2, Theorem I.17.8]) we have the following identity in $R_0(H_K)$:

(2) $$[P(M)_K] = \sum_{V \in \mathrm{Irr}(H_K)} d_{V,M}\,[V] \qquad \text{for all } M \in \mathrm{Irr}(H_k).$$

Now let $E(M)_*$ be as in (2.5). Using (1.1) and the equality $a_{E(M)_*} = a_M$, we can also write

(3) $$[P(M)_K] = [E(M)_*] + \sum_{V \in \mathrm{Irr}(H_K),\, a_V \geq a_M} d_{V,M}\,[V] \quad \text{for } M \in \mathrm{Irr}(H_k).$$

The following result shows that, in equation (3), the sum need only be extended over those V for which $a_V > a_M$.

Theorem 4.2. *Let M be a simple H_k-module and $a = a_M$. Then P_K^a is simple and isomorphic to $E(M)_*$. In $R_0(H_K)$, we have $[P(M)_K] = [P_K^a]$+sum of terms $[V]$ where V are simple H_K-modules with $a_V > a$.*

Proof. Let $P = P(M)$. We can assume that $P = H_{\mathcal{O}}e$ where $e \in H_{\mathcal{O}}$ is a primitive idempotent, so that we are in the set-up of (3.3).

Now let $i \geq 0$ be maximal such that $P = P^{\geqslant i}$. Then we have a surjective homomorphism of $H_{\mathcal{O}}$-modules $P \to P^i$. Tensoring with k also yields a surjective homomorphism of H_k-modules $P_k \to P_k^i$. The fact that P_k has a unique simple quotient implies that in any decomposition of P^i as a direct sum of $H_{\mathcal{O}}$-lattices, only one non-trivial summand can occur. Combining this with Theorem 3.4 shows that P_K^i is simple. Moreover, all composition factors of P_K other than P_K^i have a-values $\geq i+1$. Thus, equation (4.1)(2) takes the form

$$[P_K] = [P_K^i] + \sum_{V \in \mathrm{Irr}(H_K),\, a_V > i} d_{V,M}\,[V].$$

Comparison with equation (4.1)(3) shows that $i = a$ and $P_K^i \cong E(M)_*$, as desired. □

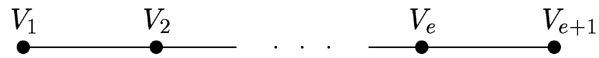
The following result was announced in [4, Remark 3.7(1)].

Corollary 4.3. *For each $M \in \mathrm{Irr}(H_k)$ there exists a unique $V \in \mathrm{Irr}(H_K)$ such that $a_M = a_V$ and $d_{V,M} \neq 0$. Consequently, property (1.3) holds. We necessarily have $d_{V,M} = 1$.*

Proof. Let $M \in \mathrm{Irr}(H_k)$ and $V \in \mathrm{Irr}(H_K)$ be such that $a_V = a_M$ and $d_{V,M} \neq 0$. Then $[V]$ occurs in the sum on the right hand side of equation (4.1)(3). But, by Theorem 4.2, there is only one such term with a-value equal to a_M, and this is $E(M)_*$. It follows that $V \cong E(M)_*$ and $d_{V,M} = 1$, and the proof is complete. □

The above results have the following application to Brauer trees, as announced in [4, Remark 4.5].

(**4.4**) Assume that k has characteristic 0, and consider a block of "defect 1" as in [4, (4.3)]. Assume that the block contains $e+1 \geq 2$ simple H_K-modules $V_1, \dots, V_{e+1}$ such that the corresponding Brauer tree is given by



where the labelling is such that $a_{V_1} \leq a_{V_{e+1}}$. We claim that then we must have

$$a_{V_1} < a_{V_2} < \cdots < a_{V_{e+1}}.$$

For the proof, let us fix some $i \in \{1, \dots, e\}$. By [4, Theorem 4.4], we already know that $a_{V_i} \leq a_{V_{i+1}}$. Hence it is enough to prove that $a_{V_i} \neq a_{V_{i+1}}$. Now the edge on the tree connecting V_i and V_{i+1} corresponds to a simple H_k-module M and, by Brauer reciprocity, we have $[P(M)_K] = [V_i] + [V_{i+1}]$. Theorem 4.2 shows that V_i and V_{i+1} must have different a-values, as required.

