HIGHER TENSOR PRODUCT FOR $\mathfrak{sl}_2$ AND WEBSTER ALGEBRAS

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Abstract. We construct a model for the tensor product of the regular 2-representation of the enveloping algebra of $\mathfrak{sl}_2^+$ with the vector 2-representation, based on the ∞-categorical definition of [Rou3]. Our model contains McMillan's minimal one [Mc]. Our use of an infinite family of generators provides a simpler model that we prove is equivalent to Webster's tensor product category [We].

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1. Introduction

Higher representation theory is a version of representation theory where vector spaces are replaced by categories. In this article, we consider the case of $\mathfrak{sl}_2$, where the original theory was introduced in [ChRou], and the graded version we consider in [Lau].

In [Rou3], the second author defines a tensor product for 2-representations and conjectures that the tensor product of simple 2-representations agrees with Webster's quiver Hecke algebra 2-representations.

The tensor product of 2-representations involves only the positive part of the Lie algebra and in this article we consider the case of the one-dimensional Lie algebra $\mathfrak{sl}_2^+$. A 2-representation of $\mathfrak{sl}_2^+$ on a category over a field k is the data of an endofunctor E and of endomorphisms x of E and τ of E^2 satisfying Hecke relations (cf (1) in §3.1). Equivalently, it is the action of the monoidal category $\mathcal{U}$ generated by E and endomorphisms x and τ. The endomorphism algebras in $\mathcal{U}$ are nil affine Hecke algebras of type A. Note that $\mathcal{U}$ is a 2-representation by left tensor product. The 2-dimensional vector representation has a categorical version, $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$, with $\mathcal{L}_i$ the category with one object whose endomorphism ring is $k[y]$.

We consider the tensor product $\mathcal{U} \bigotimes \mathcal{L}$ of $\mathcal{U}$ and $\mathcal{L}$. The general theory defines this as an ∞-category with an ∞-categorical 2-representation. It also provides a dg-model for the category together with an endofunctor E, and that is our starting point. The endomorphisms x and τ exist only on the derived category. We construct a new t-structure on the derived category for which the action of E is exact and we use the compatibility $\mathcal{U} \bigotimes \mathcal{L} \twoheadrightarrow \mathcal{U} \bigotimes \mathcal{L}_0 = \mathcal{U}$ arising from the categorical version of the one-dimensional quotient of the vector representation to determine x and τ, in the new t-structure.

This t-structure was considered by McMillan [Mc] who provides an intricate description of the endomorphism ring of a small progenerator in the heart and the bimodule corresponding to E, as well as the endomorphisms x and τ. What we do here instead is consider an infinite progenerating family for this t-structure. We show that the full subcategory with those as objects has a very simple description: it is one of Webster's tensor product categories.

Our work is a step in the description of the braided monoidal category of 2-representations of $\mathfrak{sl}_2$, toward fulfilling Crane and Frenkel's program [CrFr].

2. Notations

Let $\phi : A \to B$ be a morphism of rings. Given M a left (resp. right) B-module, we denote by ϕM (resp. $M\phi$) the left (resp. right) A-module whose underlying set is M and where the action of $a \in A$ is that of $\phi(a)$.

We denote by A-mod the category of finitely generated A-modules and by $D^b(A)$ its derived category (when A-mod is abelian).

We fix a field k. By category, we mean a category enriched in k-vector spaces and all functors between such categories are assumed to be k-linear.

Given a category $\mathcal{C}$, we denote by $\mathcal{C}^i$ the full subcategory of the category of functors from $\mathcal{C}^{\mathrm{opp}}$ to the category of sets with objects the direct summands of finite direct sums of objects of the form $\mathrm{Hom}(-,c)$ for $c \in \mathcal{C}$. We identify $\mathcal{C}$ with a full subcategory of the k-linear category $\mathcal{C}^i$ via the Yoneda functor.

We denote by $\mathcal{C}[y]$ the category with the same objects as $\mathcal{C}$ and with $\mathrm{Hom}_{\mathcal{C}[y]}(c,c') = \mathrm{Hom}_{\mathcal{C}}(c,c') \otimes_{\mathbf{Z}} \mathbf{Z}[y]$.

Given an additive category $\mathcal{A}$, we denote by $\mathrm{Comp}(\mathcal{A})$ the category of complexes of objects of $\mathcal{A}$ and by $\mathrm{Ho}(\mathcal{A})$ its homotopy category.

Let $\mathcal{U}$ be a monoidal category. We denote by $\mathcal{U}^{\mathrm{rev}}$ the monoidal category equal to $\mathcal{U}$ as a category but with $u \otimes^{\mathrm{rev}} u' = u' \otimes u$.

Consider an action of $\mathcal{U}$ on a category $\mathcal{A}$ via the monoidal functor $M : \mathcal{U} \to \mathrm{End}(\mathcal{A})$. Let $\Sigma : \mathcal{U} \to \mathcal{U}$ be a monoidal functor. We denote by $\Sigma^*\mathcal{A}$ the category $\mathcal{A}$ with the action of $\mathcal{U}$ given by $M\Sigma$.

Given E a functor, we write E for $\mathrm{id}_E : E \to E$.

3. 2-Representations of $\mathfrak{sl}_2^+$

3.1. **Monoidal category.** Let $\mathcal{U}$ be the (k-enriched) monoidal category generated by an object E and arrows $x : E \to E$ and $\tau : E^2 \to E^2$ modulo relations

(1) $\tau^2 = 0,\ (\tau E)\circ(E\tau)\circ(\tau E) = (E\tau)\circ(\tau E)\circ(E\tau),\ \tau\circ(xE)-(Ex)\circ\tau = 1 = (xE)\circ\tau-\tau\circ(Ex).$

There are isomorphisms of monoidal categories

$$\omega : \mathcal{U} \xrightarrow{\sim} \mathcal{U}^{\mathrm{opp}},\ E \mapsto E,\ \tau \mapsto \tau,\ x \mapsto x$$

$$\text{and } \mathcal{U} \xrightarrow{\sim} \mathcal{U}^{\mathrm{rev}},\ E \mapsto E,\ \tau \mapsto \tau,\ x \mapsto -x.$$

Given $n \geq 0$, let H_n be the k-algebra generated by $x_1, \ldots, x_n$, $\tau_1, \ldots, \tau_{n-1}$ modulo relations

$$\tau_i^2 = 0, \tau_i\tau_j = \tau_j\tau_i \text{ if } |i-j| \neq 1 \text{ and } \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$$

$$x_r x_s = x_s x_r,\ \tau_i x_r = x_r \tau_i \text{ if } r \neq i, i+1 \text{ and } \tau_i x_i - x_{i+1}\tau_i = 1 = x_i\tau_i - \tau_i x_{i+1}$$

for $1 \leq r, s \leq n$ and $1 \leq i, j < n$.

We put $s_i = \tau_i(x_i - x_{i+1}) - 1$. There is an injective morphism of algebras $k[x_1, \ldots, x_n] \rtimes \mathfrak{S}_n \to H_n,\ x_i \mapsto x_i, (i, i+1) \mapsto s_i$.

We identify H_{n-1} with a subalgebra of H_n. The left H_{n-1}-module H_n is free with basis $(1, \tau_{n-1}, \ldots, \tau_{n-1}\cdots\tau_1)$.

We put $H_{-1} = 0$.

There is an isomorphism of algebras

$$H_n \xrightarrow{\sim} \mathrm{End}_{\mathcal{U}}(E^n)^{\mathrm{opp}},\ x_i \mapsto E^{n-i}xE^{i-1},\ \tau_i \mapsto E^{n-i-1}\tau E^{i-1}.$$

It gives rise to a fully faithful functor

$$\bigoplus_n \mathrm{Hom}(E^n,-):\mathcal{U}\to\bigoplus_{n\geq 0} H_n\text{-mod}\,.$$

3.2. **Algebra embeddings.** Given $m,n\geq 0$, $r\in\{1,\ldots,m\}\bigcup \mathbf{Z}_{>m+n}$ and $s\geq \sup(m+n,r)$, we define an injective algebra morphism

$$\iota_{m,r}:H_n[y]\to H_s,\ x_i\mapsto x_{m+i},\ \tau_i\mapsto\tau_{m+i},\ y\mapsto x_r.$$

We put $\iota_0=\iota_{0,n+1}$ and $\iota_m=\iota_{m,m}$ for $m>0$.

Note that $H_s\iota_{m,r}$ is a $(H_s,H_n[y])$-bimodule. We extend it to a $(H_s[y],H_n[y])$-bimodule by letting the left action of y be the right action of y.

Similarly, we extend the structure of $(H_n[y],H_s)$-bimodule on $\iota_{m,r}H_s$ to a structure of $(H_n[y],H_s[y])$-bimodule by letting the right action of y be the left action of y.

We define the morphism of algebras

$$\iota_m^y:H_n[y]\to H_{m+n}[y],\ x_i\mapsto x_{m+i},\ \tau_i\mapsto\tau_{m+i},\ y\mapsto y.$$

3.3. **$\mathcal{U}$-modules.** A $\mathcal{U}$-module is a category $\mathcal{V}$ endowed with an action of $\mathcal{U}$, ie, a monoidal functor $\mathcal{U}\to\mathrm{End}(\mathcal{V})$. An action of $\mathcal{U}$ on a category $\mathcal{V}$ is the same data (equivalence of 2-categories) as the data of an endofunctor E of $\mathcal{V}$ and of $x\in\mathrm{End}(E)$ and $\tau\in\mathrm{End}(E^2)$ satisfying (1).

Consider two $\mathcal{U}$-modules $(\mathcal{V},E,x,\tau)$ and $(\mathcal{V}',E',x',\tau')$. A morphism of $\mathcal{U}$-modules

$$(\mathcal{V},E,x,\tau)\to(\mathcal{V}',E',x',\tau')$$

is the data of a pair (Φ,φ) where $\Phi:\mathcal{V}\to\mathcal{V}'$ is a functor and $\varphi:\Phi E\xrightarrow{\sim}E'\Phi$ is an isomorphism such that $\varphi\circ(\Phi x)=(x'\Phi)\circ\varphi$ and $(E'\varphi)\circ(\varphi E)\circ(\Phi\tau)=(\tau'\Phi)\circ(E'\varphi)\circ(\varphi E)$.

