Block theory via stable and Rickard equivalences

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Contents

1. Introduction

We present some topics of modular representation theory of finite groups, based on functorial methods, and motivated by Broué's abelian defect group conjecture.

In the first three sections, we review some classical material. In §2, we define various types of equivalence for symmetric algebras induced by tensoring with complexes of bimodules, following some discussion of general adjunction properties of such functors. In §3, we give some basic properties of group algebras: blocks, normal p'-subgroups and the case of TI Sylow p-subgroups. Finally, in §4, we deal more specifically with (direct summands of) permutation modules, where the Brauer functor allows us to transfer local information.

In §5, we discuss Rickard equivalences in block theory. After a detailed example, we consider Broué's abelian defect conjecture and its refinements (splendidness, equivariance with respect to p'-automorphism groups, central extensions by p-groups). In §5.3, we come to a crucial result: a splendid complex induces a stable equivalence if and only if it induces (via the Brauer functor) local Rickard equivalences (from this point, we consider only principal blocks).

The results in §6 and §7 are new. We use the results of §5.3 in §6.2 to construct stable equivalences between a principal block with defect group $\mathbb{Z}/p^a \times \mathbb{Z}/p^b$ and the principal block of the normalizer of a defect group (in the case where $(a, b) \neq (1, 1)$, we need the Z_p^*-theorem which, for p odd, depends on the classification of finite simple groups). We use here a new construction of stable equivalences given by complexes. In §6.3, we go one step further to lift these stable equivalences to Rickard equivalences when, in addition, $p = 2$ (thus solving Broué's conjecture for principal blocks with defect group $\mathbb{Z}/2 \times \mathbb{Z}/2$). In §6.4, we construct stable equivalences for principal blocks with defect group elementary abelian of order 8.

§7 is devoted to the study of a locally determined category of p-permutation modules with additional structure. We explain how this can be used to glue local Rickard equivalences into a stable equivalence: as a consequence, we prove that Broué's abelian defect group conjecture would follow (inductively) from the possibility of lifting stable equivalences to Rickard equivalences. This requires additional structure on the Rickard complexes. The construction of §6.2 appears as a special case. Our belief is that these methods reduce Broué's conjecture to a problem of "representation theory of algebras" where the groups will not be useful anymore, namely the problem of lifting certain stable equivalences to Rickard equivalences.

In the appendix, we explain some aspects of the theory for nonprincipal blocks [Rou3].

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2. Symmetric algebras, functors and equivalences

In this section, we explain what standard concepts of category theory become for module categories of symmetric algebras. §2.2 is largely inspired by Broué's notes [Br3], where more details are to be found.

In §2.3, we define various notions of equivalences and gather various properties.

2.1. Notation and conventions

Let $\mathcal{O}$ be a noetherian local commutative ring (all rings are with identity) with residue field k. All $\mathcal{O}$-algebras considered will be free and finitely generated over $\mathcal{O}$. Let A an $\mathcal{O}$-algebra.

All A-modules considered will be left modules, finitely generated over the underlying coefficient ring $\mathcal{O}$. Complexes of A-modules will always be bounded. We identify the category A-mod of A-modules with the full subcategory of complexes of A-modules concentrated in degree 0.

We denote by A° the algebra opposite to A. It is A as an $\mathcal{O}$-module, but the multiplication of a and b in A° is ba. Note that a left A-module is the same as a right A°-module and if B is a B-algebra, an (A, B)-bimodule is an $(A \otimes_{\mathcal{O}} B^\circ)$-module.

Similarly, if G is a group, we define the group G° opposite to G, with the same set of elements as G but with multiplication of g and h given by hg. The group algebra $\mathcal{O}G^\circ$ of G° is the algebra $(\mathcal{O}G)^\circ$.

We will often write $\otimes$ for $\otimes_{\mathcal{O}}$. For M an A-module, we denote by M^* the A°-module $\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$.

By *the* (A, A)-bimodule A, we mean the regular bimodule given by left and right multiplication.

2.2. Functors and adjunctions

2.2.1. Basic adjunction. Let A and B be two $\mathcal{O}$-algebras and M an (A, B)-bimodule.

We have an isomorphism of $\mathcal{O}$-modules

$$\begin{aligned}\gamma_M(V,U) : \mathrm{Hom}_A(M\otimes_B V, U) &\xrightarrow{\sim} \mathrm{Hom}_B(V, \mathrm{Hom}_A(M,U))\\ f &\mapsto (v\mapsto (m\mapsto f(m\otimes v)))\\ \text{with inverse}\quad (m\otimes v\mapsto g(v)(m)) &\leftarrow g\end{aligned}$$

for any A-module U and B-module V.

Denote by Φ and Ψ respectively the functors

$$\Phi = M\otimes_B - : B\text{-mod}\to A\text{-mod} \text{ and } \Psi = \mathrm{Hom}_A(M,-) : A\text{-mod}\to B\text{-mod}.$$

We have an isomorphism functorial in U and V:

$$\mathrm{Hom}_A(\Phi(V), U) \xrightarrow{\sim} \mathrm{Hom}_B(V, \Psi(U)).$$

We say that the functor Φ is *left adjoint* to the functor Ψ (or Ψ is *right adjoint* to Φ or (Φ, Ψ) is an *adjoint pair*) when there is such an isomorphism.

2.2.2. Projective modules. Let U and V be two A-modules. Consider the $\mathcal{O}$-linear map

$$\begin{aligned}\tau_{U,V} : \mathrm{Hom}_A(U,A)\otimes_A V &\longrightarrow \mathrm{Hom}_A(U,V)\\ f\otimes v &\mapsto (u\mapsto f(u)v)\end{aligned}$$

If U or V is projective, then $\tau_{U,V}$ is an isomorphism: the result is clear when one of the modules is A, thus when it is A^n and finally when it is any direct summand of A^n.

We have a converse to this property: if $\tau_{U,V}$ is an isomorphism for every V, then U is projective. More precisely, we have

Proposition 2.1. *Let S be a simple A-module with a projective cover P_S. Then $\tau_{U,S}$ is non-zero if and only if U has a direct summand isomorphic to P_S.*

Proof. Note that $\tau_{P_S,S}\neq 0$; hence $\tau_{U,S}\neq 0$ if P_S is a direct summand of U.

Let $f : P_S \to S$ be an essential map : this is a surjective morphism whose restriction to a proper submodule of P_S is not surjective anymore. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(U,P_S) \arrow[r, "\mathrm{Hom}_A(U,f)"] & \mathrm{Hom}_A(U,S) & \\
\mathrm{Hom}_A(U,A)\otimes_A P_S \arrow[u, "\tau_{U,P_S}", "\simeq"'] \arrow[r, "1\otimes f"'] & \mathrm{Hom}_A(U,A)\otimes_A S \arrow[u, "\tau_{U,S}"'] \arrow[r] & 0 .
\end{tikzcd}$$

If $\tau_{U,S} \neq 0$, then $\mathrm{Hom}_A(U, f)$ is non-zero. Therefore there is $g : U \to P_S$ whose composite with f is non-zero, and thus surjective. Since f is essential, it follows that g is surjective and splits since P_S is projective. □

When $\mathcal{O}$ is henselian (e.g. complete), then all A-modules have projective covers.

Let M be an (A, B)-bimodule. Let U be an A-module and V a B-module. By §2.2.1, we have an isomorphism of (A, A)-bimodules

$$\mathrm{Hom}_{B^\circ}(M, \mathrm{Hom}_{\mathcal{O}}(V, U)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M \otimes_B V, U).$$

This induces an isomorphism

$$\mathrm{Hom}_{A\otimes B^\circ}(M, \mathrm{Hom}_{\mathcal{O}}(V, U)) \xrightarrow{\sim} \mathrm{Hom}_A(M \otimes_B V, U).$$

If U and V are free over $\mathcal{O}$ and M is projective (or flat) as a B°-module, then we deduce, for all i,

$$\mathrm{Ext}^i_{A\otimes B^\circ}(M, \mathrm{Hom}_{\mathcal{O}}(V, U)) \xrightarrow{\sim} \mathrm{Ext}^i_A(M \otimes_B V, U).$$

Note that we have similar statements relating $\mathrm{Hom}_{A\otimes B^\circ}$ with Hom_{B°, by considering M as a (B°, A°)-bimodule.

Lemma 2.2. *Assume $\mathcal{O} = k$ is a field and assume the centers of the endomorphism rings of the simple A-modules and the simple B-modules are separable extensions of k. Let M be an (A, B)-bimodule.*

Then M is a projective (A, B)-bimodule if and only if $M \otimes_B V$ is a projective A-module for every B-module V and $U \otimes_A M$ is a projective B°-module for every A°-module U.

Proof. The hypothesis ensures that the largest semi-simple quotients of A and B are products of central simple algebras over separable extensions of k. Now the tensor product over k of two such simple algebras is a semi-simple algebra. It follows that given S a simple A-module and T a simple B°-module, the $(A \otimes B^\circ)$-module $\mathrm{Hom}_k(T^*, S) \simeq S \otimes T$ is semi-simple (note that every simple $(A \otimes B^\circ)$-module occurs as a direct summand of such a module for some S, T). In particular, M is projective if and only if $\mathrm{Ext}^i_{A\otimes B^\circ}(M, S \otimes T) = 0$ for all $i > 0$ and S, T simple.

Assume $M \otimes_B V$ is a projective A-module for every B-module V and $U \otimes_A M$ is a projective B°-module for every A°-module U.

The case $V = B$ shows that M is projective as an A-module. Hence, for $i > 0$, we have

$$\mathrm{Ext}^i_{A_\otimes B^\circ}(M, S \otimes T) \simeq \mathrm{Ext}^i_{B^\circ}(S \otimes_A M, T) = 0.$$

It follows that M is projective.

The converse is clear. □

2.2.3. Symmetric algebras. Assume A is a *symmetric algebra*, i.e., is endowed with an $\mathcal{O}$-linear map $t = t_A : A \to \mathcal{O}$ which is a trace ($t(aa') = t(a'a)$) and such that the morphism of (A, A)-bimodules

$$\begin{aligned} \hat{t} : A &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}) \\ a &\mapsto (a' \mapsto t(aa')) \end{aligned}$$

is an isomorphism.

This last isomorphism is equivalent to the requirement that, given an $\mathcal{O}$-basis $\{a_i\}$ of A, there is another basis $\{a'_i\}$ such that $t(a_i a'_j) = \delta_{ij}$.

When $\mathcal{O}$ is a field, the algebra A is in particular self-injective, i.e., the injective modules are the projective modules.

Given an A-module U, we have an isomorphism of right A-modules

$$\begin{aligned} \hat{t}_U : \mathrm{Hom}_A(U, A) &\xrightarrow{\sim} U^* = \mathrm{Hom}_{\mathcal{O}}(U, \mathcal{O}) \\ f &\mapsto tf \\ \text{with inverse} \quad (x \mapsto \sum_i a'_i u(a_i x)) &\leftarrow\!\shortmid u. \end{aligned}$$

Lemma 2.3. *Let M be an $(A \otimes B^\circ)$-module, projective as a B°-module, and V be a B-module. Then we have an isomorphism of A°-modules*

$$\tau_{M, A\otimes V} : \mathrm{Hom}_{A\otimes B^\circ}(M, A \otimes B^\circ) \otimes_{A\otimes B^\circ} (A \otimes V) \xrightarrow{\sim} \mathrm{Hom}_{A\otimes B^\circ}(M, A \otimes V).$$

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{A\otimes B^\circ}(M, A \otimes B^\circ) \otimes_{A\otimes B^\circ} (A \otimes V) \arrow[r, "f\otimes(a\otimes x)\mapsto(t_A\otimes 1)(fa)\otimes x"] \arrow[d, "\tau_{M,A\otimes V}"'] & \mathrm{Hom}_{B^\circ}(M, B^\circ) \otimes_{B^\circ} V \arrow[d, "\tau_{\mathrm{Res}_{B^\circ} M, V}"] \\ \mathrm{Hom}_{A\otimes B^\circ}(M, A \otimes V) \arrow[r, "f\mapsto(t_A\otimes 1)f"'] & \mathrm{Hom}_{B^\circ}(M, V) \end{tikzcd}$$

where the horizontal maps are isomorphisms. Since $\tau_{\mathrm{Res}_{B^\circ} M, V}$ is an isomorphism, we are done. □

2.2.4. Exact bimodules. Assuming A symmetric and M projective as an A-module, we have constructed isomorphisms of functors

$$\Psi = \mathrm{Hom}_A(M, -) \xrightarrow{\sim} \mathrm{Hom}_A(M, A) \otimes_A - \xrightarrow{\sim} M^* \otimes_A -.$$

In particular, the functor $M \otimes_B -$ is left adjoint to $M^* \otimes_A -$. If in addition M is projective as a right B-module and B is symmetric, then $M \otimes_B -$ is right adjoint to $M^* \otimes_A -$.

We say that M is an *exact* (A, B)-bimodule if it is projective as an A-module and as a right B-module (i.e., if the functors $\mathrm{Hom}_A(M, -)$ and $\mathrm{Hom}_{B^\circ}(M, -)$ are exact).

Proposition 2.4. *If A and B are symmetric $\mathcal{O}$-algebras and M is an exact (A, B)-bimodule, then the functor $M \otimes_B -$ is left and right adjoint to $M^* \otimes_A -$.*

Note that M can be projective as an A-module and as a right B-module without being projective as an $(A \otimes B^\circ)$-module. In the special case where $A = B$ and $M = A$ is the regular bimodule, then M is indeed projective as a right and as a left A-module, but is not projective as an $(A \otimes A^\circ)$-module in general: when $\mathcal{O}$ is a field, A is projective as an $(A \otimes A^\circ)$-module if and only if A is semi-simple separable (i.e., A is a product of matrix algebras over separable field extensions of $\mathcal{O}$), cf. Lemma 2.2.

2.2.5. Units and counits. Since Φ is left adjoint to Ψ, we have an isomorphism $\mathrm{Hom}(\Phi \circ \Psi, I_A) \xrightarrow{\sim} \mathrm{Hom}(\Psi, \Psi)$, where I_A is the identity functor of A-mod. The morphism $\varepsilon : \Phi \circ \Psi \to I_A$ corresponding to the identity morphism $\Psi \to \Psi$ is called the *counit*. Using the isomorphism $\mathrm{Hom}(\Phi, \Phi) \xrightarrow{\sim} \mathrm{Hom}(I_B, \Psi \circ \Phi)$, we obtain the *unit* $\eta : I_B \to \Psi \circ \Phi$.

Note that the functor Φ is an equivalence of categories B-mod $\to$ A-mod if and only if η and ε are isomorphisms, and then Ψ is an inverse to Φ.

In terms of morphisms of bimodules, the counit is the morphism of (A, A)-bimodules

$$\varepsilon_M : M \otimes_B M^* \to A,\ m \otimes x \mapsto \hat{t}^{-1}_{\mathrm{Res}_A M}(x)(m)$$

and the unit is the morphism of (B, B)-bimodules

$$\eta_M : B \to M^* \otimes_A M,\ b \mapsto (\hat{t}_{\mathrm{Res}_A M} \otimes 1)\tau^{-1}_{\mathrm{Res}_A M, \mathrm{Res}_A M}(b \cdot 1_M).$$

2.2.6. Complexes. Let C be a complex of A-modules. We denote by d_C its differential, with degree i part $d_C^i : C^i \to C^{i+1}$.

Let C^* the complex of A°-modules given by

$$(C^*)^i = (C^{-i})^* \ \text{ and } \ d^i_{C^*} = (-1)^{i+1}(d_C^{-i-1})^*.$$

Let D be a complex of A°-modules. We denote by $C \otimes_A D$ the complex given by

$$(C \otimes_A D)^i = \bigoplus_{r+s=i} C^r \otimes D^s \ \text{ and } \ d^i_{C \otimes_A D} = \sum_{r+s=i} d^r_C \otimes 1 + (-1)^r 1 \otimes d^s_D.$$

(Let us recall that all our complexes are bounded).

A complex C of (A, B)-bimodules gives rise to a functor $C \otimes_B -$ from the category $C^b(B)$ of complexes of B-modules to the category $C^b(A)$ of complexes of A-modules.

The results of §2.2.1–2.2.5 generalize to complexes. Given a complex C of (A, B)-bimodules which are projective as A-modules, there is a canonical morphism $\varepsilon_C : C \otimes_B C^* \to A$ and a canonical morphism $\eta_C : B \to C^* \otimes_A C$, which are units and counits of the adjoint pair $(C \otimes_B -, C^* \otimes_A -)$.

2.3. Equivalences

Let A and B be two symmetric $\mathcal{O}$-algebras. We define three types of equivalence. The usual Morita equivalences are a special case of Rickard equivalences. The Rickard equivalences are in turn a special case of the even weaker type of stable equivalences.

2.3.1. Morita. Let M be an exact (A, B)-bimodule.

The following assertions are equivalent.

