FINITE GENERATION OF COHOMOLOGY OF FINITE GROUPS

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Abstract. We give a proof of the finite generation of the cohomology ring of a finite p-group over $\mathbf{F}_p$ by reduction to the case of elementary abelian groups, based on Serre's Theorem on products of Bocksteins.

1. Definitions and basic properties

Let $k = \mathbf{F}_p$. Given G a finite group, we put $H^*(G) = H^*(G, k)$. We refer to [Ev] for results on group cohomology.

Given A a ring and M an A-module, we say that M is finite over A if it is a finitely generated A-module.

Let G be a finite group and L a subgroup of G. We have a restriction map $\mathrm{res}^G_L : H^*(G) \to H^*(L)$. It gives $H^*(L)$ the structure of an $H^*(G)$-module.

We denote by $\mathrm{norm}^G_L : H^*(L) \to H^*(G)$ the norm map. If L is central in G, then we have $\mathrm{res}^G_L \mathrm{norm}^G_L(\xi) = \xi^{[G:L]}$ for all $\xi \in H^*(L)$.

When L is normal in G, we denote by $\mathrm{inf}^G_{G/L} : H^*(G/L) \to H^*(G)$ the inflation map.

Let E be an elementary abelian p-group. The Bockstein $H^1(E) \to H^2(E)$ induces an injective morphism of algebras $S(H^1(E)) \hookrightarrow H^*(E)$. We denote by $H^*_{\mathrm{pol}}(E)$ its image. Note that $H^*(E)$ is a finitely generated $H^*_{\mathrm{pol}}(E)$-module and given $\xi \in H^*(E)$, we have $\xi^p \in H^*_{\mathrm{pol}}(E)$.

2. Finite generation for finite groups

The following result is classical. We provide here a proof independent of the finite generation of cohomology rings.

Lemma 2.1. *Let G be a p-group and E an elementary abelian subgroup. Then, $H^*(E)$ is finite over $H^{\mathrm{even}}(G)$.*

Proof. The result is straightforward when G is elementary abelian. As a consequence, given G, it is enough to prove the lemma when E is a maximal elementary abelian subgroup. We prove the lemma by induction on $|G|$. Let $Z \leq Z(G)$ with $|Z| = p$. Let P be a complement to Z in E. Let $A = \mathrm{inf}^E_{E/Z}(H^*_{\mathrm{pol}}(E/Z))$. Let x be a generator of $H^2(Z) \xrightarrow{\sim} H^2(E/P)$ and $y = \mathrm{inf}^E_{E/P}(x)$. We have

$$H^*_{\mathrm{pol}}(E) = A \otimes k[y].$$

Let $\xi = \mathrm{res}^G_E(\mathrm{norm}^G_Z(x)^p)$. We have $\mathrm{res}^E_Z(\xi) = x^{p[G:Z]}$, so $\xi - y^{p[G:Z]} \in H^*_{\mathrm{pol}}(E) \cap \ker \mathrm{res}^E_Z = A^{>0} H^*_{\mathrm{pol}}(E)$. We deduce that $H^*(E)$ is finite over its subalgebra generated by A and ξ.

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By induction, $H^*(E/Z)$ is finite over $H^*(G/Z)$. We deduce that $H^*(E)$ is finite over its subalgebra generated by ξ and $\inf_{E/Z}^E \operatorname{res}_{E/Z}^{G/Z} H^*(G/Z) = \operatorname{res}_E^G \inf_{G/Z}^G H^*(G/Z)$. $\square$

Let us recall a form of Serre's Theorem on product of Bocksteins [Se]. We state the result over the integers for a useful consequence stated in Corollary 2.3.

Theorem 2.2 (Serre). *Let G be a finite p-group. Assume G is not elementary abelian. Then, there is $n \geq 2$, there are subgroups $H_1, \ldots, H_n$ of index p of G and an exact sequence of $\mathbf{Z}G$-modules*

$$0 \to \mathbf{Z} \to \operatorname{Ind}_{H_n}^G \mathbf{Z} \to \cdots \to \operatorname{Ind}_{H_1}^G \mathbf{Z} \to \mathbf{Z} \to 0$$

defining a zero class in $\operatorname{Ext}^n_{\mathbf{Z}G}(\mathbf{Z}, \mathbf{Z})$.

Proof. Serre shows there are elements $z_1, \ldots, z_m \in H^1(G, \mathbf{Z}/p)$ such that $\beta(z_1) \cdots \beta(z_m) = 0$. The element z_i corresponds to a surjective morphism $G \to \mathbf{Z}/p$ with kernel H_i, and we identify $\operatorname{Ind}_{H_i}^G \mathbf{Z}$ with $\mathbf{Z}[G/H_i] = \mathbf{Z}[\sigma]/(\sigma^p - 1)$, where σ is a generator of G/H_i. The element $\beta(z_i) \in H^2(G, \mathbf{Z}/p)$ is the image of the class $c_i \in H^2(G, \mathbf{Z})$ given by the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{1+\sigma+\cdots+\sigma^{p-1}} \operatorname{Ind}_{H_i}^G \mathbf{Z} \xrightarrow{1-\sigma} \operatorname{Ind}_{H_i}^G \mathbf{Z} \xrightarrow{\text{augmentation}} \mathbf{Z} \to 0.$$

Let $c = c_1 \cdots c_m \in H^{2m}(G, \mathbf{Z})$. The image of c in $H^{2m}(G, \mathbf{Z}/p)$ vanishes, hence $c \in pH^{2m}(G, \mathbf{Z})$. Fix r such that $|G| = p^r$. Since $|G| H^{>0}(G, \mathbf{Z}) = 0$, we deduce that $c^r = 0$. $\square$

We will only need the case $R = \mathbf{F}_p$ of the corollary below. We denote by $D^b(RG)$ the derived category of bounded complexes of finitely generated RG-modules.

