Algebraic Geometry

On the irreducibility of Deligne–Lusztig varieties

Cédric Bonnafé a, Raphaël Rouquier b,1

a Laboratoire de mathématiques de Besançon (CNRS-UMR 6623), université de Franche-Comté, 16, route de Gray, 25030 Besançon cedex, France

b Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, UK

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Abstract

Let \( G \) be a connected reductive group defined over an algebraic closure of a finite field and let \( F: G \to G \) be an endomorphism such that some power of \( F \) is a Frobenius endomorphism. Let \( P \) be a parabolic subgroup of \( G \). We prove that the Deligne–Lusztig variety \( \{ gP \mid g^{-1}F(g) \in P \cdot F(P) \} \) is irreducible if and only if \( P \) is not contained in a proper \( F \)-stable parabolic subgroup of \( G \). To cite this article: C. Bonnafé, R. Rouquier, C. R. Acad. Sci. Paris, Ser. I 343 (2006).

Résumé

Sur l’irréductibilité des variétés de Deligne–Lusztig. Soit \( G \) un groupe réductif connexe défini sur une clôture algébrique d’un corps fini et soit \( F: G \to G \) un endomorphisme dont une puissance est un endomorphisme de Frobenius. Soit \( P \) un sous-groupe parabolique de \( G \). Nous montrons que la variété de Deligne–Lusztig \( \{ gP \mid g^{-1}F(g) \in P \cdot F(P) \} \) est irréductible si et seulement si \( P \) n’est pas contenu dans un sous-groupe parabolique \( F \)-stable propre de \( G \). Pour citer cet article : C. Bonnafé, R. Rouquier, C. R. Acad. Sci. Paris, Ser. I 343 (2006).

Let \( G \) be a connected reductive group over an algebraic closure of a finite field and let \( F: G \to G \) be an endomorphism such that some power of \( F \) is a Frobenius endomorphism. Let \( \mathcal{L}: G \to G \), \( g \mapsto g^{-1}F(g) \) be the Lang map. It is surjective and étale. If \( P \) is a parabolic subgroup of \( G \), we set

\[
X_P = \{ gP \in G/P \mid \mathcal{L}(g) \in P \cdot F(P) \}.
\]

This is the Deligne–Lusztig variety associated to \( P \). The aim of this Note is to prove the following result:

**Theorem 1.** Let \( P \) be a parabolic subgroup of \( G \). Then \( X_P \) is irreducible if and only if \( P \) is not contained in a proper \( F \)-stable parabolic subgroup of \( G \).
Note that this result has been obtained independently by Lusztig (unpublished) and Digne and Michel [2, Proposition 8.4] in the case where $P$ is a Borel subgroup: both proofs are obtained by counting rational points. We present here a geometric proof (inspired by an argument of Deligne [3, proof of Proposition 4.8]) which reduces the problem to the irreducibility of the Deligne–Lusztig variety associated to a Coxeter element: this case has been treated by Deligne and Lusztig [3, Proposition 4.8].

Before starting the proof of this theorem, we first describe an equivalent statement. Let $B$ be an $F$-stable Borel subgroup of $G$, let $T$ be an $F$-stable maximal torus of $B$, let $W$ be the Weyl group of $G$ relative to $T$ and let $S$ be the set of simple reflections of $W$ with respect to $B$. We denote again by $F$ the automorphism of $W$ induced by $F$. Given $I \subseteq S$, let $W_I$ denote the standard parabolic subgroup of $W$ generated by $I$ and let $P_I = BW_I B$. We denote by $\mathcal{P}_I$ the variety of parabolic subgroups of $G$ of type $I$ (i.e. conjugate to $P_I$) and by $B$ the variety of Borel subgroups of $G$ (i.e. $B = \mathcal{P}_G$). For $w \in W$, we denote by $O_I(w)$ the $G$-orbit of $(P_I, wP_F(I))$ in $\mathcal{P}_I \times \mathcal{P}_F(I)$. Note that $O_I(w)$ depends only on the double coset $W_I w W_F(I)$. We define now

$$X_I(w) = \{ P \in \mathcal{P}_I \mid (P, F(P)) \in O_I(w) \}.$$  

The group $G^F$ acts on $X_I(w)$ by conjugation. We set $O(w) = O_G(w)$ and $X(w) = X_G(w)$.

**Theorem 2.** Let $I \subseteq S$ and let $w \in W$. Then $X_I(w)$ is irreducible if and only if $W_I w$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$.

**Remark 1.** Let us explain why Theorems 1 and 2 are equivalent. Let $P_0$ be a parabolic subgroup of $G$. Let $I$ be its type and let $g_0 \in G$ be such that $P_0 = g_0 P_I$. Let $w_0 \in W$ be such that $L(g_0) \in P_I w_P F(I)$. The pair $(I, w_0 w P_F(I))$ is uniquely determined by $P_0$. Then, the map $X_{P_0} \to X_I(w_0), gP_0 \mapsto g g_0^{-1} P_I$ is an isomorphism of varieties (indeed, it is straightforward that $L(g) \in P_0 \cdot F(P_0)$ if and only if $L(g g_0^{-1}) \in P_I w P_F(I)$).

Let $Q$ be a parabolic subgroup of $G$ containing $P$. Let $J$ be its type. Then $I \subseteq J$, $Q = g_0 P_J$ and $L(g_0) \in P_J w P_F(J)$. Now, $Q$ is $F$-stable if and only if $F(J) = J$ and $w \in W_J$. Given $I \subseteq S$ and $w \in W$, we have $L^{-1}(P_I w P_F(I)) \neq \emptyset$ and this shows the equivalence of the two theorems.