(i) We have isomorphisms

$$M \otimes_B M^* \simeq A \text{ as } (A, A)\text{-bimodules},$$

$$M^* \otimes_A M \simeq B \text{ as } (B, B)\text{-bimodules}.$$

(ii) The morphisms ε_M and η_{M^*} are isomorphisms of (A, A)-bimodules and η_M, ε_{M^*} are isomorphisms of (B, B)-bimodules

$$\eta_{M^*} : A \xrightarrow{\sim} M \otimes_B M^*, \quad \varepsilon_M : M \otimes_B M^* \xrightarrow{\sim} A,$$

$$\eta_M : B \xrightarrow{\sim} M^* \otimes_A M, \quad \varepsilon_{M^*} : M^* \otimes_A M \xrightarrow{\sim} B.$$

When these conditions are satisfied, we say that M induces a *Morita equivalence* between A and B. This is equivalent to the requirement that $M \otimes_B -$ is an equivalence between A-mod and B-mod.

2.3.2. Rickard. We now take C a complex of exact (A, B)-bimodules.

The following assertions are equivalent.

(i) We have isomorphisms

$$C \otimes_B C^* \simeq A \oplus Z_1 \text{ as complexes of } (A, A)\text{-bimodules}$$

$$C^* \otimes_A C \simeq B \oplus Z_2 \text{ as complexes of } (B, B)\text{-bimodules}$$

where A and B are viewed as complexes concentrated in degree 0 and Z_1 and Z_2 are homotopy equivalent to 0.

(ii) The morphisms η_C, η_{C^*} (resp. ε_C and ε_{C^*}) are split injections (resp. surjections) with cokernel (resp. kernel) homotopy equivalent to 0.

When these conditions are satisfied, we say that C induces a *Rickard equivalence* between A and B or that C is a *Rickard complex*. These conditions are equivalent to the requirement that $C \otimes_B -$ is an equivalence between the homotopy categories of complexes of B-modules and A-modules.

Note that if $C = C_1 \oplus C_2$ with C_2 homotopy equivalent to 0, then C induces a Rickard equivalence if and only if C_1 induces a Rickard equivalence.

2.3.3. Stable. The following assertions are equivalent.

(i) We have isomorphisms

$$C \otimes_B C^* \simeq A \oplus Z_1' \text{ as complexes of } (A, A)\text{-bimodules}$$

$$C^* \otimes_A C \simeq B \oplus Z_2' \text{ as complexes of } (B, B)\text{-bimodules}$$

where Z_1' and Z_2' are homotopy equivalent to complexes of projective bimodules.

(ii) The morphisms η_C, η_{C^*} (resp. ε_C and ε_{C^*}) are split injections (resp. surjections) with cokernel (resp. kernel) homotopy equivalent to a complex of projective bimodules.

When these conditions are satisfied, we say that C induces a *stable equivalence* between A and B.

Actually, we want a slightly more general definition: D induces a stable equivalence whenever $D \oplus A \otimes_{\mathcal{O}} B$ satisfies the equivalent conditions above; then, when $\mathcal{O}$ is a field, $D = 0$ induces a stable equivalence between any two semisimple separable $\mathcal{O}$-algebras.

Note that if $C = C_1 \oplus C_2$ with C_2 homotopy equivalent to a complex of projective bimodules, then C induces a stable equivalence if and only if C_1 induces a stable equivalence.

The situation more commonly considered, after Broué, is the case where $C = M$ is a complex with only one term in degree 0.

In that case, we can restate the equivalences as follows:

(i) We have isomorphisms

$$M \otimes_B M^* \oplus \text{ projective } \simeq A \oplus \text{ projective}$$

$$M^* \otimes_A M \oplus \text{ projective } \simeq B \oplus \text{ projective.}$$

(ii) The morphisms η_M, η_{M^*}, ε_M and ε_{M^*} are split with projective kernels and cokernels.

This implies that $M \otimes_B -$ induces an equivalence between the $\mathcal{O}$-stable categories of B-modules and A-modules. The $\mathcal{O}$-stable category of A-modules is the quotient of A-mod by the full subcategory of $\mathcal{O}$-projective A-modules (direct summands of modules $A \otimes_{\mathcal{O}} U$ for some $\mathcal{O}$-module U). Under separability assumptions (e.g., $\mathcal{O}$ is a field and centers of endomorphism algebras of simple modules are separable extensions of $\mathcal{O}$), the conditions are equivalent to the fact that $M \otimes_B -$ induces an equivalence of stable categories.

Note that if $M = M_1 \oplus M_2$ with M_2 projective, then M induces a stable equivalence if and only if M_1 induces a stable equivalence.

Note that $M \otimes_B V$ is projective or zero if and only if V is projective or zero and that $M \otimes_B V$ is the direct sum of an indecomposable non-projective module with a projective module if V is indecomposable non-projective.

The following result shows that (for $\mathcal{O}$ henselian), when M has no projective direct summand, it sends a simple module to an indecomposable module.

Proposition 2.5. *Assume $\mathcal{O}$ is a field. Let S be a simple A-module with a projective cover P_S and T a simple B-module with a projective cover P_T. Then P_S is a direct summand of $M \otimes_B T$ if and only if $P_S \otimes P_T^*$ is a direct summand of M.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{A\otimes B^\circ}(M, A\otimes B^\circ)\otimes_{A\otimes B^\circ}\mathrm{Hom}_{\mathcal{O}}(T,S) \arrow[r, "\tau_{M,\mathrm{Hom}_{\mathcal{O}}(T,S)}"] & \mathrm{Hom}_{A\otimes B^\circ}(M,\mathrm{Hom}_{\mathcal{O}}(T,S)) \arrow[ddd, "\simeq"', "\gamma_T(M,S)^{-1}"] \\
\mathrm{Hom}_{A\otimes B^\circ}(M, A\otimes B^\circ)\otimes_{A\otimes B^\circ}\mathrm{Hom}_{\mathcal{O}}(T,A)\otimes_A S \arrow[u, "\simeq"'] \arrow[d, "\tau_{M,\mathrm{Hom}_{\mathcal{O}}(T,A)}\otimes 1"'] & \\
\mathrm{Hom}_{A\otimes B^\circ}(M,\mathrm{Hom}_{\mathcal{O}}(T,A))\otimes_A S \arrow[d, "\gamma_T^{-1}(M,A)\otimes 1"', "\simeq"] & \\
\mathrm{Hom}_A(M\otimes_B T, A)\otimes_A S \arrow[r, "\tau_{M\otimes_B T,S}"'] & \mathrm{Hom}_A(M\otimes_B T, S)
\end{tikzcd}$$

where the top vertical map is induced by the product

$$\mathrm{Hom}_{\mathcal{O}}(T, A) \otimes_A S \to \mathrm{Hom}_{\mathcal{O}}(T, S).$$

We apply Lemma 2.3 to M and T^*: the map $\tau_{M,\mathrm{Hom}_{\mathcal{O}}(T,A)}$ is an isomorphism. Consequently, $\tau_{M,\mathrm{Hom}_{\mathcal{O}}(T,S)}$ is non-zero if and only if $\tau_{M\otimes_B T,S}$ is non-zero and the proposition follows from Proposition 2.1. $\square$

2.3.4. Composition. Let A' be a symmetric $\mathcal{O}$-algebra, M an exact (A, B)-bimodule and N an exact (B, A')-bimodule. If M and N induce Morita equivalences, then $M \otimes_B N$ induces a Morita equivalence between A and A'. Rickard equivalences and stable equivalences can be similarly composed.

2.3.5. Comparison. If M induces a Morita equivalence, then it induces a Rickard equivalence. If C induces a Rickard equivalence, then it induces a stable equivalence.

Let M be an exact (A, B)-bimodule inducing a stable equivalence. Assume A and B have no projective direct summands as bimodules. Then M induces a Morita equivalence if and only if $M \otimes_B S$ is simple for every simple B-module S (for then we know that B is a direct summand of $M^* \otimes_A M$ and $M^* \otimes_A M \otimes_B S$ is indecomposable for every simple B-module S, so $M^* \otimes_A M \simeq B$. We have $M \otimes_B M^* \simeq A \oplus Z$, whence $M \otimes_B M^* \otimes_A Z = 0$; thus $Z = 0$).

Let C be a complex of exact (A, B)-bimodules. Assume that all terms of C are projective but C^r and that C^r induces a stable equivalence. Then C induces a stable equivalence.

Let $\Omega_{A\otimes A^\circ}$ be the kernel of the multiplication map $A \otimes A \to A$. This is an (A, A)-bimodule inducing a self-stable equivalence of A.

For U an A-module, we denote by $\Omega_A U$ (or ΩU) an A-module without $\mathcal{O}$-projective direct summand such that $\Omega_{A\otimes A^\circ} \otimes_A U = \Omega U \oplus \mathcal{O}$-projective. Let $f : P_U \to U$ be a surjective map with P_U an $\mathcal{O}$-projective A-module. If f splits as a morphism of $\mathcal{O}$-modules, then $\ker f \oplus \mathcal{O}$-projective $\simeq \Omega U \oplus \mathcal{O}$-projective.

We define inductively $\Omega^n U$ as $\Omega(\Omega^{n-1}U)$ for n positive. Similarly, using $\Omega^{-1}_{A\otimes A^\circ} = \Omega^*_{A\otimes A^\circ}$, we define $\Omega^{-1}U$ and $\Omega^{-n}U$ for n positive. Finally, $\Omega^0 U = U$.

For M an exact (A, B)-bimodule, the (A, B)-bimodule $(A \otimes A^\circ) \otimes_A M$ is projective. Hence

$$\Omega_{A\otimes A^\circ} \otimes_A M \oplus \text{ projective} \simeq \Omega_{A\otimes B^\circ} M \oplus \text{ projective}.$$

Let M be an exact (A, B)-bimodule. Then

$$\Omega^n_{A\otimes A^\circ} A \otimes_A M \oplus \text{ projective} \simeq \Omega^n_{A\otimes B^\circ} M \oplus \text{ projective} .$$

So, if M induces a stable equivalence, then $\Omega^n_{A\otimes B^\circ} M$ also induces a stable equivalence.

The next proposition explains how to construct a Rickard equivalence from a Morita equivalence by truncating a projective resolution of the bimodule.

Proposition 2.6. *Let M be an exact (A, B)-bimodule. Let C be an complex of exact (A, B)-bimodules with homology only in degree 0, isomorphic to M, with zero terms outside $\{0, \ldots, r-1, r\}$ and with projective terms in degrees $0, \ldots, r-1$ where r is an integer.*

If M induces a Morita equivalence, then C induces a Rickard equivalence.

Proof. Assume first r is non-positive. We have

$$C^r \oplus \text{ projective} \simeq \Omega^{-r}_{A\otimes B^\circ} M \oplus \text{ projective};$$

therefore C^r induces a stable equivalence. Since C^i is projective for $i \neq r$, it follows that C induces a stable equivalence. In particular, the kernel of ε_C is homotopy equivalent to a complex of projective modules Z.

The homology of C is projective over B; thus the homology of $C \otimes_B C^*$ is isomorphic to $H^0(C) \otimes_B H^0(C)^*$ (in degree 0). Since $H^0(C) \simeq M$ induces a Morita equivalence, it follows that $C \otimes_B C^*$ has homology only in degree 0, isomorphic to A. More precisely, the kernel of ε_C has zero homology.

The complex Z is a (bounded) complex of projective modules with zero homology, whence it is homotopy equivalent to 0.

Similarly, one shows that the kernel of ε_{C^*} is homotopy equivalent to 0.

The case where r is positive follows from the negative case by replacing A, B, M, C and r by B, A, M^*, C^* and $-r$. $\square$

2.3.6. Extension of scalars. Let $\mathcal{O}'$ be a commutative $\mathcal{O}$-algebra. Let $A' = \mathcal{O}' \otimes_{\mathcal{O}} A$, $B' = \mathcal{O}' \otimes_{\mathcal{O}} B$: these are symmetric $\mathcal{O}'$-algebras.

An isomorphism $A \xrightarrow{\sim} B$ gives rise to an isomorphism $A' \xrightarrow{\sim} B'$ by extending scalars from $\mathcal{O}$ to $\mathcal{O}'$. More generally, given an (A, B)-bimodule M inducing a Morita equivalence between A and B, the (A', B')-bimodule $A' \otimes_A M \otimes_B B'$ induces a Morita equivalence between A' and B'. We have similar statements for Rickard and stable equivalence.

3. Some steps in block theory

3.1. The group algebra

Let us start gathering some properties that do not involve blocks. We take special care to provide explicit isomorphisms when studying the TI case in §3.1.5. This way, we avoid use of the Krull–Schmidt Theorem and we can work over a non-complete ring $\mathcal{O}$.

3.1.1. Symmetric algebra structure. We have an $\mathcal{O}$-linear trace on the group algebra

$$t : \mathcal{O}G \to \mathcal{O},\ g \mapsto \delta_{1g} \text{ for } g \in G.$$

Since $t(g'g^{-1}) = \delta_{gg'}$, the form is symmetrizing. The basis dual to $\{g\}_{g\in G}$ is $\{g^{-1}\}_{g\in G}$.

3.1.2. Let H be a subgroup of G and $M = \mathcal{O}G$ the exact $(\mathcal{O}H, \mathcal{O}G)$-bimodule where the actions are given by multiplication. The functor $\mathrm{Res}_H^G = M \otimes_{\mathcal{O}G} -$ is the restriction functor from $\mathcal{O}G$-mod to $\mathcal{O}H$-mod. It is an exact functor.

We have an isomorphism $\hat{t} : \mathcal{O}G \xrightarrow{\sim} M^*$, where $\mathcal{O}G$ is the $(\mathcal{O}G, \mathcal{O}H)$-bimodule with actions given by multiplication. The corresponding functor $\mathrm{Ind}_H^G = M^* \otimes_{\mathcal{O}H} -$ is the induction functor from $\mathcal{O}H$-mod to $\mathcal{O}G$-mod. It is also an exact functor and Ind_H^G is left and right adjoint to Res_H^G.

3.1.3. The counit ε_{M^*} is the surjective morphism given by multiplication

$$\varepsilon_{M^*} : \mathcal{O}G \otimes_{\mathcal{O}H} \mathcal{O}G \to \mathcal{O}G,\ \ g \otimes g' \mapsto gg'.$$

Assume $[G : H]$ is invertible in $\mathcal{O}$. Then

$$\mathcal{O}G \to \mathcal{O}G \otimes_{\mathcal{O}H} \mathcal{O}G,\ g \mapsto \frac{1}{[G : H]} g \sum_{x \in G/H} x \otimes x^{-1}$$

is a splitting to the surjection, i.e., the morphism of functors $\varepsilon : \mathrm{Ind}_H^G \mathrm{Res}_H^G \to 1_G$ is a split surjection.

Now let U and V be two $\mathcal{O}G$-modules. Then U is a direct summand of $\operatorname{Ind}_H^G \operatorname{Res}_H^G U$. So

$$\operatorname{Ext}^1_{\mathcal{O}G}(U, V) \leq \operatorname{Ext}^1_{\mathcal{O}G}(\operatorname{Ind}_H^G \operatorname{Res}_H^G U, V) \simeq \operatorname{Ext}^1_{\mathcal{O}H}(\operatorname{Res}_H^G U, \operatorname{Res}_H^G V),$$

where the isomorphism comes from the fact that Ind_H^G is an exact functor which is a left adjoint to the exact functor Res_H^G.

As a special case, let us take for H a Sylow p-subgroup of G and for $\mathcal{O}$ a field k of characteristic p.

If H is trivial, we deduce that all Ext^1-groups are zero in kG-mod. So we obtain Maschke's theorem.

Proposition 3.1. *If k is a field and $|G|$ is invertible in k, then kG is semi-simple.*

More generally the "homological complexity" of kG-mod is measured by a Sylow p-subgroup P:

- kG has finite representation type (i.e., there are only finitely many isomorphism classes of indecomposable modules) if and only if kP has finite representation type; this is known to happen exactly when P is cyclic.

- kG is tame (i.e., indecomposable modules are in some sense classifiable) if and only if kP is tame; that happens exactly when $p = 2$ and P is a dihedral, semidihedral or generalized quaternion group.

Note that in all other cases kG is wild. So instead of pursuing the unreasonable task of describing kG-mod completely, we will try to compare it with module categories of smaller groups.

3.1.4. Mackey's formula. Composition of an induction functor followed by a restriction functor is described by Mackey's formula as a sum of compositions of a restriction functor followed by an induction functor.

Let H and H' be two subgroups of G. Then $\mathcal{O}G = \bigoplus_{g \in H' \backslash G/H} \mathcal{O}H'gH$ is a decomposition as $(\mathcal{O}H', \mathcal{O}H)$-bimodules.