Corollary 2.3. *Let G be a finite group and R a discrete valuation ring with residue field of characteristic p or a field of charactetistic p. Assume $x^{p-1} = 1$ has $p-1$ solutions in R.*

Let $\mathcal{I}$ be the thick subcategory of $D^b(RG)$ generated by modules of the form $\operatorname{Ind}_E^G M$, where E runs over elementary abelian subgroups of G and M runs over one-dimensional representations of E over R.

We have $\mathcal{I} = D^b(RG)$.

Proof. Assume first G is an elementary abelian p-group. Let L be a finitely generated RG-module. Consider a projective cover $f : P \to L$ and let $L' = \ker f$. The R-module L' is free, so L' is an extension of RG-modules that are free of rank 1 as R-modules. So $L' \in \mathcal{I}$ and similarly $P \in \mathcal{I}$, hence $L \in \mathcal{I}$. As a consequence, the corollary holds for G elementary abelian.

Assume now G is a p-group that is not elementary abelian. We proceed by induction on $|G|$. Let L be a finitely generated RG-module. By induction, $\operatorname{Ind}_H^G \operatorname{Res}_H^G(L) \in \mathcal{I}$ whenever H is a proper subgroup of G. Applying $L \otimes_{\mathbf{Z}G} -$ to the exact sequence of Theorem 2.2, we obtain an exact sequence

$$0 \to L \to \operatorname{Ind}_{H_n}^G \operatorname{Res}_{H_n}^G(L) \to \cdots \to \operatorname{Ind}_{H_1}^G \operatorname{Res}_{H_1}^G(L) \to L \to 0.$$

Since that sequence defines the zero class in $\operatorname{Ext}^n(L, L)$, it follows that L is a direct summand of $0 \to \operatorname{Ind}_{H_n}^G \operatorname{Res}_{H_n}^G(L) \to \cdots \to \operatorname{Ind}_{H_1}^G \operatorname{Res}_{H_1}^G(L) \to 0$ in $D^b(RG)$. We deduce that $L \in \mathcal{I}$.

Finally, assume G is a finite group. Let P be a Sylow p-subgroup of G and let L a finitely generated RG-module. We know that $\operatorname{Ind}_P^G \operatorname{Res}_P^G(L) \in \mathcal{I}$. Since L is a direct summand of $\operatorname{Ind}_P^G \operatorname{Res}_P^G(L)$, we deduce that $L \in \mathcal{I}$. $\square$

Theorem 2.4 (Golod, Venkov, Evens)**.** *Let G be a finite p-group. The ring $H^*(G)$ is finitely generated. Given M a finitely generated kG-module, then $H^*(G, M)$ is a finitely generated $H^*(G)$-module.*

Note that the case where G is an arbitrary finite group follows easily, cf [Ev].

Proof. Let S be a finitely generated subalgebra of $H^{\mathrm{even}}(G)$ such that $H^*(E)$ is a finitely generated S-module for every elementary abelian subgroup E of G. Such an algebra exists by Lemma 2.1.

Let $\mathcal{J}$ be the full subcategory of $D^b(kG)$ of complexes C such that the S-module $H^*(G, C) = \bigoplus_i \mathrm{Hom}_{D^b(kG)}(k, C[i])$ is finitely generated.

Let $C_1 \to C_2 \to C_3 \rightsquigarrow$ be a distinguished triangle in $D^b(kG)$. We have a long exact sequence

$$\cdots \to H^i(C_1) \to H^i(C_2) \to H^i(C_3) \to H^{i+1}(C_1) \to \cdots$$

Assume $C_1, C_3 \in \mathcal{J}$. Let I be a finite generating set of $H^*(C_1)$ as an S-module and J a finite generating set of $\ker(H^*(C_3) \to H^{*+1}(C_1))$ as an S-module. Let I' be the image of I in $H^*(C_2)$ and let J' be a finite subset of $H^*(C_2)$ with image J. Then, $I' \cup J'$ generates $H^*(C_2)$ as an S-module, hence $C_2 \in \mathcal{J}$.

Note that if $C \oplus C' \in \mathcal{J}$, then $C \in \mathcal{J}$. We deduce that $\mathcal{J}$ is a thick subcategory of $D^b(kG)$.

Let E be an elementary abelian subgroup of G. Since $H^*(G, \mathrm{Ind}_E^G(k)) \simeq H^*(E, k)$ is a finitely generated S-module, we deduce that $\mathrm{Ind}_E^G(k) \in \mathcal{J}$.

We deduce from Corollary 2.3 that $\mathcal{J} = D^b(kG)$. □

References

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