SOME EXAMPLES OF RICKARD COMPLEXES

RAPHAËL ROUQUIER

Talk given at the Constanta Conference on Representation Theory
October 6, 1995

Abstract. After a presentation of Broué's conjecture for principal blocks with an abelian defect group, we describe a Rickard complex for $GL_2(q)$ arising from the ℓ-adic cohomology of a Deligne-Lusztig variety, in accordance with the explicit form given by Broué to his conjecture in the case of Chevalley groups in non natural characteristic.

1. Overview of Broué's conjecture

Let G be a finite group and ℓ a prime number. Let P be a Sylow ℓ-subgroup of G and assume P is abelian. Let $H = N_G(P)$. Let $\mathcal{O}$ be the ring of integers of a finite unramified extension K of $\mathbf{Q}_\ell$, such that KG and KH are split. Let A and B be the principal blocks of G and H over $\mathcal{O}$. Let us denote by H° the group opposite to H. Similarly, B° denotes the algebra opposite to B. We put $\Delta P = \{(x, x^{-1})|x \in P\} \leq G \times H^\circ$. The sign $\otimes$ means $\otimes_\mathcal{O}$. Finally, if M is an $\mathcal{O}$-module, we put $KM = K \otimes M$.

Conjecture 1. *The blocks A and B are Rickard equivalent. More precisely, there is a complex C of (left) $A \otimes B^\circ$-modules which are direct summands of relatively ΔP-projective permutation modules such that :*

$$C^* \otimes_A C \simeq B \text{ in } K^b(B \otimes B^\circ)$$
$$C \otimes_B C^* \simeq A \text{ in } K^b(A \otimes A^\circ).$$

For the sake of simplicity and for the lack of a final form of the conjecture in the general case, we have stated the conjecture for principal blocks only. The original statement of the conjecture [Br1] makes no assumption on C. That C should be of this special type (“ splendid ”) appeared in [Ri4].

The conjecture is known (to the author) to hold in the following cases :

- P cyclic [Ri1, Li, Rou] ;
- $G = \mathbf{G}(\mathbf{F}_q)$ the group of rational points of a connected reductive algebraic group, when $\ell | q-1$ but ℓ does not divide the order of the Weyl group [Pu] ;
- $G = A_5$ and $\ell = 2$ [Ri4] ;
- $G = SL_2(8)$ and $\ell = 2$ [Rou] ;
- G is ℓ-solvable.

For Chevalley groups, when ℓ is not the natural characteristic of the group, there is a very precise conjecture of Broué giving a candidate for C, in terms of ℓ-adic cohomology of certain Deligne-Lusztig varieties [Br-Ma]. The aim of the second part is to present the simplest case of this conjecture.

2. A Geometrical construction for $GL_2(q)$, $\ell | q+1$

Let q be a prime power. Consider the affine curve X with equation $(xy^q - x^q y)^{q-1} = -1$ over an algebraic closure $\bar{\mathbf{F}}_q$ of $\mathbf{F}_q$. The group $G = GL_2(\mathbf{F}_q)$[1] acts naturally on the affine plane over $\bar{\mathbf{F}}_q$ and this induces an action of G on X. There is also an action of the group of rational points $T \simeq \mathbf{F}_{q^2}^*$ of a Coxeter torus of $GL_2(\bar{\mathbf{F}}_q)$ by scalar multiplication (in the isomorphism above, F acts on T as $x \mapsto x^q$ on $\mathbf{F}_{q^2}^*$). Finally, the variety X is defined over $\mathbf{F}_q$, with corresponding Frobenius endomorphism F.

This variety is actually the Deligne-Lusztig variety associated to the non trivial element of the Weyl group[2] [De-Lu, 2.2].

Let $\ell | q+1$ be an odd prime and $\mathcal{O}$ be the ring of integers of a finite unramified extension K of $\mathbf{Q}_\ell$, such that KG and KH are split, where $H = N_G(T) = T \rtimes W$ and $|W| = 2$. Let P be the ℓ-Sylow subgroup of T.

Our object of study is the complex $\mathrm{R}\Gamma_c(X, \mathcal{O})$ ($\mathcal{O}$ is the constant ℓ-adic sheaf) giving rise to the compact support ℓ-adic cohomology : this is an object in the derived category of $(\mathcal{O}G) \otimes (\mathcal{O}T)^\circ$-modules. Actually,

[1] In the talk, the case of $SL_2(q)$ had been considered, where the same methods apply.

[2] This is a Coxeter variety, *i.e.*, the variety associated to a Coxeter element of the Weyl group. These varieties should be studied by the author in a future paper.

we will consider the finer invariant $C = \Lambda_c(X,\mathcal{O})$ in the *homotopy* category of $(\mathcal{O}G)\otimes(\mathcal{O}T)^\circ$-modules, as defined by J.Rickard [Ri3]. This is a complex of direct summands of permutation $\mathcal{O}(G\times T^\circ)$-modules. Note that there is an action of the Frobenius F on C, giving rise to a right action of $\mathcal{O}T\rtimes F$.