Let $K = H' \cap gHg^{-1}$. Then we have an isomorphism of $(\mathcal{O}H', \mathcal{O}H)$-bimodules

$$\mathcal{O}H' \otimes_{\mathcal{O}K} (\mathcal{O}H)^g \xrightarrow{\sim} \mathcal{O}H'gH,\ x \otimes y \mapsto xgy$$

where $(\mathcal{O}H)^g = \mathcal{O}H$ as a right $\mathcal{O}H$-module and the action of $a \in \mathcal{O}K$ is given by left multiplication by $g^{-1}ag$.

So we have constructed an isomorphism of $(\mathcal{O}H', \mathcal{O}H)$-bimodules

$$\mathcal{O}G \xrightarrow{\sim} \bigoplus_{g \in H' \backslash G/H} \mathcal{O}H' \otimes_{\mathcal{O}(H' \cap gHg^{-1})} (\mathcal{O}H)^g.$$

In terms of functors, this is the usual Mackey's formula

$$\mathrm{Res}^G_{H'} \circ \mathrm{Ind}^G_H \xrightarrow{\sim} \bigoplus_{g \in H' \backslash G/H} \mathrm{Ind}^{H'}_{H' \cap gHg^{-1}} \circ \mathrm{Res}^{gHg^{-1}}_{H' \cap gHg^{-1}} \circ g^*$$

where $g^* : \mathcal{O}H\text{-mod} \to \mathcal{O}(gHg^{-1})\text{-mod}$ is restriction via the isomorphism

$$gHg^{-1} \to H, \quad x \mapsto g^{-1}xg.$$

3.1.5. TI Sylow *p*-subgroups. We will see here our first comparison result.

Let us assume that G has trivial intersection (TI) Sylow p-subgroups: given two distinct Sylow p-subgroups P and Q, then $P \cap Q = \{1\}$.

Let P be a Sylow p-subgroup of G and $H = N_G(P)$. We denote by $\mathbb{Z}_p$ the localization of $\mathbb{Z}$ at the prime p. Let N be the $(\mathbb{Z}_pG, \mathbb{Z}_pH)$-bimodule $\mathbb{Z}_pG$. Then $N^* = \mathbb{Z}_pG$ (the identification is made via $\hat{t}$).

We have a split exact sequence of $(\mathbb{Z}_pH, \mathbb{Z}_pH)$-bimodules (cf. §3.1.4)

$$0 \to \mathbb{Z}_pH \to \mathbb{Z}_pG \to \bigoplus_g \mathbb{Z}_pH \otimes_{\mathbb{Z}_pK} (\mathbb{Z}_pH)^g \to 0,$$

where $g \in H \setminus G/H$, $g \notin H$ and $K = H \cap gHg^{-1}$ and where the first map is the inclusion.

For $g \in G$, $g \notin H$, we have $P \cap gPg^{-1} = \{1\}$; thus K is a p'-group. It follows that $\mathbb{Z}_pH$ is a projective $(\mathbb{Z}_pH, \mathbb{Z}_pK)$-bimodule and $(\mathbb{Z}_pH)^g$ a projective $(\mathbb{Z}_pK, \mathbb{Z}_pH)$-bimodule, whence $\mathbb{Z}_pH \otimes_{\mathbb{Z}_pK} (\mathbb{Z}_pH)^g$ is a projective $(\mathbb{Z}_pH, \mathbb{Z}_pH)$-bimodule.

Since the counit

$$\eta_N : \mathbb{Z}_pH \to N^* \otimes_{\mathbb{Z}_pG} N = \mathbb{Z}_pG$$

is the inclusion, we have a split exact sequence of $(\mathbb{Z}_pH, \mathbb{Z}_pH)$-bimodules

$$0 \to \mathbb{Z}_pH \xrightarrow{\eta_N} N^* \otimes_{\mathbb{Z}_pG} N \to Z \to 0,$$

where Z is projective.

Since $[G : H]$ is invertible in $\mathbb{Z}_p$, the surjection $\varepsilon_N : N \otimes_{\mathbb{Z}_pH} N^* \to \mathbb{Z}_pG$ splits. Let Z' be its kernel.

The composition

$$N^* \xrightarrow{\eta_N \otimes 1} N^* \otimes_{\mathbb{Z}_pG} N \otimes_{\mathbb{Z}_pH} N^* \xrightarrow{1 \otimes \varepsilon_N} N^*$$

is the identity. So

$$N^* \otimes_{\mathbb{Z}_pG} Z' = \ker(1 \otimes \varepsilon_N) \simeq \mathrm{coker}(\eta_N \otimes 1) = Z \otimes_{\mathbb{Z}_pH} N^*,$$

whence

$$N \otimes_{\mathbb{Z}_pH} N^* \otimes_{\mathbb{Z}_pG} Z' \simeq N \otimes_{\mathbb{Z}_pH} Z \otimes_{\mathbb{Z}_pH} N^*.$$

As $\mathbb{Z}_p G$ is a direct summand of $N \otimes_{\mathbb{Z}_p H} N^*$, it follows that Z' is a direct summand of $N \otimes_{\mathbb{Z}_p H} Z \otimes_{\mathbb{Z}_p H} N^*$. That last module is projective since Z is projective: this shows the projectivity of Z'.

We have obtained the following isomorphisms

$$N \otimes_{\mathbb{Z}_p H} N^* \simeq \mathbb{Z}_p G \oplus \text{ projective (as } (\mathbb{Z}_p G, \mathbb{Z}_p G)\text{-bimodules)}$$

$$N^* \otimes_{\mathbb{Z}_p G} N \simeq \mathbb{Z}_p H \oplus \text{ projective (as } (\mathbb{Z}_p H, \mathbb{Z}_p H)\text{-bimodules).}$$

So we have

Proposition 3.2. *The bimodule $\mathbb{Z}_p G$ induces a stable equivalence between $\mathbb{Z}_p G$ and $\mathbb{Z}_p H$.*

3.2. Blocks

The representation theory of $\mathcal{O}G$ reduces naturally to the study of the representations of the blocks of $\mathcal{O}G$. Some blocks can have a much simpler structure than others. Furthermore, most interesting equivalences arise between blocks, not between the whole group algebras.

3.2.1. A *block idempotent* b of $\mathcal{O}G$ is a primitive idempotent of the center $Z(\mathcal{O}G)$ of $\mathcal{O}G$: $b^2 = b \neq 0$ and there do not exist idempotents b_1 and b_2 of $Z(\mathcal{O}G)$ with $b_1 b_2 = 0$ and $b = b_1 + b_2$. Let $\mathcal{B}$ be the set of block idempotents of $\mathcal{O}G$. Then we have

$$Z(\mathcal{O}G) = \bigoplus_{b \in \mathcal{B}} bZ(\mathcal{O}G).$$

This is the unique decomposition of $Z(\mathcal{O}G)$ as a direct sum of local rings. Note that $1 = \sum_b b$.

We now have the block decomposition of the group algebra

$$\mathcal{O}G = \bigoplus_b b\mathcal{O}G.$$

This is the unique decomposition of $\mathcal{O}G$ as a direct sum of indecomposable $\mathcal{O}$-algebras or, equivalently, the unique decomposition of $\mathcal{O}G$ as a direct sum of indecomposable $(\mathcal{O}G, \mathcal{O}G)$-bimodules. The (non-unitary) subalgebras $b\mathcal{O}G$ of $\mathcal{O}G$ are the *blocks* of $\mathcal{O}G$.

We now have a decomposition $\mathcal{O}G\text{-mod} = \bigoplus_b b\mathcal{O}G\text{-mod}$: every $\mathcal{O}G$-module M splits as $M = \bigoplus_b bM$ where $bM = b\mathcal{O}G \otimes_{\mathcal{O}G} M$. In particular, a non-zero indecomposable module belongs to a unique block. The *principal block* of $\mathcal{O}G$ is the block containing the trivial $\mathcal{O}G$-module.

Assume $|G|$ is invertible in $\mathcal{O}$. Then $e_1 = \frac{1}{|G|}\sum_{g\in G} g$ is the principal block idempotent and $e_1\mathcal{O}G \simeq \mathcal{O}$. If in addition $\mathcal{O}$ is a field, then the blocks are simple algebras.

3.2.2. Normal p'-subgroups. Let $H = O_{p'}(G)$, the largest normal subgroup of G whose order is prime to p and $e = \frac{1}{|H|}\sum_{h\in H} h$. Then e is an idempotent of $Z(\mathbb{Z}_p G)$.

Let $\bar{G} = G/H$. We have an isomorphism

$$e\mathbb{Z}_p G \xrightarrow{\sim} \mathbb{Z}_p \bar{G}, \quad eg \mapsto gH.$$

We have compared $\mathbb{Z}_p\bar{G}$ with a direct summand $e\mathbb{Z}_p G$ (i.e., a sum of blocks) of $\mathbb{Z}_p G$. This is compatible with blocks.

Hypothesis 1. For the remainder of the article, we assume that the residue field k of $\mathcal{O}$ has characteristic p.

The isomorphism $e\mathcal{O}G \xrightarrow{\sim} \mathcal{O}\bar{G}$ induces a bijection between the set of block idempotents of $\mathcal{O}\bar{G}$ and the set of those block idempotents b of $\mathcal{O}G$ such that $be = b$. We then have an isomorphism between the corresponding blocks of $\mathcal{O}G$ and $\mathcal{O}\bar{G}$. For example, we obtain an isomorphism between the principal blocks of $\mathcal{O}G$ and $\mathcal{O}\bar{G}$.

Note that the discussion above remains unchanged if we take for H any normal subgroup of G with order prime to p.

As a special case, assume G is p-nilpotent, i.e., $G = H \rtimes P$ where P is a Sylow p-subgroup. Then we have an isomorphism between $\mathcal{O}P$ and the principal block of $\mathcal{O}G$.

3.2.3. Blockwise version of the TI equivalence. We go back to the assumption of §3.1.5 that G has TI Sylow p-subgroups and p divides the order of G. Let f be a block idempotent of $\mathcal{O}H$ (recall that $H = N_G(P)$ where P is a Sylow p-subgroup of G). There is a unique block idempotent e of $\mathcal{O}G$ such that $eNf = e\mathcal{O}Gf$ is not a projective $(\mathcal{O}G, \mathcal{O}H)$-bimodule.

Then

Proposition 3.3. *The bimodule $e\mathcal{O}Gf$ induces a stable equivalence between $e\mathcal{O}G$ and $f\mathcal{O}H$.*

Note that the blocks of $\mathcal{O}G$ which do not correspond to blocks of $\mathcal{O}H$ are stably equivalent to 0.

In general, $e\mathcal{O}G$ and $f\mathcal{O}H$ are not Morita equivalent (i.e., you cannot get rid of the projective “remainder”), although they might be in some exceptional cases. Let us give two such cases.

Assume $p = 3$, $G = \mathfrak{S}_4$ and e is the principal block idempotent of $\mathbb{Z}_p G$. Then $P \simeq \mathbb{Z}/3$ and $H \simeq \mathfrak{S}_3$. By §3.2.2, we have an isomorphism $e\mathbb{Z}_p G \simeq \mathbb{Z}_p H$.

Assume now $p = 3$, $G = \mathfrak{S}_5$ and e is the principal block idempotent. We have $P \simeq \mathbb{Z}/3$, $H \simeq \mathfrak{S}_3 \times \mathbb{Z}/2$ and $f\mathbb{Z}_p H \simeq \mathbb{Z}_p\mathfrak{S}_3$. One can check that $e\mathbb{Z}_p G$ is Morita equivalent to $f\mathbb{Z}_p H$, but the algebras are not isomorphic (they have different $\mathbb{Z}_p$-ranks!); there is a direct summand M of $e\mathbb{Z}_p Gf$ inducing such a Morita equivalence.

3.2.4. The TI case suggests that isomorphisms or Morita equivalences are too narrow concepts in order to compare blocks. On the other hand, it is difficult to deduce much numerical information from the existence of a stable equivalence although it is expected that the number of non-projective simple modules will be invariant when $\mathcal{O}$ is a field (Auslander's conjecture).

4. The Brauer functor

The Brauer functor is a fundamental tool to pass from global to local data.

4.1. *p*-permutation modules

4.1.1. Let Q be a p-subgroup of G. We denote by Br_Q the *Brauer functor* $\mathrm{Br}_Q : \mathcal{O}G\text{-mod} \to kN_G(Q)\text{-mod}$, defined as follows.

For U an $\mathcal{O}G$-module, define

$$\mathrm{Br}_Q(U) = U^Q \Big/ \Big(\Big(\sum_{P<Q} \mathrm{Tr}_P^Q\, U^P \Big) + \mathfrak{p}U^Q \Big),$$

where the trace map $\mathrm{Tr}_P^Q : U^P \to U^Q$ between fixed point sets is given by $v \mapsto \sum_{g\in Q/P} gU$, and where $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$.

We will also consider the extension of Br_Q to the category of complexes of $\mathcal{O}G$-modules.

4.1.2. Let Ω be a G-set and $\mathcal{O}\Omega$ the corresponding permutation $\mathcal{O}G$-module. Then the inclusion $\mathcal{O}(\Omega^Q) \hookrightarrow (\mathcal{O}\Omega)^Q$ induces an isomorphism $k(\Omega^Q) \xrightarrow{\sim} \mathrm{Br}_Q(\mathcal{O}\Omega)$.

Let H be a subgroup of G. Then $(G/H)^Q \neq \emptyset$ if and only if Q is conjugate to a subgroup of H. So $\mathrm{Br}_Q(\mathrm{Ind}_H^G \mathcal{O}) \neq 0$ if and only if Q is conjugate to a subgroup of H.

Let V be a $\mathcal{O}N_G(Q)/Q$-module. We have (by adjunction) a morphism

$$\mathrm{Res}_{N_G(Q)}^{N_G(Q)/Q} V \to (\mathrm{Ind}_{N_G(Q)}^G \mathrm{Res}_{N_G(Q)}^{N_G(Q)/Q} V)^Q$$

and hence, by composition, a morphism

$$\mathrm{Res}_{N_G(Q)}^{N_G(Q)/Q} V \to \mathrm{Br}_Q \mathrm{Ind}_{N_G(Q)}^G \mathrm{Res}_{N_G(Q)}^{N_G(Q)/Q} V.$$

This gives a morphism of endofunctors of $\mathcal{O}N_G(Q)/Q$-mod

$$1_{\mathcal{O}N_G(Q)/Q\text{-mod}} \to \mathrm{Br}_Q \mathrm{Ind}_{N_G(Q)}^G \mathrm{Res}_{N_G(Q)}^{N_G(Q)/Q}.$$

Furthermore, this is an isomorphism when applied to projective $\mathcal{O}N_G(Q)/Q$-modules.

4.1.3. The Brauer functor is of particular interest when applied to *p-permutation modules* (direct summands of permutation modules).

If U is an indecomposable p-permutation $\mathcal{O}G$-module, then there is a minimal subgroup P of G such that the surjection $\mathcal{O}G \otimes_{\mathcal{O}P} \mathcal{O}G \otimes_{\mathcal{O}G} U \to U$ splits. This is a p-subgroup of G, called a *vertex* of U. It is unique up to conjugation. It is also characterized (up to conjugation) as the minimal subgroup of G such that U is a direct summand of a module induced from P or as the maximal subgroup of G such that $\mathrm{Br}_P(U) \neq 0$.

4.2. The Brauer morphism

For H a group, we define a subgroup ΔH of $H \times H^\circ$ by $\Delta H = \{(x, x^{-1}) | \ x \in H\}$. Let Q be a p-subgroup of G. The surjection

$$\mathrm{br}_Q : (\mathcal{O}G)^{\Delta Q} \to \mathrm{Br}_{\Delta Q}(\mathcal{O}G) = kC_G(Q)$$

is the *Brauer morphism*. This is a morphism of $((\mathcal{O}G)^{\Delta Q}, (\mathcal{O}G)^{\Delta Q})$-bimodules, and hence a morphism of algebras. It restricts to a (not necessarily surjective) morphism

$$\mathrm{br}_Q : Z(\mathcal{O}G) \to Z(kC_G(Q)).$$

Let $z \in Z(\mathcal{O}G)$. Then multiplication by z defines an endomorphism of any $\mathcal{O}G$-module U and the corresponding endomorphism of $\mathrm{Br}_Q(U)$ is multiplication by $\mathrm{br}_Q(z)$.

For example, if b is an idempotent of $Z(\mathcal{O}G)$, then we can consider the $b\mathcal{O}G$-module $bU = b\mathcal{O}G \otimes_{\mathcal{O}G} U$. We have

$$\mathrm{Br}_Q(bU) = \mathrm{br}_Q(b) \cdot \mathrm{Br}_Q(U).$$

If b is the principal block idempotent of $\mathcal{O}G$, then $\mathrm{br}_Q(b)$ is the principal block idempotent of $kC_G(Q)$.

4.3. Defect of blocks

It is now time to turn to defect groups of blocks!