A right $\mathcal{U}$-module is defined to be a $\mathcal{U}^{\mathrm{rev}}$-module.

3.4. **Regular 2-representation.** The left and right actions of $\mathcal{U}$ on itself by tensor product give rise to commuting left and right actions of $\mathcal{U}$ on the abelian category $\mathcal{A}=\bigoplus_{n\geq 0}H_n$-mod. Let us describe those actions.

For the left (resp. right) action, E acts as the direct sum of functors $H_{n+1}\iota_l\otimes_{H_n}-:H_n\text{-mod}\to H_{n+1}\text{-mod}$ with $l=0$ (resp. $l=1$). The endomorphism x acts as right multiplication by x_{n+1} (resp. x_1) on H_{n+1}. Via the isomorphism $H_{n+2}\iota_l\otimes_{H_{n+1}}H_{n+1}\xrightarrow{\sim}H_{n+2},\ a\otimes b\mapsto a\iota_l(b)$, the endomorphism τ acts on H_{n+2} as right multiplication by τ_{n+1} (resp. τ_1).

3.5. **Vector 2-representation.** Let $\mathcal{L}_r=k[y]$-mod for $r=0,1$. There is an action of $\mathcal{U}$ on $\mathcal{L}=\mathcal{L}_0\oplus\mathcal{L}_1$. The functor E acts as $\mathrm{Id}:\mathcal{L}_0\to\mathcal{L}_1$ and x acts as multiplication by y.

4. Description of the tensor product

4.1. **Tensor product.** We consider the tensor product $\mathcal{L}\bigotimes\mathcal{A}$ for the action of $\mathcal{U}$ [Rou3]. This is an abelian category with an ∞-categorical action of $\mathcal{U}$ on $D^b(\mathcal{L}\bigotimes\mathcal{A})$. We will only use here the classical action.

We will make a key use of the fact that the canonical morphism of $\mathcal{U}$-modules $\mathcal{L}\to\mathcal{L}_0$ (where E acts by 0 on $\mathcal{L}_0$) induces a morphism of $\mathcal{U}$-modules

$$D^b(\mathcal{L}\bigotimes\mathcal{A})\to D^b(\mathcal{L}_0\bigotimes\mathcal{A}). \tag{2}$$

We will describe the category $\mathcal{B}=\mathcal{L}\bigotimes\mathcal{A}$ and the action of E following [Rou3]. The actions of x and τ will be described later using (2).

Remark 4.1. The general tensor product construction for $\mathcal{U}$ provides a dg-category. Because the 2-representation of $\mathfrak{sl}_2^{>0}$ extends to $\mathfrak{sl}_2^{\geq 0}$, that dg-category is canonically equivalent to categories of complexes for an algebra with zero differential, cf [Rou3]. This is the model we will use for $\mathcal{B}$.

4.2. **Underlying category.** Given $n\geq 0$, let $\mathcal{B}_n$ be the category with objects $\begin{bmatrix} M\\ \gamma\uparrow\\ N\end{bmatrix}$, where M is an $H_n[y]$-module, N an $H_{n-1}[y]$-module, and $\gamma: N\to M$ is a morphism of $H_{n-1}[y]$-modules such that $(y-x_n)\gamma(m)=0$ for all $m\in N$. We define

$$\mathrm{Hom}_{\mathcal{B}_n}(\begin{bmatrix} M\\ \gamma\uparrow\\ N\end{bmatrix},\begin{bmatrix} M'\\ \gamma'\uparrow\\ N'\end{bmatrix})$$

to be the set of pairs (f,g) in $\mathrm{Hom}_{H_n[y]}(M,M')\oplus\mathrm{Hom}_{H_{n-1}[y]}(N,N')$ such that $f\circ\gamma=\gamma'\circ g$.

Let

$$P_n^+=\begin{bmatrix} H_n[y]\\ \uparrow\\ 0\end{bmatrix} \text{ and } P_n^-=\begin{bmatrix} H_n\iota_0\\ \mathrm{can}\uparrow\\ H_{n-1}[y]\end{bmatrix}.$$

We have

$$\mathrm{Hom}(P_n^+,\begin{bmatrix} M\\ \gamma\uparrow\\ N\end{bmatrix})\xrightarrow{\sim} M,\ (f,g)\mapsto f(1)$$

$$\mathrm{Hom}(P_n^-,\begin{bmatrix} M\\ \gamma\uparrow\\ N\end{bmatrix})\xrightarrow{\sim} N,\ (f,g)\mapsto g(1).$$

So $P_n^+\oplus P_n^-$ is a progenerator of $\mathcal{B}_n$. Right multiplication provides isomorphisms of algebras

$$H_n[y]\xrightarrow{\sim}\mathrm{End}(P_n^+)^{\mathrm{opp}} \text{ and } H_{n-1}[y]\xrightarrow{\sim}\mathrm{End}(P_n^-)^{\mathrm{opp}}.$$

We put $\mathcal{B}=\bigoplus_{n\geq 0}\mathcal{B}_n$.

4.3. **Left action.** We describe here the functor providing the action of the generator E of $\mathcal{U}$ on $\mathcal{B}$, following [Rou3].

Proposition 4.2. *We have a functor* $E : \mathcal{B}_n \to \mathrm{Comp}^b(\mathcal{B}_{n+1})$ *given by*

$$E\left(\begin{bmatrix} M \\ \gamma\uparrow \\ N \end{bmatrix}\right) = \left[\begin{array}{ccc} 0 \to H_{n+1}[y] \otimes_{H_n[y]} M & \xrightarrow{\text{can}} & H_{n+1}\iota_0 \otimes_{H_n[y]} M \to 0 \\ {\scriptstyle a\otimes m \mapsto a(x_n - y)\tau_n \otimes \gamma(m)}\uparrow & & \uparrow{\scriptstyle m \mapsto 1\otimes m} \\ 0 \to H_n[y] \otimes_{H_{n-1}[y]} N & \xrightarrow[a\otimes m \mapsto a\gamma(m)]{} & M \to 0 \end{array}\right]$$

where the non-zero terms of the complexes are in degrees 0 *and* 1 *and*

$$E\left(\begin{bmatrix} f \\ g \end{bmatrix}\right) = \begin{bmatrix} 1\otimes f & 1 \otimes f \\ 1 \otimes g & f \end{bmatrix}.$$

The functor E *is exact.*

Proof. Note first that the map

$$u : H_n[y] \otimes_{H_{n-1}[y]} N \to H_{n+1}[y] \otimes_{H_n[y]} M,\ a \otimes m \mapsto a(x_n - y)\tau_n \otimes \gamma(m)$$

is well defined and it is a morphisn of $H_n[y]$-modules. We have

$$(y - x_{n+1})u(a \otimes m) = a(x_n - y)(y - x_{n+1})\tau_n \otimes \gamma(m) = a\tau_n(y - x_{n+1}) \otimes (x_n - y)\gamma(m) = 0.$$

Let us show that E defines a complex. Given $a \in H_n[y]$ and $m \in N$, we have

$$u(a \otimes m) = a(\tau_n(x_{n+1} - y) + 1) \otimes \gamma(m),$$

and this has image $a \otimes \gamma(m)$ in $H_{n+1}\iota_0 \otimes_{H_n[y]} M$. It follows that E defines a complex of objects of $\mathcal{B}_{n+1}$.

Since H_{n+1} is projective as a right $H_n[x_{n+1}]$-module, it follows that $H_{n+1}\iota_0$ is projective as a right $H_n[y]$-module. We deduce that E is exact. □

We extend E to an exact functor $E : \mathrm{Comp}^b(\mathcal{B}_n) \to \mathrm{Comp}^b(\mathcal{B}_{n+1})$ by taking total complexes and we still denote by E the endofunctor of $\mathrm{Comp}^b(\mathcal{B})$ obtained as the sum of the functors E for each n.

Note that the action of E on $\mathrm{Comp}^b(\mathcal{B})$ does not extend to an action of $\mathcal{U}$. It only gives rise to an action of $\mathcal{U}$ on $D^b(\mathcal{B})$.

4.4. **Compatibility with the filtration of $\mathcal{L}$.** We have a canonical morphism of $\mathcal{U}$-modules $\mathcal{L} \to \mathcal{L}_0$ given by the projection. It induces a morphism of $\mathcal{U}$-modules $D^b(\mathcal{L} \bigotimes \mathcal{A}) \to D^b(\mathcal{L}_0 \bigotimes \mathcal{A})$. Via the canonical equivalence $\mathcal{A}[y] \xrightarrow{\sim} \mathcal{L}_0 \bigotimes \mathcal{A}$, we obtain a $\mathcal{U}$-module functor $D^b(\mathcal{L} \bigotimes \mathcal{A}) \to D^b(\mathcal{A}[y])$. Let us describe this explicitely.

Consider $\Upsilon, \Upsilon_- : \mathcal{B} \to \mathcal{A}[y]$ given by

$$\Upsilon\left(\begin{bmatrix} M \\ \gamma\uparrow \\ N \end{bmatrix}\right) = M \text{ and } \Upsilon_-\left(\begin{bmatrix} M \\ \gamma\uparrow \\ N \end{bmatrix}\right) = N.$$

There is a quasi-isomorphism φ_+

$$\begin{array}{ccc} H_{n+1}[y]\otimes_{H_n[y]}\Upsilon(\begin{bmatrix} M \\ \gamma\uparrow \\ N\end{bmatrix}) = H_{n+1}[y]\otimes_{H_n[y]} M & \longrightarrow & 0 \\ \varphi_+\Big\downarrow \qquad\qquad \Big\downarrow{\scriptstyle a\otimes m\mapsto a(x_{n+1}-y)\otimes m} & & \\ \Upsilon E(\begin{bmatrix} M \\ \gamma\uparrow \\ N\end{bmatrix}) = H_{n+1}[y]\otimes_{H_n[y]} M & \xrightarrow[\text{can}]{} & H_{n+1}\iota_0\otimes_{H_n[y]} M \end{array}$$

because

$$0\to H_{n+1}[y]\xrightarrow{a\mapsto a(x_{n+1}-y)} H_{n+1}[y]\xrightarrow{\text{can}} H_{n+1}\iota_0\to 0$$

is a split exact sequence of projective right $H_n[y]$-modules.