Let e be the sum of the ℓ-blocks with positive defect of $\mathcal{O}G$. Define $A=\mathcal{O}Ge$ and $B=\mathcal{O}H$.

Proposition 1. [3] *The action of $\mathcal{O}T\rtimes F$ on C factors through an action of an algebra isomorphic to B. The action of $\mathcal{O}G$ on C factors through an action of A : the complex C is then a complex of direct summands of relatively ΔP-projective permutation modules. We have*

$$C^*\otimes_A C\simeq B \text{ in } K^b(B\otimes B^\circ) \text{ and}$$

$$C\otimes_B C^*\simeq A \text{ in } K^b(A\otimes A^\circ).$$

Proof. Since X is an affine curve, the cohomology groups $H^i_c(X,\mathcal{O})$ are zero for $i=0$ and $i>2$. Since X is in addition smooth, the cohomology groups $H^1_c(X,\mathcal{O})$ and $H^2_c(X,\mathcal{O})$ are free as $\mathcal{O}$-modules [SGA4$\frac{1}{2}$, Arcata, III.§3].

Since both G and T act freely on X, the complex C is perfect (*i.e.* isomorphic to a bounded complex of projective modules) as an object of $\mathcal{D}^b(\mathcal{O}G)$ and as an object of $\mathcal{D}^b(\mathcal{O}T)$ [De-Lu, (proof of) 3.5].

The representation of $G\times T^\circ$ on $H^2_c(X,\mathcal{O})$ is isomorphic to the permutation representation on the connected components of X. Its character is

$$\sum_{\alpha\in\mathrm{Irr}(\mathbf{F}_q^*)}\mathrm{det}_\alpha\otimes\alpha$$

where det_α is the character $\alpha\circ\det$ of G. The Frobenius morphism F acts with the eigenvalue q on $H^2_c(X,\mathcal{O})$.

The character of the $KG\otimes(KT)^\circ$-module $H^1_c(X,K)$ is [Di-Mi2, 15.9] :

$$\sum_{\alpha\in\mathrm{Irr}(\mathbf{F}_q^*)}\mathrm{St}_\alpha\otimes\alpha+\sum_{\substack{\omega\in\mathrm{Irr}(\mathbf{F}^*_{q^2})/W,\\ \omega^{q-1}\neq 1}}[q-1]_\omega\otimes(\omega+\omega^q)$$

where $\mathrm{St}_\alpha=\mathrm{St}\cdot\mathrm{det}_\alpha$, St is the Steinberg character of G and

$$\{[q-1]_\omega\}_{\omega\in\mathrm{Irr}(\mathbf{F}^*_{q^2})/W,\ \omega^{q-1}\neq 1}$$

[3]The proposition actually holds for $\mathcal{O}$ replaced by $\mathbf{Z}_\ell$, as suggested by K.W.Roggenkamp.

is the set of irreducible characters of G with degree $q-1$. The Frobenius F acts with the eigenvalue 1 on the G-isotypic component with character St_α and with eigenvalues $\sqrt{-q}$ and $-\sqrt{-q}$ on the component with character $[q-1]_\omega$ (this is a consequence of Lefschetz formula, [Di-Mi1, V.1.3]). From this description of the character of $H^*(C)$, it follows that $\mathcal{O}G$ acts on C through A.

Let $\sigma \in KT \rtimes F$ defined by

$$\sigma = \frac{i}{q-1}(2F-1-q) + \frac{i-1}{\sqrt{-q}}F$$

where $i = \frac{1}{q^2-1}\sum_{t\in T} t^{q-1}$ and where $\sqrt{-q} \in \mathcal{O}$ is chosen such that $\ell | 1-\sqrt{-q}$. Note that σ is actually in $\mathcal{O}T \rtimes F$, since

$$\sigma = \frac{-1}{(q-1)^2}\sum_{t\in T} t^{q-1} - \frac{1}{\sqrt{-q}}F + \frac{1-\sqrt{-q}}{\sqrt{-q}(q-1)^2(1+\sqrt{-q})}\sum_{t\in T} t^{q-1}F.$$

For $t \in T$, we have $\sigma t = t^q\sigma$ since $Ft = t^qF$. Now, we see that σ acts trivially on $H^2_c(X,K)$, with eigenvalue -1 on the G-isotypic components of $H^1_c(X,K)$ with character St_α and with eigenvalues 1 and -1 on the G-isotypic components with character $[q-1]_\omega$; in particular, σ^2 acts trivially on $H^*_c(X,K)$. Finally, the image in $\mathrm{End}_{K^b(KA)}(KC)$ of the sub-algebra of $KT \rtimes F$ generated by T and σ is isomorphic to KH. But it is clear that $\mathrm{End}_{K^b(KA)}(KC)$ is isomorphic to KH : this means that the image in $\mathrm{End}_{K^b(KA)}(KC)$ of the sub-algebra of $KT \rtimes F$ generated by T and σ is actually the image of $KT \rtimes F$. Hence, we have proven the analog of the proposition where scalars are extended to K.