Let e be a block idempotent of $\mathcal{O}G$. A *defect group* of $e\mathcal{O}G$ is a subgroup D of G such that ΔD is a vertex of the $\mathcal{O}(G \times G^\circ)$-module $e\mathcal{O}G$ — i.e., this is a subgroup of G minimal with respect to the property that the multiplication map

$$e\mathcal{O}G \otimes_{\mathcal{O}D} \mathcal{O}G \to e\mathcal{O}G$$

splits. This is also a subgroup of G maximal with respect to the property that $\mathrm{br}_D(e) \neq 0$.

If $e\mathcal{O}G$ is the principal block, then D is a Sylow p-subgroup.

We can now refine the discussion of §3.1.3: the complexity of ekG is accounted for by D (ekG is semi-simple (and then simple) if and only if $D = 1$, has finite representation type if and only if D is cyclic, etc.).

The following conditions for a block are equivalent:

(i) $D = 1$;

(ii) $e\mathcal{O}G$ is a projective $(\mathcal{O}G, \mathcal{O}G)$-bimodule;

(iii) $e\mathcal{O}G$ is stably equivalent to 0.

When $\mathcal{O} = k$, this is furthermore equivalent to the fact that ekG is a simple algebra.

Defining the (numerical) defect of $e\mathcal{O}G$ to be $\log_p |D|$, we see that the blocks fulfilling those conditions are the blocks with defect 0.

5. Rickard equivalences

From here on, we will consider the usual setting for modular representation theory.

Hypothesis 2. We assume $\mathcal{O}$ is a discrete valuation ring containing all $|I|$-th roots of unity, for all the finite groups I to be considered.

5.1. An example: A_5 in characteristic 2

Let G be the alternating group A_5, $p = 2$ and D be a Sylow 2-subgroup of G ($D \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$). Let $H = N_G(D)$: we have $H \simeq A_4$. Let E be a cyclic subgroup of order 3 of H. Then $H = D \rtimes E$. The algebra $\mathcal{O}H$ is indecomposable.

Let e be the principal block idempotent of $\mathcal{O}G$. Then $(1-e)\mathcal{O}G$ is a block of defect zero — it is actually a 5-dimensional matrix algebra over $\mathcal{O}$. The Sylow 2-subgroups of G are TI; thus we know from §3.2.3 that the bimodule $M = e\mathcal{O}G$ induces a stable equivalence between $A = e\mathcal{O}G$ and $B = \mathcal{O}H$.

The non-trivial simple B-modules lift to B-modules free over $\mathcal{O}$, whereas the non-trivial simple A-modules V_1 and V_2 do not lift to $\mathcal{O}$-free A-modules; in particular, A and B are not Morita equivalent (the algebras $k \otimes A$ and $k \otimes B$ are not Morita equivalent either: they have distinct Cartan matrices).

The module $M^* \otimes_A V_i = \operatorname{Res}^G_H V_i$ is an indecomposable two-dimensional B-module: let S_i be its unique simple submodule. Then S_1 and S_2 are the non-trivial simple B-modules.

Lemma 5.1. *A projective cover of M is*

$$P_M = P_{k_A} \otimes P^*_{k_B} \oplus P_{V_1} \otimes P^*_{S_1} \oplus P_{V_2} \otimes P^*_{S_2},$$

where we denote by P_L a projective cover of the module L.

Proof. Let V be a simple A-module and S a simple B-module. We have an isomorphism of (B, B)-bimodules (§2.2.1)

$$\mathrm{Hom}_k(V^* \otimes_A M, S^*) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_k(V^*, S^*)),$$

whence

$$\mathrm{Hom}_{B^\circ}(V^* \otimes_A M, S^*) \simeq \mathrm{Hom}_{A\otimes B^\circ}(M, \mathrm{Hom}_k(V^*, S^*)) \simeq \mathrm{Hom}_{A\otimes B^\circ}(M, V \otimes S^*).$$

Finally,

$$\mathrm{Hom}_{B^\circ}(V^* \otimes_A M, S^*) \simeq \mathrm{Hom}_B(S, M^* \otimes_A V). \qquad \square$$

Let $f : P_M \to M$ be a surjection and let δ be its restriction to $R = P_{V_1} \otimes P^*_{S_1} \oplus P_{V_2} \otimes P^*_{S_2}$.

Let C be the complex

$$C = 0 \to R \xrightarrow{\delta} M \to 0,$$

where M is in degree 0.

As shown by Rickard, we have

Proposition 5.2. *The complex C induces a Rickard equivalence between A and B.*

Proof. Let us consider the double complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \otimes_B M^* \arrow[r, "\delta\otimes 1"] \arrow[d, "1\otimes\delta^*"'] & M \otimes_B M^* \arrow[r] \arrow[d, "1\otimes\delta^*"] & 0 \\
0 \arrow[r] & R \otimes_B R^* \arrow[r, "\delta\otimes 1"'] \arrow[d] & M \otimes_B R^* \arrow[r] \arrow[d] & 0 \\
 & 0 & 0\,. &
\end{tikzcd}$$

We have

$$R \otimes_B M^* \simeq M \otimes_B R^* \simeq \bigoplus_{i,j} P_{V_i} \otimes P^*_{V_j},$$

$$R \otimes_B R^* \simeq \bigoplus_{i} P_{V_i} \otimes P^*_{V_i} \oplus \bigoplus_{i,j} P_{V_i} \otimes P^*_{V_j}$$

$$\text{and } M \otimes_B M^* \simeq A \oplus \bigoplus_{i\neq j} P_{V_i} \otimes P^*_{V_j}.$$

We have a split surjection $f \otimes 1 : P_M \otimes_B R^* \to M \otimes_B R^*$. Since $(P_M/R) \otimes_B R^*$ and $M \otimes_B R^*$ have no common non-zero direct summand, it follows that the map $\delta \otimes 1 : R \otimes_B R^* \to M \otimes_B R^*$ is still a split surjection.

Similarly, $1 \otimes \delta^* : R \otimes_B M^* \to R \otimes_B R^*$ is a split injection.

Let us consider now the complex $C \otimes_B C^*$, i.e., the total complex associated to the double complex above,

$$C \otimes_B C^* = 0 \to R \otimes_B M^* \xrightarrow{\delta\otimes 1 - 1\otimes\delta^*} M \otimes_B M^* \oplus R \otimes_B R^* \xrightarrow{1\otimes\delta^* + \delta\otimes 1} M \otimes_B R^* \to 0.$$

This complex is homotopy equivalent to its 0-th homology and

$$H^0(C \otimes_B C^*) \oplus R \otimes_B M^* \oplus M \otimes_B R^* \simeq R \otimes_B R^* \oplus M \otimes_B M^*.$$

It follows that $H^0(C \otimes_B C^*) \simeq A$; thus $C \otimes_B C^*$ is homotopy equivalent to A.

A similar proof shows that $C^* \otimes_A C$ is homotopy equivalent to B. □

This means that we have been able to get rid of the projective "remainder" by suitably modifying M into C. In order to achieve this, we had to move from modules to complexes of modules — more precisely, to the homotopy category of complexes of modules.

5.2. Broué's conjecture

We present here the abelian defect conjecture of Broué and its expected compatibilities with p'-outer automorphism groups and central extensions by p-groups.

5.2.1. Let us now fix our objects of study.

Hypothesis 3. From now on, we assume Hypothesis 2 and we denote by G a finite group, by e a block idempotent of $\mathcal{O}G$ and by D a defect group of $e\mathcal{O}G$. We put $H = N_G(D)$ and we denote by f the block idempotent of $\mathcal{O}H$ corresponding to e (it is the unique block idempotent with the property that $e\mathcal{O}G$ is a direct summand of $\mathrm{Ind}_{H\times H^\circ}^{G\times G^\circ} f\mathcal{O}H$). We put $A = e\mathcal{O}G$ and $B = f\mathcal{O}H$.

Following Rickard, we say that a complex C of $(e\mathcal{O}G, f\mathcal{O}H)$-bimodules is *splendid* if its components are p-permutation modules whose indecomposable summands have vertices contained in ΔD (note that the components are then exact bimodules). The relevance of this definition will appear in §5.3.

We can now state

Conjecture 5.3 (Broué). *Assume D is abelian. Then $e\mathcal{O}G$ and $f\mathcal{O}H$ are splendidly Rickard equivalent.*

Some remarks.

- See the Appendix for comments on the notion of splendidness.
- It is unclear whether there should be a more natural equivalence.

- Not every equivalence is splendid.
- The form of the conjecture given here is a refinement due to Rickard.
- The conjecture is known to fail when D is not abelian, even if the Sylow p-subgroups of G are TI, as in the case $G = \mathrm{Sz}(8)$ and $p = 2$. It remains an open problem to find an extension of the conjecture to blocks with non-abelian defect groups.

When e and f are principal block idempotents, then it is conjectured that there is a splendid Rickard complex C with $C \otimes_{f\mathcal{O}H} \mathcal{O} \simeq \mathcal{O}$. Such an equivalence is called a *normalized equivalence*. For example, the construction of §5.1 gives a positive answer for $G = A_5$, $p = 2$ and e the principal block idempotent.

5.2.2. Let us try to give the current status of the conjecture.

The conjecture holds for

- D cyclic [Ri1, Li1, Rou2] and $D \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ ([Ri4, Li2, Li3, Rou3] and §6.3 for principal blocks)
- G p-solvable [Da1, Pu1, HaLi]
- G a connected reductive algebraic group over $\mathbf{F}_q$ and p divides $q-1$ but p does not divide the order of the Weyl group [Pu2]
- G a symmetric group and $D \simeq \mathbb{Z}/p \times \mathbb{Z}/p$ [Ch1]
- several more cases where G a symmetric group, e.g., when $D \simeq (\mathbb{Z}/p)^r$ with $r \leq 5$ [Ri2] and [ChKe]
- $G = \mathrm{GL}_n(q)$, $p \nmid q$ and D has p-rank 2 [HiMi, Tu] and [BoRou],

for the principal blocks of

- $G = {}^2G_2(q)$ and $p = 2$ [Ok1]
- $G = \mathrm{Sp}_4(q)$, $q \equiv 2, 5 \pmod 9$ and $p = 3$ [Ok1]
- $G = \mathrm{PSU}_3(q^2)$, $q \equiv 2, 5 \pmod 9$ and $p = 3$ [KoKu1]
- $G = \mathrm{PSL}_3(q)$, $q \equiv 4, 7 \pmod 9$ and $p = 3$ [Ku]
- $G = A_7$, A_8, M_{11}, M_{22}, M_{23}, $\mathrm{PSL}_3(4)$ and HS, $p = 3$ [Ok1]
- $G = J_2$ and $G = \mathrm{Sp}_4(4)$ and $p = 5$ [Holl]
- $G = J_1$ and $p = 2$ [GoOk]
- any group G with $D \simeq \mathbb{Z}/3 \times \mathbb{Z}/3$ [KoKu2]
- $G = \mathrm{PSL}_2(p^n)$ ([Ch2] for $n = 2$, [Rou1] for $p^n = 8$ and [Ok2] in general)

- $G = \mathrm{GL}_4(q)$ and $G = \mathrm{GL}_5(q)$, $q \equiv 2, 5 \pmod 9$ and $p = 3$ [KoMi]
- $G = \mathrm{SU}_3(q^2)$, $p > 3$ and $p | q + 1$ [KuWa],

and for the non-principal blocks of

- $G = ON$ and $p = 3$ [KoKuWa]
- $G = HS$ and $p = 3$ [Holm, KoKuWa]
- $G = 2.J_2$ and $p = 5$ [Holl]
- $G = \mathrm{SL}_2(p^2)$ [Holl].

5.2.3. We now consider automorphisms.

Hypothesis 3′. Hypothesis 3 holds and we let X be a finite group containing G as a normal subgroup and $Y = N_X(D)$. We assume that e is X-invariant. Then $X/G = Y/H$, and we assume that this group F is a p'-group. We put $\Delta = \{(g, h) \in X \times Y^\circ \mid (gG, hH^\circ) \in \Delta F\}$.

Then it is conjectured that there is a complex C of $\mathcal{O}\Delta$-modules whose restriction to $e\mathcal{O}G \otimes (f\mathcal{O}H)^\circ$ is a splendid Rickard complex. By Marcus [Ma] (or [Rou2, Lemma 2.8]), the complex $\mathrm{Ind}_\Delta^{X\times Y^\circ} C$ is then a splendid Rickard complex.

Remark 5.4. If F is not a p'-group, the same proof shows only that $\mathrm{Ind}_\Delta^{X\times Y^\circ} C$ will induce an equivalence between the derived categories of $e\mathcal{O}X$ and $f\mathcal{O}Y$.

Let us state some simple facts related to the extension problem.

Let M be an indecomposable direct summand of the $(e\mathcal{O}G \otimes (f\mathcal{O}H)^\circ)$-module $e\mathcal{O}Gf$ with vertex ΔD and M' with $e\mathcal{O}Gf = M \oplus M'$. Then the indecomposable summands of M' have vertices strictly contained in ΔD. The action of Δ on $\mathcal{O}X$ restricts to an action on $e\mathcal{O}Gf$ extending the natural action of $G \times H^\circ$. It follows that M and M' extend uniquely to $\mathcal{O}\Delta$-modules $\tilde{M}$ and $\tilde{M}'$ with $e\mathcal{O}Gf \simeq \tilde{M} \oplus \tilde{M}'$.

Let M be an $\mathcal{O}\Delta$-module. If $f : P_M \to M$ is a projective cover of M, then $\mathrm{Res}^\Delta_{G\times H^\circ} f$ is a projective cover of $\mathrm{Res}^\Delta_{G\times H^\circ} M$ (if $\mathrm{Res}^\Delta_{G\times H^\circ} \ker f$ has a projective direct summand, then so does $\ker f$).

5.2.4. Finally, we come to central extensions.

Hypothesis 3″. Hypothesis 3′ holds and we consider $\tilde{X}$ a finite group with a normal p-subgroup P such that $\tilde{X}/P = X$. Let $\tilde{G}$ be the inverse image of G in $\tilde{X}$. We assume P is central in $\tilde{G}$. The block idempotent e of $\mathcal{O}G$ lifts to a block idempotent $\tilde{e}$ of $\mathcal{O}\tilde{G}$ (Hensel's lemma in $Z(\mathcal{O}\tilde{G})$). Let $\tilde{Y}$ (resp. $\tilde{H}$) be the inverse image of Y (resp. H) in

$\tilde{X}$. Let $\tilde{f}$ be the block idempotent of $\mathcal{O}\tilde{H}$ lifting f. Let $\tilde{\Delta}$ be the inverse image of Δ in $\tilde{X} \times \tilde{Y}^{\circ}$. Note that ΔP is normal in $\tilde{\Delta}$.

We have a commutative diagram:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & P \arrow[r, equal] \arrow[d] & P \arrow[d] & & \\
1 \arrow[r] & \tilde{G} \arrow[r] \arrow[d] & \tilde{X} \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & F \arrow[r] & 1 \\
 & 1 & 1\,. & &
\end{tikzcd}$$

It is conjectured that there exists a complex $\tilde{C}$ of $\mathcal{O}(\tilde{\Delta}/\Delta P)$-modules that are projective for $\mathcal{O}\tilde{G}$ and $\mathcal{O}\tilde{H}^{\circ}$ such that $\mathrm{Res}_{G\times H^{\circ}}(\tilde{C} \otimes_{P\times P^{\circ}} \mathcal{O})$ is a splendid Rickard complex.

By [Rou2, Lemma 2.11], the complex $\mathrm{Res}^{\tilde{\Delta}/\Delta P}_{\tilde{G}\times\tilde{H}^{\circ}} \tilde{C}$ is then a splendid Rickard complex for $(\tilde{e}\mathcal{O}\tilde{G}, \tilde{f}\mathcal{O}\tilde{H})$ and $\mathrm{Ind}^{\tilde{X}\times\tilde{Y}^{\circ}}_{\tilde{\Delta}} \mathrm{Res}^{\tilde{\Delta}/\Delta P}_{\tilde{\Delta}} \tilde{C}$ a splendid Rickard complex for $(\tilde{e}\mathcal{O}\tilde{X}, \tilde{f}\mathcal{O}\tilde{Y})$.

Remark 5.5. This refined conjecture holds when D is cyclic or $D \simeq (\mathbb{Z}/2)^2$.

5.3. Splendid stable equivalences and local Rickard equivalences

The following result, which is a variation on a classical theme, is a cornerstone to our approach. It is the generalization from the case of local Morita equivalences to the case of local Rickard equivalences of [Br2, Theorem 6.3]. The first implication is due to Rickard and was the motivation for the introduction of the special class of spendid complexes. Given a global splendid stable equivalence, we obtain local Rickard equivalences. The second half shows that in order to check that a global splendid complex induces a stable equivalence, it suffices to check that the associated local complexes induce Rickard equivalences. This follows quickly from a result of Bouc.

From now on (except in the Appendix), we will consider only principal blocks. For the general case as well as for more details, see the Appendix and [Rou3].

