LIE GROUPS AND LIE ALGEBRAS 229B

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Contents

1. Lie algebras

1.1. Generalities.

1.1.1. *Definitions.* Let k be a field.

A *Lie algebra* (over k) is a k-vector space $\mathfrak{g}$ endowed with a bilinear map $[-,-]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that $[a,b]=-[b,a]$ and

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0$$

for all $a,b,c\in\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a k-subspace of $\mathfrak{g}$. We say that $\mathfrak{h}$ is

- a *Lie subalgebra* of $\mathfrak{g}$ if $[a,b]\in\mathfrak{h}$ for all $a,b\in\mathfrak{h}$.
- an *ideal* of $\mathfrak{g}$ if $[a,b]\in\mathfrak{h}$ for all $a\in\mathfrak{g}$ and $b\in\mathfrak{h}$.

Let A be a k-algebra. This is a Lie algebra with $[a,b]=ab-ba$. This gives a functor from algebras to Lie algebras. It has a left adjoint, the universal enveloping algebra functor $\mathfrak{g}\mapsto U(\mathfrak{g})$.

Example 1.1. Let V be a vector space. The Lie algebra $\mathrm{End}_k(V)$ is denoted by $\mathfrak{gl}(V)$. When $V=k^n$, we put $\mathfrak{gl}_n(k)=\mathfrak{gl}(k^n)$. We denote by $\mathfrak{sl}_n(k)$ the Lie subalgebra of $\mathfrak{gl}_n(k)$ of matrices with trace 0.

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Example 1.2. Let A be a k-vector space endowed with a bilinear map $A \times A \to A$, $(a,b) \mapsto a \cdot b$. A *derivation* of A is a k-linear endomorphism D of A such that $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$. The set $\mathrm{Der}(A)$ of derivations of A is a Lie subalgebra of $\mathfrak{gl}(A)$.

Proposition 1.3. *Let $\mathfrak{g}$ be a Lie algebra. Given $x \in \mathfrak{g}$, define $\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x,y]$. This is a derivation of $\mathfrak{g}$. The corresponding map $\mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is a morphism of Lie algebras.*

Given $\mathfrak{g}$ a Lie algebra and V, V' two k-subspaces of $\mathfrak{g}$, we denote by $[V, V']$ the k-subspace of $\mathfrak{g}$ generated by elements $[v, v']$ with $v \in V$ and $v' \in V'$.

We denote by $\mathfrak{n}_{\mathfrak{g}}(V)$ the set of elements $x \in \mathfrak{g}$ such that $[x, v] \in V$ for all $v \in V$. This is a Lie subalgebra of $\mathfrak{g}$.

Lemma 1.4. *Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Then $[\mathfrak{g}, \mathfrak{h}]$ is an ideal of $\mathfrak{g}$.*

We write $Z(\mathfrak{g}) = \{x \in \mathfrak{g} | \,\mathrm{ad}\, x = 0\}$ for the *center* of $\mathfrak{g}$.

We say that $\mathfrak{g}$ is *abelian* if $[x,y] = 0$ for all $x, y \in \mathfrak{g}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, then $[x,y] = 0$ for all $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$, *i.e.*, $\mathfrak{g}$ is the direct sum (and the direct product) of its ideals $\mathfrak{a}$ and $\mathfrak{b}$.

1.1.2. *Representations.* Given a k-vector space V, a *representation* of $\mathfrak{g}$ on V is a morphism of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$.

Let $V = \mathfrak{g}$. The *adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is defined by $\mathrm{ad}(g) : g' \mapsto [g, g']$. Its kernel is $Z(\mathfrak{g})$.

1.2. **Nilpotent Lie algebras.** From now on, all Lie algebras to be considered will be assumed to be finite-dimensional.

1.2.1. Let $\mathfrak{g}$ be a Lie algebra. The descending central series are the ideals defined by $C^1\mathfrak{g} = \mathfrak{g}$ and $C^n\mathfrak{g} = [\mathfrak{g}, C^{n-1}\mathfrak{g}]$ for $n \geq 2$.

TFAE:

(1) there is n such that $C^n\mathfrak{g} = 0$
(2) there is n such that $(\mathrm{ad}\, x_1) \cdots (\mathrm{ad}\, x_n) = 0$ for all $x_1, \ldots, x_n \in \mathfrak{g}$.
(3) there is a chain of ideals $0 = \mathfrak{a}_0 \subset \cdots \subset \mathfrak{a}_n = \mathfrak{g}$ such that $\mathfrak{a}_i/\mathfrak{a}_{i-1} \subset Z(\mathfrak{g}/\mathfrak{a}_{i-1})$ for all i (iterated central extension of abelian Lie algebras).

A Lie algebra satisfying these equivalent conditions is called *nilpotent*.

Exercise 1.1. The Lie algebra $\mathfrak{g}$ of strictly upper triangular matrices in $\mathfrak{gl}_n$ is nilpotent. Determine the ideals $C^i\mathfrak{g}$.

1.2.2. Let V be a finite-dimensional vector space over k. A full flag in V is a sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $\dim V_i = i$.