This gives rise to a morphism of $\mathcal{U}$-modules

$$D^b(\mathcal{B})\xrightarrow{(\Upsilon,\varphi_+^{-1})} D^b(\mathcal{A}[y]).$$

4.5. **Right action.** The right action of $\mathcal{U}$ on itself by tensor product induces a right action of $\mathcal{U}$ on $\mathcal{L}\circledS\mathcal{U}$, hence on $\mathcal{B}$. It is given by

$$(\begin{bmatrix} M \\ \gamma\uparrow \\ N\end{bmatrix})E = \begin{bmatrix} H_{n+1}[y]\iota_1^y\otimes_{H_n[y]} M \\ {\scriptstyle 1\otimes\gamma}\uparrow \\ H_n[y]\iota_1^y\otimes_{H_{n-1}[y]} N\end{bmatrix}.$$

The endomorphism x acts by

$$\begin{bmatrix} a\otimes m \\ a'\otimes m'\end{bmatrix}\mapsto\begin{bmatrix} ax_1\otimes m \\ a'x_1\otimes m'\end{bmatrix}.$$

Multiplication gives an isomorphism

$$(\begin{bmatrix} M \\ \gamma\uparrow \\ N\end{bmatrix})E^2\xrightarrow{\sim}\begin{bmatrix} H_{n+2}[y]\iota_2^y\otimes_{H_n[y]} M \\ {\scriptstyle 1\otimes\gamma}\uparrow \\ H_{n+1}[y]\iota_2^y\otimes_{H_{n-1}[y]} N\end{bmatrix}$$

and τ acts by

$$\begin{bmatrix} a\otimes m \\ a'\otimes m'\end{bmatrix}\mapsto\begin{bmatrix} a\tau_1\otimes m \\ a'\tau_1\otimes m'\end{bmatrix}.$$

Note that all the functors constructed in §4.4 are compatible with the right action of $\mathcal{U}$.

4.6. **Commutation of the actions.** There is an isomorphism from

$$\big(E(\begin{bmatrix} M \\ \gamma\uparrow \\ N\end{bmatrix})\big)E =$$

$$= \begin{bmatrix} H_{n+2}[y]\iota_1^y \otimes_{H_{n+1}[y]} H_{n+1}[y] \otimes_{H_n[y]} M & \xrightarrow{\text{can}} & H_{n+2}[y]\iota_1^y \otimes_{H_{n+1}[y]} H_{n+1}\iota_0 \otimes_{H_n[y]} M \\ {\scriptstyle a\otimes b\otimes m\mapsto a\otimes b(x_n-y)\tau_n\otimes\gamma(m)}\Big\uparrow & & \Big\uparrow{\scriptstyle a\otimes m\mapsto a\otimes 1\otimes m} \\ H_{n+1}[y]\iota_1^y \otimes_{H_n[y]} H_n[y] \otimes_{H_{n-1}[y]} N & \xrightarrow[a\otimes b\otimes m\mapsto a\otimes b\gamma(m)]{} & H_{n+1}[y]\iota_1^y \otimes_{H_n[y]} M \end{bmatrix}$$

to

$$L = \begin{bmatrix} H_{n+2}[y]\iota_1^y \otimes_{H_n[y]} M & \xrightarrow{\text{can}} & H_{n+2}\iota_{1,n+2} \otimes_{H_n[y]} M \\ {\scriptstyle a\otimes m\mapsto a(x_{n+1}-y)\tau_{n+1}\otimes\gamma(m)}\Big\uparrow & & \Big\uparrow{\scriptstyle a\otimes m\mapsto a\otimes m} \\ H_{n+1}[y]\iota_1^y \otimes_{H_{n-1}[y]} N & \xrightarrow[a\otimes m\mapsto a\otimes\gamma(m)]{} & H_{n+1}[y]\iota_1^y \otimes_{H_n[y]} M \end{bmatrix}$$

given by

$$\begin{bmatrix} a_1\otimes b_1\otimes m_1 & a_2\otimes b_2\otimes m_2 \\ a_3\otimes b_3\otimes m_3 & a_4\otimes m_4 \end{bmatrix} \mapsto \begin{bmatrix} a_1\iota_1^y(b_1)\otimes m_1 & a_2\iota_1^y(b_2)\otimes m_2 \\ a_3\iota_1^y(b_3)\otimes m_3 & a_4\otimes m_4 \end{bmatrix}$$

We have also an isomorphism from

$$E\Big(\Big(\begin{bmatrix} M \\ \gamma\Big\uparrow \\ N \end{bmatrix}\Big)E\Big) =$$

$$= \begin{bmatrix} H_{n+2}[y] \otimes_{H_{n+1}[y]} H_{n+1}[y]\iota_1^y \otimes_{H_n[y]} M & \xrightarrow{\text{can}} & H_{n+2}\iota_0 \otimes_{H_{n+1}[y]} H_{n+1}[y]\iota_1^y \otimes_{H_n[y]} M \\ {\scriptstyle a\otimes b\otimes m\mapsto a(x_{n+1}-y)\tau_{n+1}\otimes b\otimes\gamma(m)}\Big\uparrow & & \Big\uparrow{\scriptstyle a\otimes m\mapsto 1\otimes a\otimes m} \\ H_{n+1}[y] \otimes_{H_n[y]} H_n[y]\iota_1^y \otimes_{H_{n-1}[y]} N & \xrightarrow[a\otimes b\otimes m\mapsto ab\otimes\gamma(m)]{} & H_{n+1}[y]\iota_1^y \otimes_{H_n[y]} M \end{bmatrix}$$

to L given by

$$\begin{bmatrix} a_1\otimes b_1\otimes m_1 & a_2\otimes b_2\otimes m_2 \\ a_3\otimes b_3\otimes m_3 & a_4\otimes m_4 \end{bmatrix} \mapsto \begin{bmatrix} a_1b_1\otimes m_1 & a_2b_2\otimes m_2 \\ a_3b_3\otimes m_3 & a_4\otimes m_4 \end{bmatrix}.$$

Composing the second isomorphism with the inverse of the first one, we obtain an isomorphism of functors

$$E\big((?)E\big) \xrightarrow{\sim} \big(E(?)\big)E.$$

5. New t-structure

5.1. Hecke calculations. Fix $n \geq 0$. We put $\iota_1 H_0 = 0$. We define

$$\tilde{\Delta}_n = \sum_{1\leq r\leq n} \tau_r \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{r-1} \in H_n[y] \otimes_{H_{n-1}[y]} \iota_1 H_n$$

and we denote by Δ_n the image of $\tilde{\Delta}_n$ in $H_n\iota_0 \otimes_{H_{n-1}[y]} \iota_1 H_n$.

Lemma 5.1. *We have $a\Delta_n = \Delta_n a$ for all $a \in H_n$.*

Proof. Given $i \in \{1, \dots, n-1\}$, we have

$$\tau_i\Delta_n = \sum_{1\le r<i} \tau_r \cdots \tau_{n-1}\tau_{i-1} \otimes \tau_1 \cdots \tau_{r-1} + 0 + \tau_i \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_i + \sum_{i+1<r\le n} \tau_r \cdots \tau_{n-1}\tau_i \otimes \tau_1 \cdots \tau_{r-1}$$

$$= \sum_{1\le r<i} \tau_r \cdots \tau_{n-1} \otimes \tau_i\tau_1 \cdots \tau_{r-1} + \tau_i \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_i + \sum_{i+1<r\le n} \tau_r \cdots \tau_{n-1} \otimes \tau_{i+1}\tau_1 \cdots \tau_{r-1}$$

$$= \sum_{1\le r<i} \tau_r \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{r-1}\tau_i + \tau_i \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_i + \sum_{i+1<r\le n} \tau_r \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{r-1}\tau_i$$

$$= \Delta_n\tau_i.$$

Consider now $i \in \{1, \dots, n\}$. Given $n \ge d \ge j \ge 1$, we have

$$x_j\tau_j \cdots \tau_{d-1} = \tau_j \cdots \tau_{d-1}x_d + \sum_{j\le s\le d-1} \tau_j \cdots \tau_{s-1}\tau_{s+1} \cdots \tau_{d-1},$$

hence

$$x_i\Delta_n = \sum_{1\le r<i} \tau_r \cdots \tau_{n-1}x_{i-1} \otimes \tau_1 \cdots \tau_{r-1} - \sum_{1\le r<i} \tau_r \cdots \tau_{i-2}\tau_i \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{r-1}$$

$$+\tau_i \cdots \tau_{n-1}x_n \otimes \tau_1 \cdots \tau_{i-1} + \sum_{s=i}^{n-1} \tau_i \cdots \tau_{s-1}\tau_{s+1} \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{i-1} + \sum_{r=i+1}^{n} \tau_r \cdots \tau_{n-1}x_i \otimes \tau_1 \cdots \tau_{r-1}.$$

We have

$$\sum_{r=1}^{i-1} \tau_r \cdots \tau_{n-1}x_{i-1} \otimes \tau_1 \cdots \tau_{r-1} = \sum_{r=1}^{i-1} \tau_r \cdots \tau_{n-1} \otimes x_i\tau_1 \cdots \tau_{r-1} = \sum_{r=1}^{i-1} \tau_r \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{r-1}x_i,$$

$$\sum_{r=i+1}^{n} \tau_r \cdots \tau_{n-1}x_i \otimes \tau_1 \cdots \tau_{r-1} = \sum_{r=i+1}^{n} \tau_r \cdots \tau_{n-1} \otimes x_{i+1}\tau_1 \cdots \tau_{r-1}$$

$$= \sum_{r=i+1}^{n} \tau_r \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{r-1}x_i - \sum_{r=i+1}^{n} \tau_r \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{i-1}\tau_{i+1} \cdots \tau_{r-1}$$

and

$$\tau_i \cdots \tau_{n-1}x_n \otimes \tau_1 \cdots \tau_{i-1} = \tau_i \cdots \tau_{n-1} \otimes x_1\tau_1 \cdots \tau_{i-1}$$

$$= \tau_i \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{i-1}x_i + \sum_{s=1}^{i-1} \tau_i \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{s-1}\tau_{s+1} \cdots \tau_{i-1}.$$