Hypothesis 4. Hypothesis 3 holds and we assume furthermore than e is the principal block idempotent of $\mathcal{O}G$.

Now, D is a Sylow p-subgroup of G.

We take a subgroup K of G containing D and controlling the fusion of p-subgroups in G (i.e., for $P \leq D$ and $g \in G$ such that $gPg^{-1} \leq D$, then there exists $h \in K$ and $z \in C_G(P)$ such that $g = hz$). By Burnside's lemma, when D is abelian, we can take $K = H = N_G(D)$.

Let us denote by b the principal block idempotent of $\mathcal{O}K$. For $Q \leq D$, we denote also by e_Q (resp. b_Q) the principal block idempotent of $kC_G(Q)$ (resp. $kC_K(Q)$).

Theorem 5.6. *Let C be a splendid complex of $(e\mathcal{O}G, b\mathcal{O}K)$-bimodules. The following assertions are equivalent.*

(i) *C induces a stable equivalence between $e\mathcal{O}G$ and $b\mathcal{O}K$.*

(ii) *For every non-trivial subgroup Q of D, the complex $\mathrm{Br}_{\Delta Q}(C)$ induces a Rickard equivalence between $e_Q kC_G(Q)$ and $b_Q kC_K(Q)$.*

(ii′) *For every subgroup Q of order p in D, the complex $\mathrm{Br}_{\Delta Q}(C)$ induces a Rickard equivalence between $e_Q kC_G(Q)$ and $b_Q kC_K(Q)$.*

Proof. Let us recall the results of Rickard [Ri4, proof of Theorem 4.1].

- The components of the complex $C \otimes_{\mathcal{O}K} C^*$ are relatively ΔD-projective.

- For $Q \leq D$, we have an isomorphism of complexes of $(kC_G(Q), kC_K(Q))$-bimodules

$$\mathrm{Br}_{\Delta Q}(C \otimes_{\mathcal{O}K} C^*) \simeq C_Q \otimes_{kC_K(Q)} C_Q^*$$

  where $C_Q = \mathrm{Br}_{\Delta Q}(C)$ is a splendid complex of $(e_Q kC_G(Q), f_Q kC_H(Q))$-bimodules. More precisely, let X_Q be the cone of the adjunction morphism $C_Q \otimes_{kC_K(Q)} C_Q^* \to e_Q kC_G(Q)$. Then $X_Q \simeq \mathrm{Br}_{\Delta Q}(X)$, where X is the cone of the adjunction morphism $C \otimes_{\mathcal{O}K} C^* \to e\mathcal{O}G$.

By [Bou, Proposition 7.9] (cf. [Rou3] for the extension from k to $\mathcal{O}$), a complex Z is homotopy equivalent to a complex of projective modules if and only if, for every non-trivial subgroup Q of D, the complex $\mathrm{Br}_{\Delta Q}(Z)$ is homotopy equivalent to 0 (using that $\mathrm{Br}_P(Z) = 0$ if P is not contained in ΔD up to conjugacy).

We have a similar statement concerning $C^* \otimes_{\mathcal{O}G} C$ and the equivalence between (i) and (ii) follows.

The implication (ii′) $\Rightarrow$ (ii) follows by induction from (ii) $\Rightarrow$ (i) and from the isomorphism

$$\mathrm{Br}_Q(\mathrm{Br}_P(V)) \simeq \mathrm{Br}_Q(V)$$

when $P \lhd Q$. □

6. Blocks with defect group $\mathbb{Z}/p^a \times \mathbb{Z}/p^b$

In this section we assume that Hypothesis 4 holds. We recall that $A = e\mathcal{O}G$ and $B = f\mathcal{O}H$.

6.1. Cyclic defect groups

In this section, we recall the construction of splendid Rickard complexes for principal blocks with cyclic defect groups. Let us assume in §6.1 that D cyclic.

6.1.1. Let $\pi : P_{e\mathcal{O}Gf} \to e\mathcal{O}Gf$ be a projective cover of $e\mathcal{O}Gf$.

In [Rou1, Theorem 4.1], we have constructed a direct summand N of $P_{e\mathcal{O}Gf}$ with the following property. Let ϕ be the restriction of π to N and $C = 0 \to N \xrightarrow{\phi} e\mathcal{O}Gf \to 0$, where $e\mathcal{O}Gf$ is in degree 0. Then C induces a normalized splendid Rickard equivalence between $e\mathcal{O}G$ and $f\mathcal{O}H$.

Assume now Hypothesis 3′ of §5.2.3. We have [Ma, §5.5]:

Lemma 6.1. *The complex C extends to a complex of $\mathcal{O}\Delta$-modules.*

Proof. Let us decompose the $\mathcal{O}\Delta$-module $e\mathcal{O}Gf$ as $e\mathcal{O}Gf = M \oplus M'$ with M' projective and M without projective direct summand. By §5.2.3, the module $\mathrm{Res}^{\Delta}_{G\times H^\circ} M$ has no projective direct summand. Let P_M be a projective cover of M. Then $\mathrm{Res}^{\Delta}_{G\times H^\circ} P_M$ is a projective cover of $\mathrm{Res}^{\Delta}_{G\times H^\circ} M$. The direct summand N of $P_{e\mathcal{O}Gf} \simeq \mathrm{Res}^{\Delta}_{G\times H^\circ}(P_M \oplus M')$ arises as $P_0 \oplus \mathrm{Res}^{\Delta}_{G\times H^\circ} M'$, where P_0 is a direct summand of $\mathrm{Res}^{\Delta}_{G\times H^\circ} P_M$. The construction of P_M, using the Brauer tree, shows it is invariant under Δ, whence there is a direct summand N_0 of P_M with $\mathrm{Res}^{\Delta}_{G\times H^\circ} N_0 \simeq P_0$. $\square$

6.1.2. We assume finally Hypothesis 3″ of §5.2.4.

There is a projective $\mathcal{O}(\tilde{\Delta}/\Delta P)$-module $\tilde{N}$ such that $\mathcal{O} \otimes_{\mathcal{O}(P\times P^\circ)} \tilde{N} = N$. The composition of $\phi : N \to e\mathcal{O}Gf$ with the canonical map $\tilde{N} \to N$ factors through the canonical map $\tilde{e}\mathcal{O}\tilde{G}\tilde{f} \to e\mathcal{O}Gf$ as $\tilde{\phi} : \tilde{N} \to \tilde{e}\mathcal{O}\tilde{G}\tilde{f}$. This last map lifts ϕ, i.e., $\phi = 1 \otimes \tilde{\phi}$. By restriction, we obtain a morphism $\tilde{\phi} : \tilde{N} \to \tilde{e}\mathcal{O}\tilde{G}\tilde{f}$ of $\mathcal{O}\tilde{\Delta}$-modules.

We now define the complex of $\mathcal{O}\tilde{\Delta}$-modules

$$\tilde{C} = 0 \to \tilde{N} \xrightarrow{\tilde{\phi}} \tilde{e}\mathcal{O}\tilde{G}\tilde{f} \to 0$$

where $\tilde{e}\tilde{G}\tilde{f}$ is in degree 0. Then $\mathcal{O} \otimes_{\mathcal{O}(P\times P^\circ)} \tilde{C} \simeq C$.

So by §5.2.4, we have

Theorem 6.2. *The complex $\mathrm{Res}^{\tilde{\Delta}}_{\tilde{G}\times\tilde{H}^\circ} \tilde{C}$ induces a normalized splendid Rickard equivalence between $\tilde{e}\mathcal{O}\tilde{G}$ and $\tilde{e}\mathcal{O}\tilde{H}$.*

6.2. Blocks with abelian defect group of rank 2

6.2.1. In this subsection, assume that D is elementary abelian of order p^2.

Let P be a subgroup of order p of D. We have $N_G(P)/C_G(P) = N_H(P)/C_H(P)$; this is a p'-group.

We are in the setting of §6.1.2. We add an index $?_P$ to an object ? from this section to avoid confusion. We have $G_P = C_G(P)/P$, $H_P = C_H(P)/P$, $X_P = N_G(P)/P$, $Y_P = N_H(P)/P$, $\tilde{G}_P = C_G(P)$, $\tilde{H}_P = C_H(P)$, $\tilde{X}_P = N_G(P)$ and $\tilde{Y}_P = N_H(P)$. We nevertheless denote by e_P and f_P the principal block idempotents of $\mathcal{O}C_G(P)$ and $\mathcal{O}C_H(P)$. We have $\tilde{\Delta}_P = N_{G\times H^\circ}(\Delta P)$. We have a projective $\mathcal{O}(\tilde{\Delta}_P/\Delta P)$-module $\tilde{N}_P$ and a map $\tilde{\phi}_P : \tilde{N}_P \to e_P\mathcal{O}C_G(P)f_P$ with the property that the complex $\tilde{C}_P = 0 \to \tilde{N}_P \xrightarrow{\tilde{\phi}_P} e_P\mathcal{O}C_G(P)f_P \to 0$ induces a Rickard equivalence between $e_P\mathcal{O}C_G(P)$ and $f_P\mathcal{O}C_H(P)$.

Let $V_P = \mathrm{Ind}_{\tilde{\Delta}_P}^{G\times H^\circ} \mathrm{Res}_{\tilde{\Delta}_P}^{\tilde{\Delta}_P/\Delta P} \tilde{N}_P$, where we denote by $\mathrm{Res}_{\tilde{\Delta}_P}^{\tilde{\Delta}_P/\Delta P}$ the restriction through the canonical map $\mathcal{O}\tilde{\Delta}_P \to \mathcal{O}(\tilde{\Delta}_P/\Delta P)$, also called inflation. The morphism

$$e_P\mathcal{O}C_G(P)f_P \to \mathrm{Res}_{\tilde{\Delta}_P}^{G\times H^\circ} e\mathcal{O}Gf$$

(coming from the inclusion $e_P\mathcal{O}C_G(P)f_P \to \mathcal{O}G$) induces by adjunction a morphism

$$\alpha_P : \mathrm{Ind}_{\tilde{\Delta}_P}^{G\times H^\circ} e_P\mathcal{O}C_G(P)f_P \to e\mathcal{O}Gf$$

and $\mathrm{Br}_{\Delta P}(\alpha_P)$ is an isomorphism. Let $\psi_P = \alpha_P \,\mathrm{Ind}_{\tilde{\Delta}_P}^{G\times H^\circ}(\tilde{\phi}_P) : V_P \to e\mathcal{O}Gf$.

Let

$$C = 0 \to \bigoplus_P V_P \xrightarrow{\sum_P \psi_P} e\mathcal{O}Gf \to 0$$

where P runs over the subgroups of order p of D up to H-conjugacy (the term $e\mathcal{O}Gf$ is in degree 0).

Theorem 6.3. *The complex C induces a normalized splendid stable equivalence between $e\mathcal{O}G$ and $f\mathcal{O}H$.*

Proof. The complex C is splendid since V_P is a sum of p-permutation modules with vertex ΔP and $e\mathcal{O}Gf$ is a p-permutation module induced from ΔD. Let P be a subgroup of order p of D. For Q a subgroup of order p of D, we have $\mathrm{Br}_{\Delta P}(V_Q) = 0$ unless ΔQ is $(G \times H^\circ)$-conjugate to ΔP, i.e., Q is H-conjugate to P. Now we have $\mathrm{Br}_{\Delta P}(\psi_P) = \tilde{\phi}_P$ (cf. §4.1.2).

It follows that $\mathrm{Br}_{\Delta P}(C) \simeq k \otimes \tilde{C}_P$ induces a Rickard equivalence between $e_P kC_G(P)$ and $f_P kC_H(P)$.

Since C is splendid, the theorem follows now from Theorem 5.6, (ii$'$) $\Rightarrow$ (i). □

Remark 6.4. A similar construction works for nonprincipal blocks (cf. the Appendix and [Rou3]).

6.2.2. In this subsection, assume that D is abelian and has p-rank 2.

Let Q be a subgroup of order p of D. We have $D = (D \cap Z(C_H(Q))) \times [D, C_H(Q)]$. Let $P = D \cap Z(C_H(Q))$. Then $Q \leq P \leq D$, D/P is cyclic and $C_G(P)$ controls fusion of p-subgroups in $C_G(Q)$. This implies $C_G(Q) = O_{p'}C_G(Q) \cdot C_G(P)$, by the Z_p^*-theorem (given a finite group G and a p-subgroup P such that $C_G(P)$ controls fusion of p-subgroups in G, then $G = O_{p'}G \cdot C_G(P)$). This implies also $N_G(Q) = O_{p'}C_G(Q) \cdot N_G(P)$.

When $P = D$ (the only possible case when $p = 2$), the results above are easy since $C_G(Q)$ is p-nilpotent by Burnside's Theorem. Let e_Q and f_Q be the principal block idempotents of $\mathcal{O}C_G(Q)$ and $\mathcal{O}C_H(Q)$. The construction of §3.2.2 provides an extension of $\mathrm{Res}^{\tilde{\Delta}_P/\Delta P}_{\tilde{\Delta}_P/\Delta Q} \tilde{N}_P$ to an $(e_Q \otimes f_Q)\mathcal{O}(\tilde{\Delta}_Q/\Delta Q)$-module $\tilde{N}_Q$ and $\tilde{\phi}_P$ lifts to a morphism $\tilde{\phi}_Q : \tilde{N}_Q \to e_Q\mathcal{O}C_G(Q)f_Q$. Now, we continue as in §6.2.1 and construct V_Q, α_Q and ψ_Q.

We put

$$C = 0 \to \bigoplus_Q V_Q \xrightarrow{\sum_Q \psi_Q} e\mathcal{O}Gf \to 0$$

where Q runs over the subgroups of order p in D up to H-conjugacy (the term $e\mathcal{O}Gf$ is in degree 0).

The same proof as in Theorem 6.3 leads to the following result.

Theorem 6.5. *The complex C induces a normalized splendid stable equivalence between $e\mathcal{O}G$ and $f\mathcal{O}H$.*

Remark 6.6. When $N_G(D)/C_G(D)$ acts freely on $D - \{1\}$, then this result is due to Puig [Pu3, Corollary 6.7]. This is always the case when $p = 2$.

6.3. Blocks with Klein four defect groups

6.3.1. In this part we will make more explicit the constructions of §6.2 for the case $D = \mathbb{Z}/2 \times \mathbb{Z}/2$, where some simplifications occur. Then we will show how to construct a Rickard equivalence from the stable equivalence. The reason why the method does not apply for any other D of rank 2 is that there are too many indecomposable kD-modules (the type is wild).

Throughout §6.3, we assume D is elementary abelian of order 4.

6.3.2. Stable equivalence. Let P be a subgroup of order 2 of D. The complex C_P of §6.2.1 (i.e., the complex C of §6.1.1 constructed for the group $C_G(P)/P$) has homology only in degree 0: this homology is a direct summand of $\bar{e}_P\mathcal{O}(C_G(P)/P)\bar{f}_P$ and it induces a Morita equivalence between $\bar{e}_P\mathcal{O}(C_G(P)/P)$ and $\bar{f}_P\mathcal{O}(C_H(P)/P)$ (here, $\bar{e}_P$ and $\bar{f}_P$ are the principal block idempotents of $\mathcal{O}(C_G(P)/P)$ and $\mathcal{O}(C_H(P)/P)$).

The complex $\tilde{C}_P$ inducing a Rickard equivalence between $e_P\mathcal{O}C_G(P)$ and $f_P\mathcal{O}C_H(P)$ then has homology only in degree 0, i.e., there is a direct summand of $e_P\mathcal{O}C_G(P)f_P$ inducing a Morita equivalence between $e_P\mathcal{O}C_G(P)$ and $f_P\mathcal{O}C_H(P)$.

Finally, the complex C constructed in §6.2.1 has homology only in degree 0: this is a bimodule N isomorphic to a direct summand of $e\mathcal{O}Gf$. Let M be a direct summand of N such that $N = M \oplus$ projective and M has no projective direct summand. Then M induces a stable equivalence between $A = e\mathcal{O}G$ and $B = f\mathcal{O}H$.

Let us state this as

Proposition 6.7. *There is a direct summand of $e\mathcal{O}Gf$ inducing a normalized splendid stable equivalence between $e\mathcal{O}G$ and $f\mathcal{O}H$.*

6.3.3. The following result now solves Conjecture 5.3 for $D \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

Theorem 6.8. *There is a normalized splendid Rickard equivalence between $e\mathcal{O}G$ and $f\mathcal{O}H$ induced by a complex C such that C^{-1} is projective, $C^0 = M$ and $C^i = 0$ for $i \neq 0, -1$.*

Remark 6.9. For non-principal blocks, see the Appendix and [Rou3]. The existence of a Rickard equivalence was established by Linckelmann in [Li2, Corollary 1.5], but no equivalence was constructed.

The rest of this section is devoted to the proof of this theorem.