The following result was announced in [4, Remark 3.7(2)].

Proposition 4.5. *Let M be a simple H_k-module and set $a = a_M$. Consider the (J_k, H_k)-bimodule H_k^a and let $\tilde{M}_J := H_k^a \otimes_{H_k} M$. Then $\tilde{M}_J$ is a simple J_k-module isomorphic to $E(M)$ (see (2.5)).*

Proof. Note that the above definition of $\tilde{M}_J$ is the one used in Theorem 2.4. Hence, by definition, $E(M)$ is a composition factor of $\tilde{M}_J$. So it remains to show that $\tilde{M}_J$ is a simple J_k-module. Again, as in the proof of Theorem 4.2, we use the compatible left actions of J_k and H_k on H_k^a.

Let $e \in H_{\mathcal{O}}$ be a primitive idempotent such that $P(M) \cong H_{\mathcal{O}}e$. As in (3.3), we have $P(M)^a \cong H_{\mathcal{O}}^a e$. We claim that if we set

$$Q := H_{\mathcal{O}}^a \otimes_{H_{\mathcal{O}}} H_{\mathcal{O}}e,$$

then Q_k is simple when regarded as a (left) J_k-module. This can be seen as follows.

We consider the natural $H_{\mathcal{O}}$-module structure on Q given by left multiplication. Now observe that the canonical map $Q \to H_{\mathcal{O}}^a e$ given by multiplication is an isomorphism. (To see this, also work with the idempotent $1-e$.) Hence we have in fact $Q \cong P(M)^a$. Using Theorem 4.2, we conclude that Q_K is a simple H_K-module. Now the left action of $H_{\mathcal{O}}$ on $H_{\mathcal{O}}^a$ (and, hence, also on Q), factors through the action of $J_{\mathcal{O}}$, via Lusztig's homomorphism $\phi_{\mathcal{O}}$ (see (3.1)(2')). Since Φ_K is an isomorphism (see (2.2)(1)), we deduce that Q_K is also simple when regarded as a J_K-module. Using (2.2)(2), it follows that Q_k is a simple J_k-module, as claimed.

Now, by definition, $P(M)_k$ is a projective indecomposable H_k-module with M as a simple quotient, and we have

$$Q_k \cong (H_{\mathcal{O}}^a \otimes_{H_{\mathcal{O}}} P(M))_k \cong H_k^a \otimes_{H_k} P(M)_k.$$

The quotient map $P(M)_k \to M$ induces a surjective homomorphism of J_k-modules $Q_k \to H_k^a \otimes_{H_k} M = \tilde{M}_J$. Since Q_k is simple, this map must be an isomorphism. So $\tilde{M}_J$ is also simple. □

(**4.6**) Finally, we remark that the above results also yield results about projective modules and decomposition numbers of finite Weyl groups. For this purpose, we have to consider a prime ideal $\mathfrak{p} \subset A$ such that the image of v in k is 1. Then H_k is nothing but the group algebra $k[W]$.

If k has characteristic $p > 0$, then d_k^H coincides with the usual p-modular decomposition map for W; see [3, Example 3.2].

The results in Section 3 yield the following. Let P be a projective $k[W]$-module. Then P has a filtration

$$\{0\} = P^{\geqslant N} \subseteq P^{\geqslant N-1} \subseteq \cdots \subseteq P^{\geqslant 1} \subseteq P^{\geqslant 0} = P, \quad \text{where} \quad P^{\geqslant i} := k[W]_k^{\geqslant i} P.$$

Theorem 4.2 shows that the quotients $P^i = P^{\geqslant i}/P^{\geqslant i+1}$ are direct sums of $k[W]$-modules which are p-modular reductions of simple modules of W over a field of characteristic 0.

The existence of such filtrations on projective modules seems to be a new result, even in the case where W is of type A_n, i.e., a symmetric group.

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