We have

$$\sum_{s=i}^{n-1} \tau_i \cdots \tau_{s-1}\tau_{s+1} \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{i-1} = \sum_{s=i}^{n-1} \tau_{s+1} \cdots \tau_{n-1}\tau_i \cdots \tau_{s-1} \otimes \tau_1 \cdots \tau_{i-1}$$

$$= \sum_{s=i}^{n-1} \tau_{s+1} \cdots \tau_{n-1} \otimes \tau_{i+1} \cdots \tau_s\tau_1 \cdots \tau_{i-1}$$

$$= \sum_{r=i+1}^{n} \tau_r \cdots \tau_{n-1}\tau_1 \cdots \tau_{i-1} \otimes \tau_{i+1} \cdots \tau_{r-1}$$

and

$$\begin{aligned}\sum_{s=1}^{i-1} \tau_i \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{s-1}\tau_{s+1} \cdots \tau_{i-1} &= \sum_{s=1}^{i-1} \tau_i \cdots \tau_{n-1} \otimes \tau_{s+1} \cdots \tau_{i-1}\tau_1 \cdots \tau_{s-1} \\ &= \sum_{s=1}^{i-1} \tau_i \cdots \tau_{n-1}\tau_s \cdots \tau_{i-2} \otimes \tau_1 \cdots \tau_{s-1} \\ &= \sum_{r=1}^{i-1} \tau_r \cdots \tau_{i-2}\tau_i \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{r-1}.\end{aligned}$$

We deduce that $x_i\Delta_n = \Delta_n x_i$ □

We define a morphism of left $H_n[y]$-modules

$$\begin{aligned}\nu_n : \iota_1 H_{n+1} &\to H_n[y] \oplus H_n[y] \otimes_{H_{n-1}[y]} \iota_1 H_n \\ \iota_1(a)\tau_1 \cdots \tau_r &\mapsto \begin{cases} (a, a\tilde{\Delta}_n) & \text{if } r = n \\ (0, a(y - x_n) \otimes \tau_1 \cdots \tau_r) & \text{if } r < n. \end{cases}\end{aligned}$$

where $a \in H_n[y]$ and $r \in \{0, \ldots, n\}$ and a morphism of left $H_{n+1}[y]$-modules

$$\begin{aligned}\nu'_n : H_{n+1}\iota_0 \otimes_{H_n[y]} \iota_1 H_{n+1} &\to H_{n+1}\iota_0 \oplus H_{n+1}\iota_{0,n} \otimes_{H_{n-1}[y]} \iota_1 H_n \\ a \otimes \tau_1 \cdots \tau_r &\mapsto \begin{cases} (a, -a\sum_{s=1}^{n} \tau_s \cdots \tau_{n-1}s_n \otimes \tau_1 \cdots \tau_{s-1}) & \text{if } r = n \\ (0, a(x_n - x_{n+1})s_n \otimes \tau_1 \cdots \tau_r) & \text{if } r < n. \end{cases}\end{aligned}$$

where $a \in H_{n+1}$ and $r \in \{0, \ldots, n\}$.

Note that there is a commutative diagram
(3)

$$\begin{array}{ccc} H_{n+1}\iota_0 \otimes_{H_n[y]} \iota_1 H_{n+1} & \xrightarrow{\nu'_n} & H_{n+1}\iota_0 \oplus H_{n+1}\iota_{0,n} \otimes_{H_{n-1}[y]} \iota_1 H_n \\ \downarrow{\scriptstyle 1\otimes\nu_n} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} & 0 \\ 0 & a\otimes b \mapsto -as_n \otimes b \end{pmatrix}} \\ H_{n+1}\iota_0 \otimes_{H_n[y]} H_n[y] \oplus H_{n+1}\iota_0 \otimes_{H_n[y]} H_n[y] \otimes_{H_{n-1}[y]} \iota_1 H_n & \xrightarrow[\text{mult}]{\sim} & H_{n+1}\iota_0 \oplus H_{n+1}\iota_0 \otimes_{H_{n-1}[y]} \iota_1 H_n \end{array}$$

Lemma 5.2. *There is an exact sequence of $H_n[y]$-modules*

$$0 \to \iota_1 H_{n+1} \xrightarrow{\nu_n} H_n[y] \oplus H_n[y] \otimes_{H_{n-1}[y]} \iota_1 H_n \xrightarrow{(a \mapsto -a\Delta_n, \mathrm{can})} H_n\iota_0 \otimes_{H_{n-1}[y]} \iota_1 H_n \to 0$$

and an exact sequence of $H_{n+1}[y]$-modules

$$0 \to H_{n+1}\iota_0 \otimes_{H_n[y]} \iota_1 H_{n+1} \xrightarrow{\nu'_n} H_{n+1}\iota_0 \oplus H_{n+1}\iota_{0,n} \otimes_{H_{n-1}[y]} \iota_1 H_n \xrightarrow{(a \mapsto -a\Delta_n, \mathrm{can})} \bar{H}_{n+1} \otimes_{H_{n-1}[y]} \iota_1 H_n \to 0$$

where $\bar{H}_{n+1} = H_{n+1}\iota_0 / H_{n+1}(x_{n+1} - x_n)$.

Proof. There is an exact sequence of $(H_n[y], H_{n-1}[y])$-bimodules

$$0 \to H_n[y] \xrightarrow{a \mapsto a(y-x_n)} H_n[y] \xrightarrow{\text{can}} H_n\iota_0 \to 0.$$

Tensoring by the free $H_{n-1}[y]$-module $\iota_1 H_n$, we obtain an exact sequence of $H_n[y]$-modules

$$0 \to H_n[y] \otimes_{H_{n-1}[y]} \iota_1 H_n \xrightarrow{a\otimes b \mapsto a(y-x_n)\otimes b} H_n[y] \otimes_{H_{n-1}[y]} \iota_1 H_n \xrightarrow{\text{can}} H_n\iota_0 \otimes_{H_{n-1}[y]} \iota_1 H_n \to 0.$$

That exact sequence is a subcomplex of the complex of $H_n[y]$-modules

$$(4) \quad 0 \to H_n[y] \oplus H_n[y] \otimes_{H_{n-1}[y]} \iota_1 H_n \xrightarrow{\begin{pmatrix} \text{id} & 0 \\ a \mapsto a\tilde{\Delta}_n & a\otimes b \mapsto a(y-x_n)\otimes b \end{pmatrix}} H_n[y] \oplus H_n[y] \otimes_{H_{n-1}[y]} \iota_1 H_n \xrightarrow{(a \mapsto -a\Delta_n, \text{can})} H_n\iota_0 \otimes_{H_{n-1}[y]} \iota_1 H_n \to 0$$

and the quotient is acyclic. We deduce that (4) is acyclic.

We have an isomorphism of $H_n[y]$-modules

$$\iota_1 H_{n+1} \xrightarrow{\sim} H_n[y] \oplus H_n[y] \otimes_{H_{n-1}[y]} \iota_1 H_n$$
$$\iota_1(a)\tau_1 \cdots \tau_r \mapsto \begin{cases} (a, 0) & \text{if } r = n \\ (0, a \otimes \tau_1 \cdots \tau_r) & \text{if } r < n \end{cases}$$

where $a \in H_n[y]$. The composition of this map with the map $\begin{pmatrix} \text{id} & 0 \\ a \mapsto a\tilde{\Delta}_n & a\otimes b \mapsto a(y-x_n)\otimes b \end{pmatrix}$ of (4) is equal to ν_n. We deduce that the first sequence of the lemma is exact.

Tensoring the first exact sequence of the lemma with $H_{n+1}\iota_0$ gives an exact sequence of $H_{n+1}[y]$-modules

$$0 \to H_{n+1}\iota_0 \otimes_{H_n[y]} \iota_1 H_{n+1} \xrightarrow{\nu''_n} H_{n+1}\iota_0 \oplus H_{n+1}\iota_0 \otimes_{H_{n-1}[y]} \iota_1 H_n \xrightarrow{(a \mapsto -a\Delta_n, \text{can})} \bar{H}_{n+1} \otimes_{H_{n-1}[y]} \iota_1 H_n \to 0$$

where

$$\nu''_n(a \otimes \tau_1 \cdots \tau_r) = \begin{cases} (a, \sum_{s=1}^n \tau_s \cdots \tau_{n-1} \otimes \tau_1 \cdots \tau_{s-1}) & \text{if } r = n \\ (0, a(y-x_n) \otimes \tau_1 \cdots \tau_r) & \text{if } r < n. \end{cases}$$

for $a \in H_{n+1}$ and $r \in \{0, \ldots, n\}$. There is an isomorphism of $(H_{n+1}[y], H_{n-1}[y])$-bimodules

$$f : H_{n+1}\iota_0 \xrightarrow{\sim} H_{n+1}\iota_{0,n}, \ a \mapsto -as_n.$$

We have $\nu'_n = (\text{id}, f \otimes 1) \circ \nu''_n$ (commutative diagram (3)) and we deduce that the second sequence of the lemma is exact. $\square$

Lemma 5.3. *The following diagram commutes*

$$\begin{array}{ccc} H_{n+1}[y] & \xrightarrow{\begin{pmatrix}\text{can}\\ a\mapsto a\Delta_n\end{pmatrix}} & H_{n+1}\iota_0 \oplus H_{n+1}\iota_{0,n}\otimes_{H_{n-1}[y]}\iota_1 H_n \\ \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \nu'_n} \\ H_{n+1}[y] & \xrightarrow[a\mapsto a\Delta_{n+1}]{} & H_{n+1}\iota_0\otimes_{H_n[y]}\iota_1 H_{n+1} \end{array}$$

Proof. We have

$$\begin{aligned}\nu'_n(\Delta_{n+1}) &= \sum_{r=1}^{n+1}\nu'_n(\tau_r\cdots\tau_n\otimes\tau_1\cdots\tau_{r-1})\\ &= (1,-\sum_{s=1}^{n}\tau_s\cdots\tau_{n-1}s_n\otimes\tau_1\cdots\tau_{s-1}) + \sum_{r=1}^{n}(0,\tau_r\cdots\tau_n(x_n-x_{n+1})s_n\otimes\tau_1\cdots\tau_{r-1})\\ &= (1,\sum_{r=1}^{n}\tau_r\cdots\tau_{n-1}s_n^2\otimes\tau_1\cdots\tau_{r-1}).\end{aligned}$$

□

Lemma 5.4. *We have* $s_{n-1}\cdots s_1(x_n-y)\cdots(x_2-y)\cdot\Delta_n = 1\otimes 1$ *in* $H_n\iota_0\otimes_{H_{n-1}[y]}\iota_1 H_n$.