The group $E = H/C_G(D)$ is a p'-subgroup of the automorphism group of D, whence it is the trivial group or a cyclic group of order 3.

There is a normalized Morita equivalence between B and $B' = \mathcal{O}D \rtimes E$ (cf. for example [Rou2, Proposition 2.15]). Let W be a (B, B')-bimodule inducing such an equivalence. Let $L = M \otimes_B W$: this is an indecomposable (A, B')-bimodule inducing a stable equivalence.

Note that $L \otimes_{B'} \mathcal{O} \simeq \mathcal{O}$.

6.3.4. The nilpotent case. Assume $E = 1$. Then L induces a Morita equivalence between A and B' (cf. §2.3.5), whence $C = M$ induces a Morita equivalence and hence a Rickard equivalence between A and B.

6.3.5. The A_4-case. Let us now consider the case $|E| = 3$. Then $D \rtimes E \simeq A_4$.

Let S be a non-trivial simple B'-module. We have

$$\mathrm{Hom}_A(L \otimes_{B'} S, k) \simeq \mathrm{Hom}_{B'}(S, L^* \otimes_A k) = \mathrm{Hom}_{B'}(S, k) = 0.$$

Let V_1 be a simple quotient of $L \otimes_{B'} S$ (this is not the trivial module).

We have an isomorphism $\mathrm{Hom}_{B'}(L^* \otimes_A V_1, k) \simeq \mathrm{Hom}_A(V_1, k) = 0$, and similarly $\mathrm{Hom}_{B'}(k, L^* \otimes_A V_1) = 0$. It follows that k is not a composition factor of $L^* \otimes_A V_1$ (this module has no projective direct summand and hence has Loewy length at most 2). Consequently, $L^* \otimes_A V_1$ has a unique simple quotient S_1. Let S_2 be a simple

B'-module not isomorphic to k or S_1. Then $L^* \otimes_A V_1 = S_1$ or $L^* \otimes_A V_1$ is an extension of S_1 by S_2.

Furthermore, we have $\mathrm{Hom}_A(V_1, L \otimes_{B'} S_2) \simeq \mathrm{Hom}_{B'}(L^* \otimes_A V_1, S_2) = 0$ and $\mathrm{Hom}_A(k, L \otimes_{B'} S_2) = 0$. So there is a simple submodule V_2 of $L \otimes_{B'} S_2$ that is not isomorphic to k or V_1.

If $L^* \otimes_A V_1 = S_1$, then we have $L^* \otimes_A V_2 = S_2$, so $L \otimes_{B'} -$ send simple modules to simple modules, whence L induces a Morita equivalence between A and B' (cf. 2.3.5). So in this case a solution to Theorem 6.8 is provided by $C = M$.

Assume $L^* \otimes_A V_1$ is an extension of S_1 by S_2. Then $L^* \otimes_A V_2$ is an extension of S_2 by S_1. Now we are in a situation similar to 5.1: a projective cover of L is

$$P_L = P_{k_A} \otimes P_{k_B}^* \oplus P_{V_1} \otimes P_{S_2}^* \oplus P_{V_2} \otimes P_{S_1}^*.$$

Let δ be the restriction of a surjective map $P_L \to L$ to $R = P_{V_1} \otimes P_{S_1}^* \oplus P_{V_2} \otimes P_{S_2}^*$ and C' be the complex

$$C' = 0 \to R \xrightarrow{\delta} L \to 0$$

with L in degree 0.

The same proof as that of Proposition 5.2 shows that C' induces a Rickard equivalence between A and B'. So

$$C = C' \otimes_{B'} W^* = 0 \to R \otimes_{B'} W^* \to M \to 0$$

provides a solution to Theorem 6.8. □

6.3.6. As in §6.1.1 and 6.1.2, one checks that the construction can be done compatibly with p'-outer automorphism groups and central extensions by p-groups, as conjectured in §5.2.3 and 5.2.4.

6.4. Blocks with defect $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$

In this subsection, we assume that Hypothesis 4 holds with $D \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$.

Let M be the unique indecomposable direct summand of $e\mathcal{O}Gf$ with vertex ΔD.

A construction similar to that in §6.2.1 (using now §6.3 instead of §6.1) provides, for every subgroup P of order 2 of D, a relatively ΔP-projective p-permutation $(e\mathcal{O}G \otimes (f\mathcal{O}H)^\circ)$-module V_P and a map $\psi_P : V_P \to M$. Consider now the complex

$$C = 0 \to \bigoplus_P V_P \xrightarrow{\sum_P \psi_P} M \to 0$$

where P runs over the subgroups of order 2 of D up to H-conjugacy (the term M is in degree 0).

A proof analogous to that of Theorem 6.3 shows

Theorem 6.10. *The complex C induces a normalized splendid stable equivalence between $e\mathcal{O}G$ and $f\mathcal{O}H$.*

Note that we have a block of wild type (unlike the case where D has order 4) and we do not know how to lift this to a Rickard equivalence (without using a case by case proof based on the classification of finite simple groups with 2-Sylow subgroups elementary abelian of order at most 8).

7. Local constructions

In this section, we develop a formalism for gluing Rickard complexes and apply it here only to principal blocks. In §5.3, we constructed local Rickard equivalences from a global stable equivalence, using the Brauer functor. Here, we try to provide some converse, namely the construction of a global stable equivalence from a suitable family of local Rickard equivalences. This generalizes the construction of §6.2.

A more detailed study is being conducted in [Rou3] (cf. also the Appendix), where various categories of "sheaves" over the poset of p-subgroups will be considered, giving rise for example to a local construction of the stable category of all modules or of p-permutation modules.

Our approach here consists of constructing locally a "subcategory" of the category of p-permutation modules.

7.1. Gluing G-sets

Let G be a finite group and k a field of characteristic $p > 0$.

Let $\mathcal{T} = \mathcal{T}_G$ be the category of p-subgroups of G, with maps the inclusions. There is an action of G by conjugation on $\mathcal{T}$.

Let $\mathcal{F}$ be a G-stable full subcategory of $\mathcal{T}$.

7.1.1. We define the category $\mathcal{E}(\mathcal{F}) = \mathcal{E}_G(\mathcal{F})$ as the full subcategory of the category $\mathcal{E}$ of G-sets of objects whose point-stabilizers are in $\mathcal{F}$.

Given X, Y in $\mathcal{E}(\mathcal{F})$, then a morphism $f : X \to Y$ is an isomorphism if and only if, for every P in $\mathcal{F}$, the morphism $f^P : X^P \to Y^P$ is an isomorphism.

7.1.2. Sheaves. Assume now that $\mathcal{F}$ is closed above (i.e., if P is in $\mathcal{F}$ and Q is a p-subgroup containing P, then Q is in $\mathcal{F}$).

Let us denote by $\mathcal{L}(\mathcal{F})$ the category of G-equivariant presheaves of (finite) sets over $\mathcal{F}$, i.e., the category of G-equivariant contravariant functors $\mathcal{F} \to$ sets.

Its objects are families $\Omega = (\Omega_P, \phi(Q, R), [g, S])_{P,Q,R,S,g}$ where P, Q, R, S run over the objects of $\mathcal{F}$ with $Q \subseteq R$ and g over G. Here, Ω_P is a set, $\phi(Q, R)$ is a map from Ω_R to Ω_Q and $[g, S]$ is an isomorphism $\Omega_S \xrightarrow{\sim} \Omega_{{}^gS}$. Furthermore, the following conditions should be fulfilled:

(i) $\phi(R, R) = 1_{\Omega_R}$;

(ii) $\phi(Q,R)\phi(R,S) = \phi(Q,S)$;

(iii) $[g, {}^hP][h,P] = [gh,P]$;

(iv) we have a commutative diagram

$$\begin{tikzcd}
\Omega_Q \arrow[r, "\phi(P,Q)"] \arrow[d, "{[g,Q]}"'] & \Omega_P \arrow[d, "{[g,P]}"] \\
\Omega_{{}^gQ} \arrow[r, "\phi({}^gP,{}^gQ)"'] & \Omega_{{}^gP}\ .
\end{tikzcd}$$

Note that the maps $[g,P]$ for $g \in N_G(P)$ give a structure of a $N_G(P)$-set to Ω_P: we have a functor $?_P : \mathcal{L}(\mathcal{F}) \to N_G(P)$sets.

We say that Ω is a *sheaf* if, for H a subgroup of G and P an object of $\mathcal{F}$ normal in H, then

$$\phi(P,H) \text{ is an isomorphism } \Omega_H \xrightarrow{\sim} (\Omega_P)^H \text{ if } H \text{ is a } p\text{-group}$$

$$\text{and } (\Omega_P)^H = 0 \text{ otherwise.}$$

We denote by $\mathcal{S}(\mathcal{F})$ the full subcategory of $\mathcal{L}(\mathcal{F})$ consisting of sheaves.

For a sheaf Ω, the group P acts trivially on Ω_P and the maps $\Phi(P,Q)$ are inclusions, since every map in $\mathcal{F}$ is a composition of normal inclusions.

7.1.3. Some functors. Let $\mathcal{G}$ be a full subcategory of $\mathcal{F}$ closed above.

We denote by $\mathrm{Res}^{\mathcal{F}}_{\mathcal{G}}$ the restriction functor $\mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{G})$.

We need to construct also a left adjoint $\mathrm{Ind}^{\mathcal{F}}_{\mathcal{G}}$ from $\mathcal{L}(\mathcal{G})$ to $\mathcal{L}(\mathcal{F})$.

For $\Omega \in \mathcal{L}(\mathcal{G})$, and $P \in \mathcal{F}$, the set $\Omega'_P = (\mathrm{Ind}^{\mathcal{F}}_{\mathcal{G}}\,\Omega)_P$ is the direct limit of the restriction of Ω to the full subcategory of $\mathcal{G}$ of objects containing P: this is the quotient of $\coprod_{Q\in\mathcal{G},\,P\subseteq Q} \Omega_Q$ by the (coarsest) equivalence relation that identifies x and $\phi(R,Q)(x)$ for $x \in \Omega_Q$ and $R \hookrightarrow Q$ in $\mathcal{G}$. We denote by $\lambda_{P,Q}$ the canonical map $\Omega_Q \to \Omega'_P$ for $P \subseteq Q$.

The equivalence relation admits the following easier description when $\Omega \in \mathcal{S}(\mathcal{G})$: we have $\lambda_{P,Q}(a) = \lambda_{P,Q'}(a')$ if and only if there exists $R \in \mathcal{G}$ and $b \in \Omega_R$ with $Q \subseteq R$ and $Q' \subseteq R$ such that $a = \phi(Q,R)(b)$ and $a' = \phi(Q',R)(b)$.

To prove this claim, it is enough to consider the case where there is S normal in Q and in Q' with $\phi(S,Q)(a) = \phi(S,Q')(a')$. Let us denote by c this element of $\Omega_S^Q \cap \Omega_S^{Q'} = \Omega_S^R$ (we put $R = \langle Q, Q'\rangle$). Since Ω is a sheaf, this shows that R is a p-group and that there is $b \in \Omega_R$ such that $c = \phi(S,R)(b)$. Now $\phi(S,Q)\phi(Q,R)(b) = \phi(S,Q)(a)$. Since $\phi(S,Q)$ is an inclusion, we obtain $a = \phi(Q,R)$. Similarly, $a' = \phi(Q,R')(b)$.

There is a morphism of functors $\mathrm{Ind}_{\mathcal{G}}^{\mathcal{F}}\,\mathrm{Res}_{\mathcal{G}}^{\mathcal{F}} \to 1$ making the following diagram commutative

$$\begin{tikzcd}
(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{F}}\,\mathrm{Res}_{\mathcal{G}}^{\mathcal{F}}\,\Omega)_P \arrow[r] & \Omega_P \\
\coprod_{Q\in\mathcal{G},\, P\subseteq Q} \Omega_Q \arrow[u, two heads] \arrow[ur, "\{\phi(P,Q)\}_Q"'] &
\end{tikzcd}$$

and a morphism of functors $1 \to \mathrm{Res}_{\mathcal{G}}^{\mathcal{F}}\,\mathrm{Ind}_{\mathcal{G}}^{\mathcal{F}}$ given by

$$\Omega_Q \xrightarrow{\lambda_{Q,Q}} (\mathrm{Res}_{\mathcal{G}}^{\mathcal{F}}\,\mathrm{Ind}_{\mathcal{G}}^{\mathcal{F}}\,\Omega)_Q.$$

They make $\mathrm{Ind}_{\mathcal{G}}^{\mathcal{F}}$ a left adjoint of $\mathrm{Res}_{\mathcal{G}}^{\mathcal{F}}$.

The functor $\mathrm{Res}_{\mathcal{G}}^{\mathcal{F}}$ clearly restricts to a functor $\mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{G})$.

Lemma 7.1. *The functor* $\mathrm{Ind}_{\mathcal{G}}^{\mathcal{F}}$ *restricts to a functor from* $\mathcal{S}(\mathcal{G})$ *to* $\mathcal{S}(\mathcal{F})$.

Proof. Let $\Omega \in \mathcal{S}(\mathcal{G})$ and $\Omega' = \mathrm{Ind}_{\mathcal{G}}^{\mathcal{F}}\,\Omega$. Let $P \in \mathcal{F}$ and $x \in \Omega'_P$. Let $Q \in \mathcal{G}$ be maximal such that $x = \lambda_{P,Q}(a)$ for some element $a \in \Omega_Q$. Let $g \in N_G(P)$ such that $g(x) = x$. Then $\lambda_{P,{}^gQ}([g, Q](a)) = \lambda_{P,Q}(a)$. It follows that there is R in $\mathcal{G}$ containing Q and Q^g and $b \in \Omega_R$ such that $\phi(Q, R)(b) = a$ and $\phi({}^gQ, R)(b) = [g, Q](a)$.

The maximality of Q shows that $R = Q = Q^g$, whence $g \in N_G(Q)$ and $a \in (\Omega_Q)^{\langle g,Q\rangle}$. So $\langle g, Q\rangle$ is a p-group and a is in the image of $\phi(Q, \langle g, Q\rangle)$. The maximality of Q shows that $g \in Q$.

Since Ω'_P is a direct limit over a transitive system of injections, the map $\lambda_{P,Q} : \Omega_Q \to \Omega'_P$ is injective. It follows that Ω' is a sheaf. $\square$

Remark 7.2.

- The adjunction between Res and Ind restricts also to sheaves.
- The adjunction morphism $1 \to \mathrm{Res}_{\mathcal{G}}^{\mathcal{F}}\,\mathrm{Ind}_{\mathcal{G}}^{\mathcal{F}}$ between functors from $\mathcal{S}(\mathcal{G})$ to itself is an isomorphism.
- For Ω in $\mathcal{S}(\mathcal{G})$ and $P \in \mathcal{F}$, the point-stabilizers of the $N_G(P)$-set $(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{F}}\,\Omega)_P$ are in $\mathcal{G}$.

Let $X \in \mathcal{E}$. Let $\Omega_P = X^P$ and, for $P \subseteq Q$, let $\phi(P, Q)$ be the inclusion $X^Q \hookrightarrow X^P$. We define $[g, P] : X^P \to X^{{}^gP}$ by $x \mapsto gx$. Then $(\Omega_P, \phi(Q, R), [g, S])$ is an element of $\mathcal{S}(\mathcal{T})$. This gives a functor $\mathrm{Br} : \mathcal{E} \to \mathcal{S}(\mathcal{T})$ which is canonically inverse to $?_1 : \mathcal{S}(\mathcal{T}) \to \mathcal{E}$.

Let $\mathrm{Br}(\mathcal{F})$ be the restriction of $\mathrm{Res}_{\mathcal{F}}^{\mathcal{T}}\,\mathrm{Br}$ to $\mathcal{E}(\mathcal{F})$.

Theorem 7.3. *The functor* $\mathrm{Br}(\mathcal{F})$ *is an equivalence of categories* $\mathcal{E}(\mathcal{F}) \xrightarrow{\sim} \mathcal{S}(\mathcal{F})$ *with inverse* $?_1 \cdot \mathrm{Ind}_{\mathcal{F}}^{\mathcal{T}}$.

Proof. To simplify notation, we put $A = \mathrm{Br}(\mathcal{F})$ and $B = ?_1 \cdot \mathrm{Ind}_{\mathcal{F}}^{\mathcal{T}}$. We know already that B is left adjoint to A and that the adjunction morphism $1 \to AB$ is an isomorphism.

Now the counit of adjunction $BA \to 1$ becomes an isomorphism after composing with A since the composite $A \to ABA \to A$ is the identity and the first map is already known to be an isomorphism. This means that, given X in $\mathcal{E}(\mathcal{F})$, the counit $BA(X) \to X$ becomes an isomorphism after taking fixed points by a subgroup in $\mathcal{F}$. Since $BA(X)$ and X are in $\mathcal{E}(\mathcal{F})$, it follows that the counit is an isomorphism (cf. §7.1.1). $\square$

Of special interest is the case where $\mathcal{F}$ consists of the non-trivial p-subgroups of G: Theorem 7.3 says that the category $\mathcal{E}(\mathcal{F})$ of G-sets whose stabilizers are non-trivial p-groups is "locally determined".