**Remark 2.** The condition “$W_I w$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$” is equivalent to “$W_I w W_F(I)$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$”.

The rest of this Note is devoted to the proof of Theorem 2. We fix a subset $I$ of $S$ and an element $w$ of $W$. We first recall two elementary facts. If $I \subseteq J$, let $\tau_{I,J} : \mathcal{P}_I \to \mathcal{P}_J$ be the morphism of varieties that sends $P \in \mathcal{P}_I$ to the unique parabolic subgroup of type $J$ containing $P$. It is surjective. Moreover,

$$\tau_{I,J} (X_I(w)) \subseteq X_J(w)$$  \hspace{1cm} (1)

and

$$\tau_{I,J}^{-1}(X_J(w)) = \bigcup_{W_{I,J} w P_F(I) \subseteq W_J w P_F(J)} X_I(w).$$  \hspace{1cm} (2)

**First step:** the “only if” part. Assume that there exists a proper $F$-stable subset $J$ of $S$ such that $W_I w \subseteq W_J$. Then, by 1, we have $\tau_{I,J}(X_I(w)) \subseteq X_J(1) = \mathcal{P}_J^F$. Since $G^F$ acts transitively on $\mathcal{P}_J^F$, we get $\tau_{I,J}(X_I(w)) = X_J(1)$. This shows that $X_I(w)$ is not irreducible.

**Second step:** reduction to Borel subgroups. By the previous step, we can concentrate on the “if” part. So, from now on, we assume that $W_I w$ is not contained in a proper $F$-stable parabolic subgroup of $W$. Then, by 2, we have

$$\tau_{G,I}^{-1}(X_I(w)) = \bigcup_{x \in W_I w W_F(I)} X(x).$$

Let $v$ denote the longest element of $W_I w W_F(I)$. Then every element $x$ of the double coset $W_I w W_F(I)$ satisfies $x \leq v$ (here, $\leq$ denotes the Bruhat order on $W$): this follows for instance from the fact that $P_I w P_F(I)$ is irreducible and is equal to $\bigcup_{x \in W_I w W_F(I)} B x B$. In particular, $v$ is not contained in a proper $F$-stable parabolic subgroup of $W$.
Now, let $X' = \bigcup_{x \in W \cdot w \cdot W_{F}(t)} X(x)$. Note that $B \cdot v \cdot B = \bigcup_{x \leq v} B \cdot x \cdot B$, hence $L^{-1}(B \cdot v \cdot B) = \bigcup_{x \leq v} L^{-1}(B \cdot x \cdot B)$ since $L$ is open. So, $\overline{X(v)} = \bigcup_{x \leq v} \overline{X(x)}$ and we deduce that

$$X(v) \subset X' \subset \overline{X(v)}.$$

So, since $\tau_{\emptyset I}(X') = X_I(w)$, it is enough to show that $X(v)$ is irreducible. In other words, we may, and we will, assume that $I = \emptyset$.

**Third step: smooth compactification.** Let $(s_1, \ldots, s_n)$ be a finite sequence of elements of $S$. Let

$$\widehat{X}(s_1, \ldots, s_n) = \{ (B_1, \ldots, B_n) \in B^n \mid (B_n, F(B_1)) \in \overline{O}(s_n) \text{ and } (B_i, B_{i+1}) \in \overline{O}(s_i) \text{ for } 1 \leq i \leq n-1 \}.$$

If $\ell(s_1 \cdots s_n) = n$, then $\widehat{X}(s_1, \ldots, s_n)$ is a smooth compactification of $X(s_1 \cdots s_n)$ (see [1, Lemma 9.11]): in this case, $X(s_1 \cdots s_n)$ is irreducible if and only if $\widehat{X}(s_1, \ldots, s_n)$ is irreducible. \hfill(3)

Note that $(B_1, \ldots, B_n) \in \widehat{X}(s_1, \ldots, s_n)$. We denote by $\widehat{X}(s_1, \ldots, s_n)$ the connected (i.e. irreducible) component of $\widehat{X}(s_1, \ldots, s_n)$ containing $(B_1, \ldots, B_n)$. Let $H(s_1, \ldots, s_n) \subset G^F$ be the stabilizer of $\widehat{X}(s_1, \ldots, s_n)$. Let us now prove the following fact:

$$\text{if } 1 \leq i_1 < \cdots < i_r \leq n, \text{ then } H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n).$$

**Proof of (4).** The map $f : \widehat{X}(s_{i_1}, \ldots, s_{i_r}) \to \widehat{X}(s_1, \ldots, s_n)$ defined by

$$f(B_1, \ldots, B_1, B_2, \ldots, B_{i_r-1}, B_r, \ldots, B_n, F(B_1), \ldots, F(B_1))$$

is a $G^F$-equivariant morphism of varieties. Moreover,

$$f(B_1, \ldots, B_1, B_2, \ldots, B_n)$$

is contained in $\widehat{X}(s_1, \ldots, s_n)$. This proves the expected inclusion between stabilizers.

**Last step: twisted Coxeter element.** The quotient variety $G^F \backslash L^{-1}(B \cdot v \cdot B) \simeq B \cdot v \cdot B$ is irreducible, hence $G^F \backslash X(w)$ is irreducible as well. So,

$$G^F \text{ permutes transitively the irreducible components of } X(w).$$

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**References**