Proof. We proceed by induction on n. The lemma is clear for $n=1$. Assume the lemma holds for n.

By Lemma 5.3, we have

$$\nu'_n(s_n\cdots s_1(x_{n+1}-y)\cdots(x_2-y)\cdot\Delta_{n+1}) = (0, s_n\cdots s_1(x_{n+1}-y)\cdots(x_2-y)\cdot\Delta_n) =$$
$$= (0, s_n(x_{n+1}-y)s_{n-1}\cdots s_1(x_n-y)\cdots(x_2-y)\cdot\Delta_n) = (0, s_n(x_{n+1}-y)(1\otimes 1)) = \nu'_n(1\otimes 1).$$

The lemma follows now from the injectivity of ν'_n (Lemma 5.2). □

Lemma 5.5. *The element* Δ_n *generates* $H_n\iota_0\otimes_{H_{n-1}[y]}\iota_1 H_n$ *as a* $H_n[y]$*-module.*

Proof. Note that $1\otimes 1$ generates $H_n\iota_0\otimes_{H_{n-1}[y]}\iota_1 H_n$ as a $(H_n[y],H_n)$-bimodule. It follows from Lemma 5.4 that Δ_n generates $H_n\iota_0\otimes_{H_{n-1}[y]}\iota_1 H_n$ as a $(H_n[y],H_n)$-bimodule, hence as a $H_n[y]$-module by Lemma 5.1. □

5.2. **Generators.** Given $n\geq 0$, we define a complex Y_n of objects of $\mathcal{B}_n$ in degrees 0 and 1

$$Y_n = \begin{bmatrix} H_n[y] & \xrightarrow{a\mapsto a\Delta_n} & H_n\iota_0\otimes_{H_{n-1}[y]}\iota_1 H_n \\ & & \uparrow{\scriptstyle a\mapsto 1\otimes a} \\ 0 & \longrightarrow & \iota_1 H_n \end{bmatrix}.$$

Thanks to Lemma 5.1, right multiplication defines a morphism

$$\gamma_n : H_n \to \mathrm{End}(Y_n)^{\mathrm{opp}}.$$

The canonical isomorphism $H_{n-1}[y]\otimes_{H_{n-1}[y]}\iota_1 H_n \xrightarrow{\sim} \iota_1 H_n$ induces an isomorphism

$$(0\to P_n^+ \to P_n^- \otimes_{H_{n-1}[y]} \iota_1 H_n \to 0) \xrightarrow{\sim} Y_n.$$

We have isomorphisms

$$H_{n+1}[y]\otimes_{H_n[y]} H_n[y] \xrightarrow[\text{mult.}]{\sim} H_{n+1}[y],\ \ H_{n+1}\iota_0\otimes_{H_n[y]} H_n[y] \xrightarrow[\text{mult.}]{\sim} H_{n+1}\iota_0,$$

$$H_{n+1}[y]\otimes_{H_n[y]} H_n\iota_0\otimes_{H_{n-1}[y]}\iota_1 H_n \xrightarrow{\sim} H_{n+1}\iota_{0,n}\otimes_{H_{n-1}[y]}\iota_1 H_n,\ \ ay^i\otimes b\otimes c\mapsto abx_n^i\otimes c$$

for $a\in H_{n+1}$, $b,c\in H_n$ and $i\geq 0$ and

$$H_{n+1}\iota_0\otimes_{H_n[y]} H_n\iota_0\otimes_{H_{n-1}[y]}\iota_1 H_n \xrightarrow{\sim} \bar{H}_{n+1}\otimes_{H_{n-1}[y]}\iota_1 H_n,\ \ a\otimes b\otimes c\mapsto ab\otimes c.$$

They give rise to an isomorphism

$$f_n' : E(Y_n) \xrightarrow{\sim}$$

$$X_n = \left[\begin{array}{ccccccc} H_{n+1}[y] & \xrightarrow{\begin{pmatrix}\text{can}\\ a\mapsto a\Delta_n\end{pmatrix}} & H_{n+1}\iota_0\oplus H_{n+1}\iota_{0,n}\otimes_{H_{n-1}[y]}\iota_1 H_n & \xrightarrow{(a\mapsto -a\Delta_n,\text{can})} & \bar{H}_{n+1}\otimes_{H_{n-1}[y]}\iota_1 H_n \\ & & \Big\uparrow {\begin{pmatrix}\iota_0 & 0\\ 0 & a\otimes b\mapsto a(x_n-y)\tau_n\otimes b\end{pmatrix}} & & \Big\uparrow \text{can} \\ 0 & \longrightarrow & H_n[y]\oplus H_n[y]\otimes_{H_{n-1}[y]}\iota_1 H_n & \xrightarrow[(a\mapsto -a\Delta_n,\text{can})]{} & H_n\iota_0\otimes_{H_{n-1}[y]}\iota_1 H_n \end{array}\right].$$

Proposition 5.6. *There is a morphism $f_n : Y_{n+1}\to X_n$ given by*

$$\begin{bmatrix}\text{id} & \nu_n' & 0\\ 0 & \nu_n & 0\end{bmatrix}.$$

The map f_n is a quasi-isomorphism.

Proof. Since $(x_n-y)\tau_n = \tau_n(x_{n+1}-x_n)+1 = -s_n$ in $H_{n+1}\iota_{0,n}$, the commutativity of the diagram (3) implies that the following diagram commutes

$$\begin{array}{ccc} H_{n+1}\iota_0\otimes_{H_n[y]}\iota_1 H_{n+1} & \xrightarrow{\nu_n'} & H_{n+1}\iota_0\oplus H_{n+1}\iota_{0,n}\otimes_{H_{n-1}[y]}\iota_1 H_n \\ {\scriptstyle a\mapsto 1\otimes a}\Big\uparrow & & \Big\uparrow {\begin{pmatrix}\iota_0 & 0\\ 0 & a\otimes b\mapsto a(x_n-y)\tau_n\otimes b\end{pmatrix}} \\ \iota_1 H_{n+1} & \xrightarrow[\nu_n]{} & H_n[y]\oplus H_n[y]\otimes_{H_{n-1}[y]}\iota_1 H_n \end{array}$$

This shows that $(f_n)^1$ defines a morphism $(Y_n)^1\to (X_n)^1$.

By Lemma 5.2, the compositions

$$\iota_1 H_n \xrightarrow{\nu_n} H_n[y]\oplus H_n[y]\otimes_{H_{n-1}[y]}\iota_1 H_n \xrightarrow{(a\mapsto -a\Delta_n,\text{can})} H_n\iota_0\otimes_{H_{n-1}[y]}\iota_1 H_n$$

and

$$H_{n+1}\iota_0\otimes_{H_n[y]}\iota_1 H_{n+1} \xrightarrow{\nu_n'} H_{n+1}\iota_0\oplus H_{n+1}\iota_{0,n}\otimes_{H_{n-1}[y]}\iota_1 H_n \xrightarrow{(a\mapsto -a\Delta_n,\text{can})} \bar{H}_{n+1}\otimes_{H_{n-1}[y]}\iota_1 H_n$$

vanish.

Together with Lemma 5.3, this shows that f_n is a morphism of complexes. Lemma 5.2 shows that f_n is a quasi-isomorphism. $\square$

We put $g_n = f_n'^{-1} \circ f_n : Y_{n+1} \to E(Y_n)$. We define inductively $h_n : Y_n \to E^n(Y_0)$ by $h_0 = \mathrm{id}$ and $h_{n+1} = E(h_n) \circ g_n$. As an immediate consequence of Proposition 5.6, we have the following corollary.

Corollary 5.7. *The map $h_n : Y_n \to E^n(Y_0)$ is a quasi-isomorphism.*

Lemma 5.8. *Let C and C' be two complexes of objects of $\mathcal{B}_n$ with*
- *C^0 and C'^0 isomorphic to a direct sum of copies of P_n^+*
- *C^1 and C'^1 isomorphic to a direct sum of copies of P_n^-*
- *$C^i = C'^i = 0$ for $i \notin \{0, 1\}$.*

The map $\Upsilon : \mathrm{Hom}_{D(\mathcal{B}_n)}(C, C')) \to \mathrm{Hom}_{D(H_n[y])}(\Upsilon(C), \Upsilon(C'))$ is injective.

Proof. The canonical map $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{B})}(C, C') \to \mathrm{Hom}_{D(\mathcal{B})}(C, C')$ is an isomorphism. Note that since $\mathrm{Hom}(C^1, C'^0) = 0$, the canonical map $\mathrm{Hom}_{\mathrm{Comp}(\mathcal{B})}(C, C') \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{B})}(C, C')$ is also an isomorphism.

Consider now $\alpha \in \mathrm{Hom}_{\mathrm{Comp}(\mathcal{B})}(C, C')$ such that $\Upsilon(\alpha)$ vanishes in $D(H_n[y])$. We have $H^0(\Upsilon(\alpha)) = 0$. Let $\kappa = \prod_{i=1}^n (y - x_i) \in k[x_1, \ldots, x_n]^{\mathfrak{S}_n}[y] \subset Z(H_n[y])$. We have $\kappa \cdot \Upsilon(P_n^-) = 0$, hence $\kappa\Upsilon(C^0) \subset H^0(\Upsilon(C))$. We deduce that the restriction of $\Upsilon(\alpha^0)$ to $\kappa\Upsilon(C^0)$ vanishes. The algebra $H_n[y]$ is free as a $k[x_1, \ldots, x_n, y]$-module, hence also as a $k[x_1, \ldots, x_n]^{\mathfrak{S}_n}[y]$-module. So the annihilator of κ in $\Upsilon(C'^0)$ is 0. It follows that $\Upsilon(\alpha)^0 = 0$.