7.1.4. Let $\mathcal{F}$ be a G-stable full subcategory of $\mathcal{T}$ and $\bar{\mathcal{F}}$ be the closure of $\mathcal{F}$, i.e., the full subcategory of $\mathcal{T}$ with objects the p-subgroups that contain some object of $\mathcal{F}$.

We define $\mathcal{S}(\mathcal{F})$ as the full subcategory of $\mathcal{S}(\bar{\mathcal{F}})$ with objects the Ω such that $\Omega_P = \emptyset$ for $P \notin \mathcal{F}$. Then $\mathrm{Br}(\bar{\mathcal{F}})$ restricts to an equivalence $\mathrm{Br}(\mathcal{F}) : \mathcal{E}(\mathcal{F}) \to \mathcal{S}(\mathcal{F})$.

7.2. *p*-permutation modules

7.2.1. Let $\tilde{\mathcal{E}}(\mathcal{F})$ be the Karoubian envelope of $\mathcal{E}(\mathcal{F})$: this is the category obtained from $\mathcal{E}(\mathcal{F})$ by k-linearizing and then adding images of idempotents. Its objects are pairs (X, e) where X is an object of $\mathcal{E}(\mathcal{F})$ and e is an idempotent of the k-algebra of the monoid $\mathrm{End}(X)$.

The space $\mathrm{Hom}((X, e), (X', e'))$ is the subspace $e'(k^{\mathrm{Hom}(X,X')})e$ of $k^{\mathrm{Hom}(X,X')}$. Similarly, we have a category $\tilde{\mathcal{S}}(\mathcal{F})$ obtained from $\mathcal{S}(\mathcal{F})$ by k-linearizing and then adding images of idempotents as above.

The functor $\mathrm{Br}(\mathcal{F})$ gives rise to a functor $\tilde{\mathcal{E}}(\mathcal{F}) \to \tilde{\mathcal{S}}(\mathcal{F})$. From Theorem 7.3 we can deduce

Corollary 7.4. *The functor* $\mathrm{Br}(\mathcal{F})$ *is an equivalence* $\tilde{\mathcal{E}}(\mathcal{F}) \xrightarrow{\sim} \tilde{\mathcal{S}}(\mathcal{F})$.

We have a faithful functor $\rho : \tilde{\mathcal{E}}(\mathcal{T}) \to kG\text{-perm}$, $(\Omega, e) \mapsto k\Omega e$. If Ω is free, then we have an isomorphism

$$\mathrm{Hom}_{\tilde{\mathcal{E}}(\mathcal{T})}((\Omega, e), (\Omega', e')) \xrightarrow{\sim} \mathrm{Hom}_{kG\text{-perm}}(k\Omega e, k\Omega' e').$$

The category $\tilde{\mathcal{E}}(\mathcal{T})$ consists of certain p-permutation modules with additional structure and the maps between them are those which can be "constructed" from maps between G-sets. A complex of objects of $\tilde{\mathcal{E}}(\mathcal{T})$ will be called a *geometrical* complex for kG.

7.2.2. Given $X, Y \in \tilde{\mathcal{E}}$ such that the G-set underlying X is free, we have an isomorphism

$$\mathrm{Hom}_{\tilde{\mathcal{E}}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{kG}(\rho(X), \rho(Y)).$$

Consequently, every projective kG-module arises as $\rho(X)$ for some $X \in \tilde{\mathcal{E}}$ with an underlying G-set free.

7.2.3. Let $\mathcal{F}$ be a G-stable full subcategory of $\mathcal{T}_G$ closed above. Let Q be a maximal p-subgroup of G outside $\mathcal{F}$. Let $\mathcal{F}'$ be the full subcategory of $\mathcal{T}_G$ with objects those of $\mathcal{F}$ together with the conjugates of Q. Let $\mathcal{H}$ be the full subcategory of $\mathcal{T}_{N_G(Q)}$ with objects those p-subgroups of $N_G(Q)$ containing Q.

Let us construct a category $\mathcal{C}$. Its objects are families $(\Omega, V, \{\phi_R\}_{Q<R})$ where $\Omega \in \mathcal{S}(\mathcal{F})$, $V \in \mathcal{E}_{N_G(Q)}(\mathcal{H})$ and $\phi_R : \Omega_R \to V$ is a map of sets satisfying

(i) given S in $\mathcal{F}$ containing R, we have $\phi_S = \phi_R \phi(R, S)$,

(ii) given $h \in N_G(Q)$, we have a commutative diagram

$$\begin{tikzcd} \Omega_R \arrow[r, "\phi_R"] \arrow[d, "{[h,R]}"'] & V \arrow[d, "h"] \\ \Omega_{{}^hR} \arrow[r, "\phi_{{}^hR}"'] & V \end{tikzcd},$$

and

(iii) given R a p-subgroup with $Q \lhd R$ and $Q \neq R$, we have an isomorphism $\phi_R : \Omega_R \xrightarrow{\sim} V^R$.

A morphism $(\Omega, V, \{\phi_R\}_{Q\le R}) \to (\Omega', V', \{\phi'_R\}_{Q\le R})$ is a pair (Λ, f) consisting of a morphism $\Lambda : \Omega \to \Omega'$ and a map $f : V \to V'$ such that, for all R containing Q, the following diagram is commutative

$$\begin{tikzcd} \Omega_R \arrow[r, "\phi_R"] \arrow[d, "\Lambda_R"'] & V \arrow[d, "f"] \\ \Omega'_R \arrow[r, "\phi'_R"'] & V' \end{tikzcd}.$$

Lemma 7.5. *The functor $\mathcal{S}(\mathcal{F}') \to \mathcal{C}$ given by $\Omega \mapsto (\mathrm{Res}^{\mathcal{F}'}_{\mathcal{F}} \Omega, \Omega_Q, \{\phi(Q, R)\})$ is an equivalence.*

Let us give a useful application of this lemma. Let $X \in \tilde{\mathcal{S}}(\mathcal{F}')$ and $Y \in \tilde{\mathcal{E}}_{N_G(Q)}(\mathcal{H})$ with $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}-\{Q\}} \mathrm{Br}\, X_Q \xrightarrow{\sim} \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}-\{Q\}} \mathrm{Br}\, Y$. Then there is $X' \in \tilde{\mathcal{S}}(\mathcal{F}')$ with $\mathrm{Res}^{\mathcal{F}'}_{\mathcal{F}} X' \xrightarrow{\sim} \mathrm{Res}^{\mathcal{F}'}_{\mathcal{F}} X$ and $X'_Q \xrightarrow{\sim} Y$.

7.3. Geometric stable and Rickard equivalences

7.3.1. We assume Hypotheses 4 (the blocks are principal) and $3'$ of §5.2.3 (cf. the Appendix and [Rou3] for nonprincipal blocks). Assume furthermore that D is abelian.

Denote by $\mathcal{F}$ the full subcategory of p-subgroups of Δ contained in ΔD up to conjugacy.

We may refine Broué's conjecture 5.3:

Conjecture 7.6. *There is a complex C in $\tilde{\mathcal{E}}_\Delta(\mathcal{F})$ such that $\operatorname{Res}^{\Delta}_{G\times H^\circ}\rho(C)$ is a Rickard complex for $A\otimes B^\circ$.*

Note that if C is such a complex, then $\operatorname{Res}^{\Delta}_{G\times H^\circ}\rho(C)$ is a splendid Rickard complex, i.e., a Rickard complex of p-permutation $k(G\times H^\circ)$-modules with vertices contained in ΔD. So Conjecture 7.6 indeed implies Broué's conjecture.

7.3.2. Let us assume Hypothesis $3''$. We denote by $\tilde{D}$ the inverse image of D in $\tilde{X}$. Let $\tilde{\mathcal{F}}$ be the full subcategory of p-subgroups of $\tilde{\Delta}$ containing ΔP and contained in $\Delta\tilde{D}$ up to conjugacy.

One can ask the following question about $(\tilde{X}, P, G)$:

Question 7.7. Let C be a complex in $\tilde{\mathcal{E}}_{\tilde{\Delta}}(\tilde{\mathcal{F}})$ such that $\operatorname{Res}^{\Delta}_{G\times H^\circ} k\Delta\otimes_{k\tilde{\Delta}}\rho(C)$ induces a stable equivalence between A and B.

Is there a complex C' in $\tilde{\mathcal{E}}_{\tilde{\Delta}}(\tilde{\mathcal{F}})$ such that

(i) $\operatorname{Res}^{\Delta}_{G\times H^\circ} k\Delta\otimes_{k\tilde{\Delta}}\rho(C')$ induces a Rickard equivalence between A and B, and

(ii) $\operatorname{Res}^{\tilde{\mathcal{F}}}_{\tilde{\mathcal{F}}-\{\Delta P\}}\operatorname{Br} C'\simeq\operatorname{Res}^{\tilde{\mathcal{F}}}_{\tilde{\mathcal{F}}-\{\Delta P\}}\operatorname{Br} C$?

This question has a positive answer when $|D|=p$ (more generally, for D cyclic) or when D is a Klein four group.

Let us justify this claim, for a particular choice of C (when this will be used in the application of Theorem 7.8, it will be possible to ensure that C is of that type). We assume C is concentrated in degree 0 and $M=\rho(C^0)$ is a direct summand of $k\tilde{X}$, as a $k\tilde{\Delta}$-module. Then it follows from §6.1 and §6.3 that there is a projective $k(\tilde{\Delta}/\Delta P)$-module R' and a morphism

$$f: R=\operatorname{Res}^{\tilde{\Delta}/\Delta P}_{\tilde{\Delta}} R'\to M$$

such that the complex $0\to R\xrightarrow{f} M\to 0$ induces a Rickard equivalence between the principal blocks of $\tilde{G}$ and $\tilde{H}$. By §7.2.2, there is an object V in $\tilde{\mathcal{E}}_{\tilde{\Delta}/\Delta P}$ and a morphism $\alpha: V\to C^0$ with $\rho(\alpha)=f$. Now $C'=0\to V\xrightarrow{\alpha} C^0\to 0$ answers Question 7.7.

Note that when D is not abelian, there are stable equivalences that do not lift to Rickard equivalences (that happens for the principal block of $G=\mathrm{Sz}(8)$ for $p=2$).

7.3.3. We can now state our main result.

Theorem 7.8. *Assume Question* 7.7 *has a positive answer for* $(N_X(Q), Q, C_G(Q)/Q)$ *for all non-trivial subgroups* Q *of* D. *Then there is a complex* C *in* $\tilde{\mathcal{E}}_\Delta(\mathcal{F})$ *such that* $\mathrm{Res}^{\Delta}_{G\times H^\circ}\rho(C)$ *induces a stable equivalence between* $e\mathcal{O}G$ *and* $f\mathcal{O}H$.

We recover the existence of stable equivalences for $D \simeq \mathbb{Z}/p \times \mathbb{Z}/p$ (§6.2) and for D elementary abelian of order 8 (§6.4).

This result readily implies

Corollary 7.9. *Assume Question* 7.7 *has a positive answer for*

$$(N_X(Q), Q, C_G(Q)/Q)$$

for all subgroups Q *of* D. *Then Conjecture* 7.6 *holds for* (X, G).

7.3.4. *Proof of Theorem* 7.8. Let $\mathcal{G}$ be a full subcategory of $\mathcal{F} - \{1\}$ closed under conjugation and such that, given Q in $\mathcal{G}$ and R in $\mathcal{F}$ with $Q \le R$, then R is in $\mathcal{G}$. We will construct by induction on $|\mathcal{G}|$ a complex $X_{\mathcal{G}}$ in $\tilde{\mathcal{S}}_\Delta(\mathcal{G})$ such that

$$k(\bar{C}_G(Q) \times \bar{C}_H(Q)^\circ) \otimes_{k(C_G(Q)\times C_H(Q)^\circ)} \mathrm{Res}^{N_\Delta(\Delta Q)}_{C_G(Q)\times C_H(Q)^\circ}\,\rho\left((X_{\mathcal{G}})_{\Delta Q}\right)$$

is a Rickard complex for the principal blocks of $\bar{C}_G(Q) = C_G(Q)/Q$ and $\bar{C}_H(Q) = C_H(Q)/Q$, for all $Q \le D$ with $\Delta Q \in \mathcal{G}$.

Assume we have a complex $X_{\mathcal{G}}$ satisfying the induction hypothesis and let $Q \ne 1$ be a maximal subgroup of D with $\Delta Q \notin \mathcal{G}$. Let $\mathcal{G}' = \mathcal{G} \cup \{(\Delta Q)^x\}_{x\in\Delta}$.

Let $Y = \mathrm{Ind}^{\mathcal{G}'}_{\mathcal{G}} X_{\mathcal{G}}$. Let $Z = Y_{\Delta Q}$, an object of $\tilde{\mathcal{E}}_{N_\Delta(\Delta Q)}(\mathcal{H})$, where $\mathcal{H}$ is the full subcategory of $\mathcal{T}_{N_\Delta(\Delta Q)}$ of p-subgroups containing ΔQ and contained in ΔD up to conjugacy.

For every p-subgroup R of D strictly containing Q, we have $\mathrm{Br}_{\Delta R}\,\rho(Z) \simeq \rho\left((X_{\mathcal{G}})_{\Delta R}\right)$, whence $\mathrm{Br}_{\Delta R/\Delta Q}\, C$ induces a Rickard equivalence between the principal blocks of $\bar{C}_G(R)$ and $\bar{C}_H(R)$, where

$$C = \mathrm{Res}^{N_\Delta(\Delta Q)/(Q\times Q^\circ)}_{\bar{C}_G(Q)\times\bar{C}_H(Q)^\circ}\, kN_\Delta(\Delta Q)/(Q\times Q^\circ) \otimes_{kN_\Delta(\Delta Q)} \rho(Z).$$

By Theorem 5.6, it follows that C induces a stable equivalence between the principal blocks of $\bar{C}_G(Q)$ and $\bar{C}_H(Q)$.

Since we are assuming a positive answer to Question 7.7 by hypothesis for $(N_X(Q), Q, C_G(Q)/Q)$, it follows that there exists a complex Z' in $\tilde{\mathcal{E}}_{N_\Delta(\Delta Q)}(\mathcal{H})$ and an isomorphism

$$\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}-\{\Delta_Q\}}\,\mathrm{Br}\, Z' \xrightarrow{\sim} \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}-\{\Delta_Q\}}\,\mathrm{Br}\, Z$$

such that $\mathrm{Res}^{N_\Delta(\Delta Q)/(Q\times Q^\circ)}_{\bar{C}_G(Q)\times\bar{C}_H(Q)^\circ}\, kN_\Delta(\Delta Q)/(Q\times Q^\circ)\otimes_{kN_\Delta(\Delta Q)}\rho(Z')$ induces a Rickard equivalence between the principal blocks of $\bar{C}_G(Q)$ and $\bar{C}_H(Q)$.

By §7.2.3, there is a complex Y' in $\tilde{\mathcal{S}}_\Delta(\mathcal{G}')$ with $Y'_{\Delta Q} \xrightarrow{\sim} Z'$ and $Y'_R \xrightarrow{\sim} (X_{\mathcal{G}})_R$ for $R \in \mathcal{G}$. Now $X_{\mathcal{G}'} = Y'$ satisfies the required properties. □

Appendix

Our aim here is to present some of the main points of the work in progress of [Rou3]. It completes the results described previously, mainly by extending most of these to nonprincipal blocks.

We will be working in the Appendix with Hypothesis 3 unless otherwise stated.

A.1. Splendid equivalences

A.1.1. Previous constructions. As explained by Broué in [Br1], a Rickard equivalence between A and B induces a perfect isometry between the blocks — a character correspondence with signs satisfying certain arithmetical properties. Now, Broué introduced also in [Br1] the class of isotypies, which consist of a compatible system (for the generalized decomposition maps) of perfect isometries for local subgroups. In [Ri4], Rickard proved that, by adding the assumption of splendidness to a Rickard equivalence, one then gets such an isotypie. Actually, Rickard proves more. Namely, he shows how to get Rickard equivalences for local subgroups (this is Theorem 5.6, (i) $\Rightarrow$ (ii)). Unfortunately, Rickard's construction was done only for principal blocks. Later, Harris [Ha] showed that, if H is a subgroup of G, then Rickard's definition of splendidness still provides Rickard equivalences for local subgroups (i.e., Theorem 5.6,(i) $\Rightarrow$ (ii) holds for nonprincipal blocks and without the assumption that K controls fusion of p-subgroups in G). As for a general definition of splendidness, this has been given by Linckelmann in [Li5].

In order to define splendid equivalences (or even isotypies), one needs an equivalence between the Brauer categories of the two blocks (we will recall later what they are). One needs also an identification of the defect groups of the blocks. Then Linckelmann's definition of splendidness is relative to a choice of source idempotents for the blocks, which makes it difficult to check on examples.