For $\alpha \in \mathrm{Irr}(\mathbf{F}_q^*)$, let e_α be the sum of the blocks of G containing the characters $[q-1]_\omega$ with $\omega^{q+1} = \alpha$ and the characters $\det_\alpha$ and St_α. Let $C_\alpha = e_\alpha C$. Since $H^i(C)$ is free over $\mathcal{O}$ and C is perfect in $\mathcal{D}^b(\mathcal{O}G)$, it is isomorphic to a complex of projective modules $0 \to C^1 \xrightarrow{\varphi} C^2 \to 0$ as an $\mathcal{O}G$-module. Let us choose C^1 such that $\mathrm{Im}\,\varphi$ has no projective direct summand. Put $C^i_\alpha = e_\alpha C^i$. Then, C^2_α is a projective cover of $\det_\alpha$, since $H^2(C_\alpha) \simeq \det_\alpha$. Now, C_α splits as

$$0 \to C'^1_\alpha \to C^2_\alpha \to 0 \oplus 0 \to C"^1_\alpha \to 0 \to 0$$

where $C'^1_\alpha \to C^2_\alpha$ is the beginning of a projective resolution of $\det_\alpha$. From this description and from the knowledge of the characters of $C'^1_\alpha, C"^1_\alpha$ and C^2_α, it follows that C is a tilting complex for A.

Let B' be the image of the sub-algebra of $\mathcal{O}T \rtimes F$ generated by T and σ in $\mathrm{End}_{K^b(A)}(C)$. The algebra B' is isomorphic to $\mathcal{O}H$ and C is perfect in $\mathcal{D}^b(B')$, since it is perfect in $\mathcal{D}^b(\mathcal{O}T)$.

A proof similar to the one above shows that C is a tilting complex for B'. Now, by [Br2, théorème 2.3], this implies that C is a two-sided tilting complex for $A \otimes B^\circ$, *i.e.*, the isomorphisms of the proposition hold in the derived categories and a priori not in the homotopy categories. Note that we have obtained that B' is the whole of $\mathrm{End}_{K^b(A)}(C)$, hence B' is the image of $\mathcal{O}T \rtimes F$ in $\mathrm{End}_{K^b(A)}(C)$.

If S is a non trivial ℓ-subgroup of $G \times T^\circ$ which is not conjugate to ΔP, then S acts freely on X, hence by [Ri3, Corollary 3.3], C is a complex of direct summands of relatively ΔP-projective permutation modules. If S is a non trivial ℓ-subgroup of $G \times T^\circ$, then the fixed points set X^S has dimension zero, hence $\Lambda_c(X^S, \mathcal{O})$ is concentrated in degree 0. Hence, by [Ri3, Theorem 4.2], C is homotopic to a bounded complex of modules which are all projective $A \otimes B^\circ$-modules, except C^0 ; this implies that the isomorphisms of the proposition hold indeed in the homotopy category [Ri2, (proof of) Corollary 5.5]. □

References

[Br1] M. Broué, *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque **181–182** (1990), 61–92.

[Br2] M. Broué, *Isométries de caractères et équivalences de Morita ou dérivées*, Publ. Math. IHES **71** (1990), 45–63.

[Br-Ma] M. Broué und G. Malle, *Zyklotomische Heckealgebren*, Astérisque **212** (1993), 119–189.

[De-Lu] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Annals of Math. **103** (1976), 103–161.

[Di-Mi1] F. Digne and J. Michel, “Fonctions L des variétés de Deligne-Lusztig et descente de Shintani”, Mémoires de la Soc. Math. de France **20**, 1985.

[Di-Mi2] F. Digne and J. Michel, “Representations of finite groups of Lie type”, London Math. Soc. Student Texts **21**, Cambridge University Press, 1991.

[Li] M. Linckelmann, *Derived equivalence for cyclic blocks over a p-adic ring*, Math. Z. **207** (1991), 293-304.

[Pu] L. Puig, *Algèbres de source de certains blocs des groupes de Chevalley*, Astérisque **181–182** (1990), 221–236.

[Ri1] J. Rickard, *Derived categories and stable equivalences*, J. Pure and Appl. Algebra **61** (1989), 303–317.

[Ri2] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. **43** (1991), 37–48.

[Ri3] J. Rickard, *Finite group actions and étale cohomology*, Publ. Math. IHES **80** (1994), 81–94.

[Ri4] J. Rickard, *Splendid equivalences : derived categories and permutation modules*, to appear in Proc. London Math. Soc.

[Rou] R. Rouquier, *From stable equivalences to Rickard equivalences for blocks with cyclic defect*, proceedings of “Groups 1993, Galway–Saint-Andrews conference" volume 2, 512–523, London Math. Soc. series 212, Cambridge University Press, 1995.

[SGA4$\frac{1}{2}$] P. Deligne et al., "Séminaire de géométrie algébrique du Bois-Marie, SGA 4$\frac{1}{2}$", Cohomologie étale, Springer Lecture Notes 569, 1977.