The composition $\Upsilon(C^1)[-1] \xrightarrow{\Upsilon(\alpha^1)[-1]} \Upsilon(C'^1)[-1] \xrightarrow{\mathrm{can}} \Upsilon(C')$ is equal to the composition $\Upsilon(C^1)[-1] \xrightarrow{\mathrm{can}} \Upsilon(C) \xrightarrow{\Upsilon(\alpha)} \Upsilon(C')$, hence it vanishes in $D(H_n[y])$. There is an exact sequence

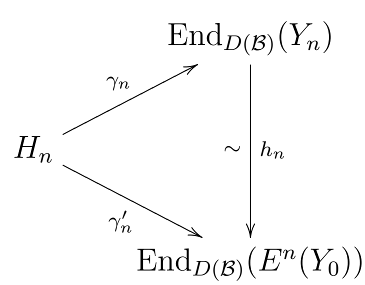
$$\mathrm{Hom}_{H_n[y]}(\Upsilon(C^1), \Upsilon(C'^0)) \to \mathrm{Hom}_{H_n[y]}(\Upsilon(C^1), \Upsilon(C'^1)) \to \mathrm{Hom}_{D(H_n[y])}(\Upsilon(C^1), \Upsilon(C')[1]).$$

Since $\kappa\Upsilon(C^1) = 0$, it follows that $\mathrm{Hom}_{H_n[y]}(\Upsilon(C^1), \Upsilon(C'^0)) = 0$. We deduce that $\Upsilon(\alpha^1) = 0$, hence $\Upsilon(\alpha) = 0$.

Since the structure map $\Upsilon_-(C'^1) \to \Upsilon(C'^1)$ is injective, it follows that $\Upsilon_-(\alpha) = 0$. So $\alpha = 0$. $\square$

The structure of $\mathcal{U}$-module on $D(\mathcal{B})$ provides a morphism of algebras $\gamma_n' : H_n \to \mathrm{End}_{D(\mathcal{B})}(E^n(Y_0))$. The following proposition shows that the action of H_n on Y_n defined earlier is compatible with γ_n'.

Proposition 5.9. *We have a commutative diagram*



Proof. Multiplication provides an isomorphism r from $(\Upsilon E^n(Y_0))^0$, the degree 0 term of $\Upsilon E^n(Y_0)$, to $H_n[y]$. By §4.4, we have a quasi-isomorphism

$$\rho : H_n[y] \to \Upsilon E^n(Y_0),\ a \mapsto r^{-1}(a(x_n - y)\cdots(x_1 - y))$$

and a commutative diagram

$$\begin{CD} \mathrm{End}_{D(\mathcal{B})}(E^n(Y_0)) @>{\Upsilon}>> \mathrm{End}_{D(H_n[y])}(\Upsilon(E^n(Y_0))) \ . \\ @A{\gamma'_n}AA @A{\sim}A{\rho}A \\ H_n[y] @>>{\text{right mult.}}> \mathrm{End}_{H_n[y]}(H_n[y]) \end{CD}$$

There is a commutative diagram

$$\begin{array}{ccc} & & \Upsilon(Y_n) \\ & \overset{a\mapsto a(x_n-y)\cdots(x_1-y)}{\nearrow} & \\ H_n[y] & & \Big\downarrow{\scriptstyle \Upsilon(h_n)} \\ & \underset{\rho}{\searrow} & \\ & & \Upsilon(E^n(Y_0)) \end{array}$$

We deduce that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathrm{End}_{D(\mathcal{B})}(Y_n) & \xrightarrow{\Upsilon} & \mathrm{End}_{D(H_n[y])}(\Upsilon(Y_n)) \\ & \overset{\gamma_n}{\nearrow} & & & \\ H_n & & & & \sim\Big\downarrow{\scriptstyle \Upsilon(h_n)} \\ & \underset{\gamma'_n}{\searrow} & & & \\ & & \mathrm{End}_{D(\mathcal{B})}(E^n(Y_0)) & \xrightarrow[\Upsilon]{} & \mathrm{End}_{D(H_n[y])}(\Upsilon E^n(Y_0)) \end{array}$$

The proposition follows from Lemma 5.8. □

5.3. **2-representation on an additive subcategory.** Given $m, n \geq 0$, we put

$$Y_{n,m} = \begin{bmatrix} H_{m+n}[y] \xrightarrow{d_{n,m}} & H_{m+n}\iota_{m,m+n} \otimes_{H_{n-1}[y]} \iota_1 H_n \\ & \uparrow{\scriptstyle \mathrm{can}} \\ 0 \longrightarrow & H_{m+n-1}[y]\iota_m^y \otimes_{H_{n-1}[y]} \iota_1 H_n \end{bmatrix}$$

where $d_{n,m}(a) = a \sum_{r=1}^n \tau_{m+r} \cdots \tau_{m+n-1} \otimes \tau_1 \cdots \tau_{r-1}$. Note that $Y_n = Y_{n,0}$.

Right multiplication provides a morphism of algebras

$$H_n \to \mathrm{End}_{\mathrm{Comp}(\mathcal{B})}(Y_{n,m})^{\mathrm{opp}}.$$

$$H_m \to \mathrm{End}_{\mathrm{Comp}(\mathcal{B})}(Y_{n,m}),\ h \mapsto \left(\begin{bmatrix} a_0 & b_1\otimes c_1 \\ 0 & b_1'\otimes c_1'\end{bmatrix} \mapsto \begin{bmatrix} a_0 h & b_1 h\otimes c_1 \\ 0 & b_1' h\otimes c_1'\end{bmatrix}\right).$$

Multiplication provides an isomorphism $(Y_n)E^m \xrightarrow{\sim} Y_{n,m}$ that is equivariant for the action of H_m. Corollary 5.7 provides a quasi-isomorphism $Y_{n,m} \to E^n(Y_{0,m})$ compatible with the given maps $H_n \to \mathrm{End}_{D(\mathcal{B})}(Y_{n,m})^{\mathrm{opp}}$ and $H_n \to \mathrm{End}_{D(\mathcal{B})}(E^n(Y_{0,m}))^{\mathrm{opp}}$ (Proposition 5.9).

We have an isomorphism given by multiplication

$$\left(0 \to P^+_{m+n} \to P^-_{m+n} \otimes_{H_{m+n-1}[y]} (H_{m+n-1}[y]\iota^y_m \otimes_{H_{n-1}[y]} \iota_1 H_n) \to 0\right) \xrightarrow{\sim} Y_{n,m}.$$

Lemma 5.10. *Given $m, n, m', n' \geq 0$, we have* $\mathrm{Hom}_{D(\mathcal{B})}(Y_{n,m}, Y_{n',m'}[i]) = 0$ *for $i \neq 0$.*

Proof. Let $r = m + n$. We have a morphism

$$P^-_r \otimes_{H_{r-1}[y]} (H_{r-1}\iota^y_m \otimes_{H_{n-1}[y]} \iota_1 H_n) \to P^-_r,\ a \otimes 1 \otimes \tau_1 \cdots \tau_{r-1} \mapsto \delta_{r,n} a \text{ for } 1 \leq r \leq n.$$

Via the isomorphism above, it gives rise to a morphism $(Y_{n,m})^1 \to P^-_r$ whose composition with $d_{m,n}$ is the canonical map $P^+_r \to P^-_r$. Since $\mathrm{Hom}(P^+_r, P^-_r)$ is generated by the canonical map as a module over $\mathrm{End}(P^-_r)$, it follows that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{B})}(Y_{n,m}, P^-_r) = 0$, hence $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{B})}(Y_{n,m}, Y_{n',m'}[1]) = 0$, as $(Y_{n',m'})^1 \simeq (P^-_{m'+n'})^{\oplus n'}$. Since $\mathrm{Hom}(P^-_r, P^+_r) = 0$, the lemma follows. □

Lemma 5.11. *The objects $Y_{0,m}$ and $Y_{1,m}$ for $m \geq 0$ generate $D^b(\mathcal{B})$ as a thick subcategory.*

Proof. The algebra $H_n[y]$ has finite global dimension since it is isomorphic to a matrix algebra over $k[x_1, \ldots, x_n]^{\mathfrak{S}_n}[y]$ (cf e.g. [Rou2, Proposition 2.21]). It follows that $\mathcal{B}_n$ has finite global dimension since $\mathrm{Hom}(P^-_n, P^+_n) = 0$ and $\mathrm{End}(P^+_n)$ and $\mathrm{End}(P^-_n)$ have both finite global dimension and $P^+_n \oplus P^-_n$ is a progenerator for $\mathcal{B}_n$.

We have $Y_{0,m} = P^+_m$ and the cone of the canonical map $Y_{1,m-1} \to P^+_m$ is isomorphic to P^-_m. It follows that $Y_{0,m} \oplus Y_{1,m-1}$ generates $D^b(\mathcal{B}_n)$. □

Let $\mathcal{T}$ be the full subcategory of $D(\mathcal{B})$ with objects the $Y_{n,m}$'s and $\mathcal{T}'$ the full subcategory of $D(\mathcal{B})$ with objects those isomorphic to $Y_{n,m}$'s.

Lemmas 5.11 and 5.10 provide an equivalence $\mathrm{Ho}^b(\mathcal{T}) \xrightarrow{\sim} D^b(\mathcal{B})$. Proposition 5.6 shows that the action of $\mathcal{U}$ on $D(\mathcal{B})$ restricts to an action of $\mathcal{T}'$. Also, the right action of $\mathcal{U}$ on $D(\mathcal{B})$ restricts to $\mathcal{T}'$. So, we obtain commuting left and right actions of $\mathcal{U}$ on $\mathcal{T}$ and we have the following theorem.