For principal blocks, the situation is simpler because of Brauer's third main theorem and because the assumption on the Brauer categories means that the groups have the same fusion of p-subgroups. More generally, Harris pointed out that it is still enough to assume some relative projectivity on the terms of the complex, provided one of the groups controls the fusion of p-subgroups in the other group — but this is not a very satisfactory assumption, being non-symmetric and involving more than just information about the Brauer categories of the blocks. There, one trick (due to Rickard) is needed, namely, that a complex of exact (A, B)-bimodules C induces a Rickard equivalence if and only if one of the two isomorphisms in §2.3.2, (i) holds.

A.1.2. A new approach. What is not natural in the previous approaches is the identification *a priori* of the defect groups of the blocks, as pointed out by Broué.

Let D be a defect group of A and D' a defect group of B. Let C be an indecomposable complex of p-permutation (A, B)-bimodules inducing a stable equivalence.

One shows that there is an isomorphism $\phi : D \xrightarrow{\sim} D'$ such that the terms of C are projective relatively to $\Delta_\phi(D) = \{(x, \phi(x)) | x \in D\}$. Now let (D, e_D) be a maximal e-subpair. There is an equivalence $F : \mathrm{Br}(e) \xrightarrow{\sim} \mathrm{Br}(f)$ with the following property:

Given a subpair $(Q, b_Q) \leq (D, b_D)$, the complex $e_Q \mathrm{Br}_{\Delta_\phi(Q)}(C)$ induces a Rickard equivalence between $e_Q kC_G(Q)$ and $f_{Q'} kC_H(Q')$, where $(Q', f_{Q'}) = F(Q, e_Q)$.

This is the generalization of Theorem 5.6, (i) $\Rightarrow$ (ii).

We recall that the Brauer category $\mathrm{Br}(e)$ of $e\mathcal{O}G$ has for objects the e-subpairs and $\mathrm{Hom}((Q, e_Q), (R, e_R))$ is the set of homorphisms $Q \to R$ which are the composition of conjugation $Q \to Q^g, x \mapsto g^{-1}xg$ followed by an inclusion $Q^g \subseteq R$, for some $g \in G$ such that $g^{-1}e_Q g = e_R$.

One can say much more about the vertices of the indecomposable terms in C. One constructs a theory of vertex-subpairs for indecomposable modules. Then one shows that the vertex-subpairs of the terms have the form $(\Delta_\phi(Q), e_Q \otimes f_{\phi(Q)})$, where $(Q, e_Q) \leq (D, e_D)$ and $(\phi(Q), f_{\phi(Q)}) = F(Q, e_Q)$. This last property finally tells us that the complex is splendid in the sense of Linckelmann.

Now, what should be a splendid complex? It depends on the data! As long as there has been no chosen isomorphism $\phi : D \xrightarrow{\sim} D'$ between defect groups of the two blocks, this should be any indecomposable complex of p-permutation modules (this makes it easy to check that the known examples of Rickard complexes are splendid). Once such an isomorphism is chosen, we should ask that the terms are projective relative to $\Delta_\phi D$. If furthermore maximal subpairs (D, e_D) and $(D', f_{D'})$ are chosen, one should put the more precise assumption on the vertex-subpairs.

All of this is compatible with the previous definitions of Rickard, Harris and Linckelmann. This should also be seen as a special (but more explicit) case of the theory of basic equivalences of Puig [Pu4].

A.2. Nilpotent blocks, p-rank 2 and Puig's finiteness conjecture

Let P be an abelian p-group. We know that given an indecomposable bimodule M inducing a stable equivalence between ekG and kP, there is an integer n such that $\Omega^n M$ induces a Rickard equivalence (it is crucial, here, to assume P is abelian). This follows from Dade's classification of endotrivial modules for abelian p-groups [Da2] (we use the fact that $M^* \otimes_{ekG} V$ is endotrivial if V is simple). A difficulty arises when n is negative: starting from a geometrical complex giving the stable equivalence, we cannot, in general, get a geometrical complex giving a Rickard equivalence. Note that when P is cyclic, we can always take $n = 0$ or $n = 1$, so we assume now P is not cyclic.

Now take for M an indecomposable p-permutation module. Assume the block ekG is self-dual. Then "the" simple ekG-module V is self-dual; hence $M^* \otimes_{ekG} V$ is also self-dual. Since $M^* \otimes_{ekG} V \simeq \Omega^{-n}k$, it follows that $\Omega^{2n}k \simeq k$. As P is not cyclic, this forces $n = 0$, so that M already induces a Morita equivalence.

Let us explain how the problem should be solved in general. We work over a semi-local ring $\mathcal{O} = \mathbb{Z}_p[\zeta]$ where ζ is a suitable root of unity. The complex conjugation induces an automorphism of $\mathcal{O}$ and the semi-linear anti-automorphism of $\mathcal{O}G$ given by $ag \mapsto \bar{a}g^{-1}$ for $a \in \mathcal{O}$ and $g \in G$ stabilizes all the blocks of $\mathcal{O}G$. Now, the extension of the theory to the base ring $\mathcal{O}$ should provide the conclusion that M is always a Morita equivalence! Note that similar considerations should solve some cases of Puig's finiteness conjecture. Examples are blocks with defect group $\mathbb{Z}/2 \times \mathbb{Z}/2$ and (non-abelian defect) nilpotent blocks, where the problem is to show that the endopermutation module involved has finite order in the Dade group.

This would give a proof of Conjecture 7.6 for nilpotent blocks with abelian defect and would show that there are stable equivalences for blocks with abelian defect groups and nilpotent local structure: the existence of such stable equivalences is due to Puig [Pu3], who shows how to glue endopermutation modules — this was a important source of inspiration for this work on local constructions.

When the defect groups are abelian with p-rank 2, we obtain a geometrical stable equivalence between A and B (here, the problem of "negative n" does not arise, since the local blocks are p-central extensions of blocks with cyclic defect). We then get a splendid Rickard equivalence for blocks with defect group $\mathbb{Z}/2 \times \mathbb{Z}/2$ (without using complex conjugation, it is not clear that this comes from a geometrical complex). Let us explain this last result more precisely.

Assume $D \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$. Let M be the unique indecomposable direct summand of $e\mathcal{O}Gf$ with vertex ΔD. Let $E = N_G(D, e_D)/C_G(D)$, where (D, e_D) is a maximal e-subpair.

If $|E| = 1$, then there is an integer n such that $\Omega^n M$ induces a Morita equivalence between $e\mathcal{O}G$ and $f\mathcal{O}H$. This can be realized using a Rickard complex of p-permutation modules, by truncating a projective resolution (or a relatively injective resolution) of M (note that by the discussion above, we should have $n = 0$).

If $|E| = 3$, then there is an integer n and a direct summand R of a projective cover of $\Omega^n M$ such that the complex $0 \to R \to \Omega^n M \to 0$ induces a Rickard equivalence. As above, this can be realized by a Rickard complex of p-permutation modules (although, again, we should already have $n = 0$).

In order to be able to construct stable equivalences for blocks with defect group $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ by gluing local Rickard equivalences coming from the construction above, we would need $n = 0$.

For nilpotent blocks with non-abelian defect groups, we have a Morita equivalence between A and kD, as shown by Puig, but this cannot be realized, in general, by a complex of p-permutation modules. A common generalization of this case and of the abelian defect case is the case where the hyperfocal subgroup of the defect group (as defined by Puig in [Pu5]) is abelian: it is tempting to ask whether there is still a Rickard equivalence between A and B. Such an equivalence would not be splendid in general, but should be basic in the sense of Puig [Pu4].

A.3. Further categories of sheaves

A.3.1. Stable category of *p*-permutation modules. In order to construct global complexes of p-permutation modules by gluing local complexes, we can look for a local description of the category of p-permutation modules. The projective modules will not be (directly) reflected locally, so we should rather look for a local description of the stable category of p-permutation kG-modules, defined as the quotient category of the category of p-permutation modules by the full subcategory of projective modules.

This can be achieved as follows: a p-permutation module gives rise, via the Brauer functor, to a family of p-permutation modules for $kN_G(Q)/Q$, where Q runs over the non-trivial p-subgroups of G, together with certain isomorphisms. Conversely, the data of a compatible family of $kN_G(Q)/Q$-modules comes from a p-permutation kG-module, unique up to a unique isomorphism in the stable category. To make all of this precise, one needs first to check various natural properties of the Brauer functor and then to define a category of "sheaves" of p-permutation modules over the p-subgroups complex, where the transitions maps account for isomorphisms $\mathrm{Br}_Q \mathrm{Br}_P V \xrightarrow{\sim} \mathrm{Br}_Q V$ when $P \lhd Q$.

We can use this construction to glue a compatible family of complexes of p-permutation modules for various $kN_G(Q)/Q$ ($Q \neq 1$). What we get is not quite a complex of p-permutation modules, but only a graded p-permutation kG-module with an endomorphism d of degree 1 such that d^2 is a projective map — it is not clear how to get a genuine complex (i.e., with $d^2 = 0$) from such a complex.

This construction has nevertheless some interesting and useful consequences. It permits to get direct sum decompositions of certain full subcategories of the stable category of p-permutation modules defined by conditions on vertex-subpairs.}

A.3.2. Complexes of geometrical origin. The idea we pursue here, following a suggestion of Alperin, is to study a category of complexes that behaves like the complexes of chains of a finite simplicial complex acted upon by G — we want nevertheless to replace the assumption that the terms are permutation modules by the fact that they are p-permutation modules. We introduce a category where the objects are p-permutation modules M with additional structure, namely, compatible splittings of the canonical morphisms $M^P \to \mathrm{Br}_P M$ for every p-subgroup M.

The induction and restriction functors can be extended to this category, and the induction is left adjoint to the restriction. Something new happens nevertheless, namely the Brauer functor has now a left adjoint! One can consider (fairly complicated) categories of presheaves of objects of such categories, over the p-subgroup complex.

Nevertheless, one can deal with an easier subcategory. This category can be given the structure of an exact category, by deciding that the exact sequences are those sequences that are split exact when we only keep the p-permutation modules underlying the objects. It turns out that the full subcategory of projective objects for this structure of exact category is related to the category $\tilde{\mathcal{E}}$ of §7.2.

A.4. p-extensions

A.4.1. Inductive approach to Broué's conjecture. In the inductive approach to Broué's conjecture described in §7.3 the problem is to give a positive answer to Question 7.7. Let us recall that inductive approach.

First start with defect 0, then proceed by induction on the order of the defect group:

- Construct liftings of Rickard equivalences through p-central extensions (in order to go from a Rickard equivalence between blocks of $C_G(P)/P$ and $C_H(P)/P$, which exists by the induction hypothesis, to one between blocks of $C_G(P)$ and $C_H(P)$ for $P \neq 1$).
- Use outer automorphism equivariance to extend the Rickard equivalence (in order to obtain a Rickard equivalence between blocks of $N_G(P)$ and $N_H(P)$).
- Gluing: construct a global stable equivalence from the compatible system of local Rickard equivalences.
- Lift the stable equivalence to a derived equivalence.

The introduction of geometrical complexes in §7 was needed in order to achieve the gluing step.

We want to explain two facts here. First, the lifting problem through central p-extensions can be handled *a priori*. Then, in order to lift the stable equivalence to a Rickard equivalence, one need not worry about geometrical complexes, as long as the lifting is of a particular type. So, in order to solve Conjecture 7.6, it is enough to give a positive answer to a weaker form of Question 7.7. Note that this works as well for non-principal blocks. It is not necessary to worry about p-central extensions. More precisely, a positive answer to the question can be deduced from the case where the central p-subgroup P of Hypothesis $3''$ is trivial and one can forget about the geometrical complexes and even work directly with stable equivalences induced by bimodules.

A.4.2. Let us review first the general problem of p-extensions. We assume Hypothesis $3''$ but, to simplify, we assume $X = G$, i.e., we forget about automorphisms.

Consider a Rickard complex C of $(e\mathcal{O}G, f\mathcal{O}H)$-bimodules. Under what condition does there exist a Rickard complex $\tilde{C}$ of $(\tilde{e}\mathcal{O}\tilde{G}, \tilde{f}\mathcal{O}\tilde{H})$-bimodules with ΔP acting trivially, such that $\mathcal{O}(G\times H^\circ)\otimes_{\mathcal{O}(\tilde{G}\times\tilde{H}^\circ)}\tilde{C} \simeq C$? As pointed out in §5.2.4, it is enough to construct a complex $\tilde{C}$ of $(\tilde{e}\mathcal{O}\tilde{G}, \tilde{f}\mathcal{O}\tilde{H})$-bimodules with ΔP acting trivially, and such that $\mathcal{O}(G \times H^\circ) \otimes_{\mathcal{O}(\tilde{G}\times\tilde{H}^\circ)} \tilde{C} \simeq C$.

The problem of lifting from $\mathcal{O}G$ to $\mathcal{O}\tilde{G}$ is similar to the one of lifting from kG to $\mathcal{O}G$: in both cases, we obtain an algebra as a quotient of the other algebra by an ideal I generated by central elements contained in the radical (the set of $x - 1$ for $x \in P$ in the first case, the radical of $\mathcal{O}$ in the second case) and the algebra is complete for the toplogy defined by that ideal. We can then apply lifting methods, similar to that of Rickard [Ri3]. The only problem is to lift the individual terms of the complex to modules that are free as left $\mathcal{O}P$-modules and are acted on trivially by ΔP. Once this is done, we lift the differential d of the complex C to get a graded endomorphism $\tilde{d}$. The square $\tilde{d}^2$ is not zero, but it so modulo I. Thanks to the vanishing of the module of homomorphisms from C to $C[2]$ in the homotopy category of complexes of $(e\mathcal{O}G, f\mathcal{O}H)$-bimodules, we can change $\tilde{d}$ to get a new lifting $\tilde{d}_1$ of d such that $\tilde{d}_1^2$ is zero modulo I^2. We go on and, since $I^n = 0$ for n large enough, we eventually get a genuine differential and the complex $\tilde{C}$ is constructed.

Note that this strategy works even if P is a normal but non-central p-subgroup.

When the complex C is splendid and the inverse image of D in $\tilde{G}$ is abelian, one shows that it is always possible to lift the individual terms of the complex. Here, we consider the case where $H = N_G(D)$ but this applies as well to the case where we consider any other finite group H and any block idempotent f of $\mathcal{O}H$. Then we only require C to be a complex of p-permutation modules and we use the results of §A.1.2.

A.4.3. Let us come to a more concrete problem. Suppose we are given a complex C of $(e\mathcal{O}G, f\mathcal{O}H)$-bimodules inducing a stable equivalence and a complex $\tilde{C}$ of $(\tilde{e}\mathcal{O}\tilde{G}, \tilde{f}\mathcal{O}\tilde{H})$-bimodules with ΔP acting trivially, such that

$$\mathcal{O}(G \times H^\circ) \otimes_{\mathcal{O}(\tilde{G} \times \tilde{H}^\circ)} \tilde{C} \simeq C.$$

Consider a (bounded) complex of exact $(e\mathcal{O}G, f\mathcal{O}H)$-bimodules all of whose non-zero terms are projective, except for the one of smallest degree d, $M = C^d$. Then M induces a stable equivalence between $e\mathcal{O}G$ and $f\mathcal{O}H$.

Suppose there is a complex Z of projective $(e\mathcal{O}G, f\mathcal{O}H)$-bimodules and a morphism $\psi : Z \to M$ (where M is seen as a complex concentrated in degree d) whose cone is a Rickard complex. This means we have been able to lift the stable equivalence induced by M to a Rickard equivalence in a particular way. Then one gets a complex Z' of projective $(e\mathcal{O}G, f\mathcal{O}H)$-bimodules and a morphism $\psi' : Z' \to C$ whose cone is a Rickard complex. Now, there is a complex $\tilde{Z}'$ of exact $(\tilde{e}\mathcal{O}\tilde{G}, \tilde{f}\mathcal{O}\tilde{H})$-bimodules with ΔP acting trivially and a map $\tilde{\psi}' : \tilde{Z}' \to \tilde{C}$ such that $1 \otimes \tilde{\psi}' = \psi'$. The cone $\tilde{C}'$ of $\tilde{\psi}'$ is now a Rickard complex of $(\tilde{e}\mathcal{O}\tilde{G}, \tilde{f}\mathcal{O}\tilde{H})$-bimodules with ΔP acting trivially.

Finally, if $\tilde{C}$ comes from a geometrical complex as in Question 7.7, then we will be able to find another geometrical complex giving rise to $\tilde{C}'$. This means that it is enough to lift, in a particular way, certain stable equivalences between $e\mathcal{O}G$ and $f\mathcal{O}H$ to Rickard equivalences in order to solve Question 7.7 — in general, all of this should be done in a way compatible with the action of X/G.

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