Theorem 5.12. *There is an equivalence of $(\mathcal{U},\mathcal{U})$-bimodules $\mathrm{Ho}^b(\mathcal{T}) \xrightarrow{\sim} D^b(\mathcal{B})$.*

By Lemma 5.8, the functor $\Upsilon : \mathcal{T} \to D(\mathcal{A}[y])$ is a faithful functor of $(\mathcal{U},\mathcal{U})$-bimodules. Since $\Upsilon(Y_{0,0}) = k[y]$, it follows that $\Upsilon(Y_{n,m})$ has homology concentrated in degree 0. Composing with the H^0 functor, we obtain a faithful morphism of $(\mathcal{U},\mathcal{U})$-bimodules

$$Q : \mathcal{T} \to \mathcal{U}[y],\ Y_{n,m} \mapsto E^{n+m}[y].$$

6. Relation with Webster algebras

6.1. Webster category. We follow [We, §4.2].

Let $\tilde{\mathcal{W}}$ be the free $(\mathcal{U},\mathcal{U})$-bimodule generated by an object $*$ and by maps $\rho : *E \to E*$ and $\lambda : E* \to *E$.

Let $\mathcal{W}$ be the $(\mathcal{U},\mathcal{U})$-bimodule quotient of $\tilde{\mathcal{W}}$ by the relations

$$\tau * \circ E\rho \circ \rho E = E\rho \circ \rho E \circ *\tau,\ \tau * \circ \lambda E \circ E\lambda = \lambda E \circ E\lambda \circ *\tau,$$

$$x * \circ \rho = \rho \circ *x,\ *x \circ \lambda = \lambda \circ x*,\ \lambda \circ \rho = *x,\ \rho \circ \lambda = x*$$

$$\rho E \circ *\tau \circ \lambda E - E\lambda \circ \tau * \circ E\rho = E * E.$$

6.2. From Webster's category to the tensor product. Let $\Sigma_y : \mathcal{U}[y] \xrightarrow{\sim} \mathcal{U}[y]$ be the monoidal self-equivalence given by

$$\Sigma_y(E) = E,\ \Sigma_y(x) = x - y \text{ and } \Sigma_y(\tau) = \tau.$$

Let $\tilde{\Phi} : \tilde{\mathcal{W}}[y] \to (\Sigma_y \otimes \Sigma_y)^*\mathcal{T}$ be the (strict) $(\mathcal{U},\mathcal{U})$-bimodules $k[y]$-enriched functor given by

$$\tilde{\Phi}(E^m * E^n) = Y_{m,n},\ \tilde{\Phi}(\rho) = \bar{\rho} \text{ and } \tilde{\Phi}(\lambda) = \bar{\lambda}.$$

Proposition 6.1. *The functor $\tilde{\Phi}$ factors through $\mathcal{W}[y]$ and induces a morphism of $(\mathcal{U},\mathcal{U})$-bimodules $\Phi : \mathcal{W}[y] \to (\Sigma_y \otimes \Sigma_y)^*\mathcal{T}$.*

Proof. Note that

$$Y_{0,0} = Y_0 = \begin{bmatrix} k[y] \\ 0 \end{bmatrix},\ Y_{1,0} = Y_1 = \begin{bmatrix} k[x_1, y] & \xrightarrow{\text{can}} & k[x_1] \\ & & \uparrow \text{id} \\ 0 & \longrightarrow & k[x_1] \end{bmatrix} \text{ and } Y_{0,1} = \begin{bmatrix} k[x_1, y] \\ 0 \end{bmatrix}.$$

The action of x on $Y_{1,0} = E(Y_0)$ (resp. on $Y_{0,1} = (Y_0)E$) is given by multiplication by x_1.

We put

$$\bar{\lambda} = \begin{bmatrix} \text{id} \\ 0 \end{bmatrix} : Y_{1,0} \to Y_{0,1} \text{ and } \bar{\rho} = \begin{bmatrix} x_1 - y \\ 0 \end{bmatrix} : Y_{0,1} \to Y_{1,0}.$$

The quasi-isomorphism $E[y] = k[x_1, y] \to \Upsilon(Y_{1,0})$ is multiplication by $x_1 - y$ in degree 0. The isomorphism $E[y] = k[x_1, y] \to \Upsilon(Y_{0,1})$ is the identity.

It follows that $Q(\bar{\lambda}) = x - y$ and $Q(\bar{\rho}) = \text{id}$.

We consider $\tilde{\Psi} = Q \circ \tilde{\Phi} : \tilde{\mathcal{W}}[y] \to (\Sigma_y \otimes \Sigma_y)^*\mathcal{U}[y]$. We have

$$\tilde{\Psi}(x*) \circ Q(\bar{\rho}) = x - y = Q(\bar{\rho}) \circ \tilde{\Psi}(*x)$$

$$Q\bar{\lambda} \circ Q(\bar{\rho}) = x - y = Q(*x)$$

$$Q(\bar{\rho}) \circ Q(\bar{\lambda}) = x - y = Q(x*)$$

$$\tilde{\Psi}(\tau*) \circ EQ(\bar{\rho}) \circ Q(\bar{\rho})E = \tau = EQ(\bar{\rho}) \circ Q(\bar{\rho})E \circ \tilde{\Psi}(*\tau)$$

$$Q(\bar{\rho})E \circ \tilde{\Psi}(*\tau) \circ Q(\bar{\lambda})E - EQ(\bar{\lambda}) \circ \tilde{\Psi}(\tau*) \circ EQ(\bar{\rho}) = \tau(xE - y) - (Ex - y)\tau = 1.$$

It follows that $\tilde{\Psi}$ factors through $\mathcal{W}[y]$. Since Q is faithful, the proposition follows. □

Theorem 6.2. *The functor* Φ *is an isomorphism of* $(\mathcal{U},\mathcal{U})$*-bimodules* $\mathcal{W}[y] \xrightarrow{\sim} (\Sigma_y \otimes \Sigma_y)^* \mathcal{T}$.

Theorem 6.2 will be deduced from its graded version (Theorem 6.5). Let us first show the faithfulness of Φ.

Lemma 6.3. *The functor* Φ *is faithful.*

Proof. We define a $k[u]$-linear functor

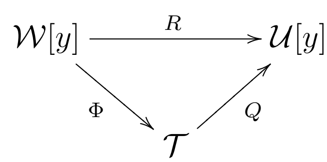
$$\tilde{R} : \tilde{\mathcal{W}}[y] \to (\Sigma_y \otimes \Sigma_y)^* \mathcal{U}[y]$$

of $(\mathcal{U},\mathcal{U})$-bimodules by

$$\tilde{R}(*) = 1, \ \tilde{R}(\lambda) = \tilde{R}(\rho) = E.$$

The functor $\tilde{R}$ factors through $\mathcal{W}[y]$ and induces a $k[y]$-linear functor $R : \mathcal{W}[y] \to (\Sigma_y \otimes \Sigma_y)^* \mathcal{U}[y]$ of $(\mathcal{U},\mathcal{U})$-bimodules. It follows from [We, Proposition 4.16] that the functor R is faithful.

The composition $Q \circ \Phi : \mathcal{W}[y] \to (\Sigma_y \otimes \Sigma_y)^* \mathcal{U}[y]$ is a $k[y]$-linear functor of $(\mathcal{U},\mathcal{U})$-bimodules. The functors $Q \circ \Phi$ and R take the same value on $*$ and on the generating arrows ρ and λ. It follows that they are equal.



Since R is faithful, it follows that Φ is faithful. ☐

6.3. **Gradings.**

6.3.1. *Generalities.* Consider a category $\mathcal{C}$ enriched in graded k-modules. We denote by $\mathcal{C}$-gr the category enriched in k-modules with objects pairs (c, n) where c an object of $\mathcal{C}$ and $n \in \mathbf{Z}$ and with $\mathrm{Hom}_{\mathcal{C}\text{-gr}}((c, n), (c', n'))$ the space of homogeneous elements of degree $n' - n$ of $\mathrm{Hom}_{\mathcal{C}}(c, c')$. This is a graded category, i.e., a category endowed with an action of $\mathbf{Z}$. We denote by $c \mapsto v^n c$ the action of $n \in \mathbf{Z}$.

For example, if M is a graded k-vector space, then $(v^n M)_i = M_{i-n}$. When the homogenous components of M are finite-dimensional, we put $\mathrm{grdim}(M) = \sum_{i \in \mathbf{Z}} v^i \dim(M_i)$. We put $q = v^2$.

Given $\mathcal{C}$ a graded k-linear category, we equip $K_0(\mathcal{C})$ with a structure of $\mathbf{Z}[v^{\pm 1}]$-module by $v[M] = [vM]$. When Hom's in $\mathcal{C}$ are finite-dimensional, we define a bilinear form

$$K_0(\mathcal{C}) \otimes_{\mathbf{Z}} K_0(\mathcal{C}) \to \mathbf{Z}[v^{\pm 1}], \ \langle [M], [N] \rangle = \mathrm{grdim}(\mathrm{Hom}(M, N)).$$

6.3.2. *Gradings of* 2*-representations.* We enrich the monoidal category $\mathcal{U}$ in graded k-vector spaces by setting $\deg(x) = 2$ and $\deg(\tau) = -2$. Similarly, we define a grading on the algebra H_n by setting $\deg(x_i) = 2$ and $\deg(\tau_i) = -2$.

We define $\deg(y) = 2$. We define $\mathcal{B}'_n$ to be the category with objects $\begin{bmatrix} M_n \\ \gamma \uparrow \\ M_{n-1} \end{bmatrix}$, where M_r is a graded $H_r[y]$-module and γ is a morphism of graded $H_{n-1}[y]$-modules $\gamma : M_{n-1} \to M_n$

such that $(y-x_n)\gamma(m)=0$ for all $m\in M_{n-1}$. We define $\mathrm{Hom}_{\mathcal{B}'_n}(M,N)$ to be the subspace of $\mathrm{Hom}_{\mathcal{B}_n}(M,N)$ of graded morphisms. We define a graded version of the left action of $\mathcal{U}$ by letting E act by multiplying by v the formula in Proposition 4.2. The graded version of the left action of $\mathcal{U}$ is obtained by using the formula in §4.5. So, we have a structure of ($\mathcal{U}$-gr,$\mathcal{U}$-gr)-bimodule on $\mathcal{B}'$.

We define a graded structure on $Y_{n,m}$ by

$$Y_{n,m}=\begin{bmatrix} H_{m+n}[y] \xrightarrow{d_{m,n}} & q^{n-1}H_{m+n}\iota_{m,m+n}\otimes_{H_{n-1}[y]}\iota_1 H_n \\ & \uparrow \mathrm{can} \\ 0 \longrightarrow & q^{n-1}H_{m+n-1}[y]\iota_m^y\otimes_{H_{n-1}[y]}\iota_1 H_n \end{bmatrix}$$

Corollary 5.7 provides a graded quasi-isomorphism $Y_{n,0}\to E^nY_{0,0}$.

There is a unique enrichment in graded k-vector spaces of $\mathcal{W}$ that makes the structure of $(\mathcal{U},\mathcal{U})$-bimodule compatible with gradings and with $\deg(\rho)=2$ and $\deg(\lambda)=0$. Note that while our choice of gradings differ from the one in [We, Definition 4.4], the categories of graded modules are equivalent.

The functor Φ gives rise to a morphism of graded ($\mathcal{U}$-gr,$\mathcal{U}$-gr)-bimodules $\mathcal{W}[y]$-gr $\to(\Sigma_y\otimes\Sigma_y)^*\mathcal{T}$-gr.

Lemma 6.4. *The functor Φ induces an isometry $K_0(\mathcal{W}[y]\text{-gr})\xrightarrow{\sim}K_0(\mathcal{T}\text{-gr})$.*

Proof. Let $\tilde{U}$ be the $\mathbf{Q}(v)$-algebra generated by $e,f,k^{\pm1}$ modulo relations

$$ke=v^2ek,\ kf=v^{-2}fk \text{ and } ef-fe=\frac{k-k^{-1}}{v-v^{-1}}.$$

We consider the coproduct on $\tilde{U}$ given by

$$\Delta(e)=e\otimes k+1\otimes e,\ \Delta(f)=f\otimes 1+k^{-1}\otimes f \text{ and } \Delta(k)=k^{-1}.$$

We put $[n]_v=\frac{v^n-v^{-1}}{v-v^{-1}}$ and $[n]_v!=[2]_v\cdots[n]_v$.

Let $R=\mathbf{Z}[v,v^{-1}]$ and let U be the R-subalgebra of $\tilde{U}$ generated by the elements $e^{(n)}=\frac{e^n}{[n]_v!}$, $f^{(n)}=\frac{f^n}{[n]_v!}$ for $n\geq 1$ and by $k^{\pm1}$.

Let U^+ be the R-subalgebra of U generated by the elements $e^{(n)}=\frac{e^n}{[n]_v!}$ for $n\geq 1$. There is an isomorphism of R-algebras

$$U^+\xrightarrow{\sim}K_0(\mathcal{U}\text{-gr}),\ e\mapsto[E].$$

As a consequence, if $\mathcal{M}$ is a graded category with a graded action of $\mathcal{U}$-gr, then $K_0(\mathcal{M})$ has a structure of U^+-module.

Consider the $(n+1)$-dimensional representation $L(n)$ of U with R-basis $(b_i)_{0\leq i\leq n}$ and

$$f(b_i)=\delta_{i\neq 0}[n-i+1]_vb_{i-1}, e(b_i)=\delta_{i\neq n}[i+1]_vb_{i+1} \text{ and } k(b_i)=v^{2i-n}.$$

Let $V=L(1)$ and put $b_-=b_0$ and $b_+=b_1$.

We follow Webster, but we swap e and f and k and k^{-1}.

The q-Shapovalov form on $L(n)\otimes V$ is the $\mathbf{Z}$-linear map $(L(n)\otimes V)\times(L(n)\otimes V)\to\mathbf{Q}(v)$ defined by

$$\langle v^a b_i\otimes e^c b_-, v^b b_j\otimes e^d b_-\rangle = v^{b-a}\delta_{i,j}\delta_{c,d}\frac{\prod_{r=1}^i(1-q^{n-r+1})}{\prod_{r=1}^i(1-q^r)}.$$

for $a,b\in\mathbf{Z}$, $i,j\in\{0,\ldots,n\}$ and $c,d\in\{0,1\}$.

We define a new form on $L(n)\otimes V$ by

$$(u,u') = \langle u, u' + (v-v^{-1})(e\otimes f)(u')\rangle.$$

So

$$(v^a b_i\otimes b_-, v^b b_j\otimes b_-) = (v^a b_i\otimes b_+, v^b b_j\otimes b_+) = v^{b-a}\delta_{i,j}\frac{\prod_{r=1}^i(1-q^{n-r+1})}{\prod_{r=1}^i(1-q^r)},$$

$$(v^a b_i\otimes b_+, v^b b_j\otimes b_-) = 0 \text{ and } (v^a b_i\otimes b_-, v^b b_j\otimes b_+) = -v^{b-a}\delta_{i-1,j}v^{-i}\frac{\prod_{r=1}^i(1-q^{n-r+1})}{\prod_{r=1}^{i-1}(1-q^r)}$$

for $a,b\in\mathbf{Z}$ and $i,j\in\{0,\ldots,n\}$.

Taking the limit as $n\to\infty$ (cf [We, Proof of Proposition 4.39]), we obtain a form on $U^+\otimes V$. This is the $\mathbf{Z}$-linear map $(U^+\otimes V)\times(U^+\otimes V)\to\mathbf{Q}(v)$ defined by

$$(v^a e^{(i)}\otimes b_-, v^b e^{(j)}\otimes b_-) = (v^a e^{(i)}\otimes b_+, v^b e^{(j)}\otimes b_+) = v^{b-a}\delta_{i,j}\frac{1}{\prod_{r=1}^i(1-q^r)},$$

$$(v^a e^{(i)}\otimes b_+, v^b e^{(j)}\otimes b_-) = 0 \text{ and } (v^a e^{(i)}\otimes b_-, v^b e^{(j)}\otimes b_+) = -v^{b-a}\delta_{i-1,j}v^{-i}\frac{1}{\prod_{r=1}^{i-1}(1-q^r)}$$

for $a,b\in\mathbf{Z}$ and $i,j\in\mathbf{Z}_{\geq 0}$.

We consider the (U^+,U^+)-bimodule $U^+\otimes V$ where the left action of U^+ is the diagonal action and the right action of U^+ is the action by right multiplication on U^+. Webster [We, Proposition 4.39] shows there is an isomorphism of (U^+,U^+)-bimodules

$$\phi : U^+\otimes V \xrightarrow{\sim} K_0((\mathcal{W}\text{-gr})^i),\ 1\otimes b_-\mapsto [*]$$

that is compatible with the bilinear forms $(-,-)$. Note that Webster's isomorphism is the composition of ϕ with the automorphism coming from the anti-involution of $\mathcal{W}$ that swaps $E^m * E^n$ with $E^n * E^m$, ρ with λ, τ with $-\tau$ and that fixes x.

There is an isomorphism of (U^+,U^+)-bimodules

$$\psi : U^+\otimes V \xrightarrow{\sim} K_0(\mathcal{B}\text{-gr}),\ 1\otimes b_-\mapsto [P_0^+],$$

where the K_0 is for $\mathcal{B}$-gr as an abelian category. We have

$$\psi(e^n\otimes b_-) = [P_n^+] \text{ and } \psi(e^n\otimes b_+) = -v^{-1}[P_{n+1}^-].$$

We have

$$\mathrm{grdim}(H_n[y]) = q^{-n(n-1)/2}\frac{(1-q)\cdots(1-q^n)}{(1-q)^{2n+1}},$$

hence

$$\langle P_i^+, P_j^+\rangle = \delta_{i,j} q^{-i(i-1)/2}\frac{(1-q)\cdots(1-q^i)}{(1-q)^{2i+1}} = \frac{1}{1-q}([i]_v!)^2(e^{(i)}\otimes b_-, e^{(j)}\otimes b_-)$$

$$\langle P_{i+1}^-, P_{j+1}^-\rangle = \delta_{i,j} q^{-i(i-1)/2}\frac{(1-q)\cdots(1-q^i)}{(1-q)^{2i+1}} = \frac{1}{1-q}([i]_v!)^2(e^{(i)}\otimes b_+, e^{(j)}\otimes b_+)$$

$$\langle P_i^+, P_j^-\rangle = \delta_{i,j} q^{-i(i-1)/2}\frac{(1-q)\cdots(1-q^i)}{(1-q)^{2i}} = \frac{-v}{1-q}[i]_v![i-1]_v!(e^{(i)}\otimes b_-, e^{(j-1)}\otimes b_+)$$

$$\langle P_i^-, P_j^+\rangle = 0$$

We deduce that for all $w, w' \in U^+\otimes V$, we have $(\psi(w),\psi(w')) = \frac{1}{1-q}(w,w')$.

The composition $\psi\circ\phi^{-1}$ is a morphism of (U^+,U^+)-bimodules sending $[*]$ to $[P_0^+]$. So, $\psi\circ\phi^{-1}$ and $[\Phi]$ agree on $[*]$. Since $1\otimes b_-$ generates $U^+\otimes V$ as a (U^+,U^+)-bimodule, we deduce that $\psi\circ\phi^{-1} = [\Phi]$ and the proposition follows. □

Theorem 6.5. *The functor Φ is an isomorphism of graded $(\mathcal{U}\text{-gr},\mathcal{U}\text{-gr})$-bimodules $\mathcal{W}[y]\text{-gr} \xrightarrow{\sim} (\Sigma_y\otimes\Sigma_y)^*\mathcal{T}\text{-gr}$.*

Proof. Lemma 6.3 shows that Φ induces an injective map between graded Hom-spaces. By Lemma 6.4, these Hom spaces have the same graded dimension. It follows that Φ is fully faithful, hence it is an equivalence. □

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