Theorem 1.5 (Engel)**.** *Consider $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation such that $\rho(x)$ is nilpotent for all $x \in \mathfrak{g}$. Then, there is a full flag $V_\bullet$ in V such that $\rho(x)(V_i) \subset V_{i-1}$ for all $x \in \mathfrak{g}$ and all i.*

Corollary 1.6. *$\mathfrak{g}$ is nilpotent iff $\mathrm{ad}(x)$ is nilpotent for all $x \in \mathfrak{g}$.*

1.3. **Solvable Lie algebras.** The derived series of $\mathfrak{g}$ are the ideals defined by $D^1\mathfrak{g} = \mathfrak{g}$ and $D^n\mathfrak{g} = [D^{n-1}\mathfrak{g}, D^{n-1}\mathfrak{g}]$ for $n \geq 2$.

TFAE:

- There is n such that $D^n\mathfrak{g} = 0$
- $\mathfrak{g}$ is a successive extension of abelian Lie algebras.

A Lie algebra satisfying these equivalent conditions is called *solvable*.

Exercise 1.2. The Lie algebra $\mathfrak{g}$ of upper triangular matrices in $\mathfrak{gl}_n$ is solvable. Determine the ideals $D^i\mathfrak{g}$.

Theorem 1.7 (Lie). *Assume k is algebraically closed and has characteristic 0. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ with V a finite-dimensional vector space. If $\mathfrak{g}$ is solvable, there is a full flag $V_\bullet$ of V such that $\rho(x)(V_i) \subset V_i$ for all $x \in \mathfrak{g}$ and all i.*

Corollary 1.8. *$\mathfrak{g}$ solvable, k arbitrary (char 0). Then, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

1.4. **Semi-simple Lie algebras.**

1.4.1. *Bilinear forms.* Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. A bilinear form $\beta : V \times V \to k$ is $\mathfrak{g}$-*invariant* if $\beta(\rho(x)v_1, v_2) = -\beta(v_1, \rho(x)v_2)$ for all $x \in \mathfrak{g}$ and $v_1, v_2 \in V$.

Remark 1.9. Assume $k = \mathbf{C}$. Let G be a complex Lie group with Lie algebra $\mathfrak{g}$ and let $\psi : G \to \mathrm{GL}(V)$ be a representation of G whose associated Lie algebra representation is ρ. The bilinear form β is G-invariant if and only if $\beta(\psi(g)v_1, \psi(g)v_2) = \beta(v_1, v_2)$ for all $g \in G$ and $v_1, v_2 \in V$. Equivalently: $\beta(\psi(g)v_1, v_2) = \beta(v_1, \psi(g^{-1})v_2)$ for all g, v_1, v_2. This equality implies the $\mathfrak{g}$-equivariance of β.

Fix a representation and a $\mathfrak{g}$-invariant bilinear form. Given $L \subset V$, let $L^\perp = \{v \in V | \beta(l, v) = 0 \ \forall l \in L\}$.

Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. A bilinear form $\alpha : \mathfrak{g} \times \mathfrak{g} \to k$ is $\mathfrak{g}$-invariant (for the adjoint representation) if and only if $\alpha([x, y], z) = \alpha(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$.

The bilinear form given by $\beta(x, y) = \mathrm{Tr}_\mathfrak{g}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$ is called the *Killing form*. It is $\mathfrak{g}$-invariant. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{a}^\perp$ is also an ideal. Note also that the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{a}$ is the Killing form of $\mathfrak{a}$.

Exercise 1.3. Show that if $\mathfrak{g}$ is nilpotent, then $\beta = 0$.

1.4.2. *Radical and semi-simple Lie algebras.* Note that given $\mathfrak{a}_1$ and $\mathfrak{a}_2$ two solvable ideals of $\mathfrak{g}$, then $\mathfrak{a}_1 + \mathfrak{a}_2$ is a solvable ideal.

Definition 1.10. *The* radical $\mathrm{rad}(\mathfrak{g})$ *is the largest solvable ideal of* $\mathfrak{g}$.

Definition 1.11. $\mathfrak{g}$ *is* semi-simple *if* $\mathrm{rad}(\mathfrak{g}) = 0$.

Note that $\mathfrak{g}$ is semi-simple if and only if it has no non-zero abelian ideal.

Theorem 1.12. *$\mathfrak{g}$ is semisimple iff the Killing form is non degenerate.*

Theorem 1.13 (Cartan). *Let V be a vector space and $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{g}$ is solvable if and only if $\mathrm{Tr}_V(xy) = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$*

Exercise 1.4. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{C})$. Show that the Killing form is $\beta(x, y) = 2n\mathrm{tr}(xy) - 2\mathrm{tr}(x)\mathrm{tr}(y)$. Deduce that $\mathfrak{sl}_n(\mathbf{C})$ is semi-simple for $n \geq 2$.

Proposition 1.14. $\mathrm{rad}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^\perp$.

Proposition 1.15. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{a}$ an ideal. Then, $\mathfrak{a}^\perp$ is an ideal and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$.*

Definition 1.16. *A Lie algebra is simple if it is non abelian and it has no non-zero proper ideal.*

Proposition 1.17. *Let $\mathfrak{g}$ be a semi-simple Lie algebras. Then, there are ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_n$ and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are simple Lie algebras. This decomposition is unique up to ordering.*

Proposition 1.18. *If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

1.4.3. *Enveloping algebras and Casimir.* The enveloping algebra of $\mathfrak{g}$ is the k-algebra defined by $U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])_{x,y \in \mathfrak{g}}$. The functor U is left adjoint to the canonical functor from algebras to Lie algebras.

If $\mathfrak{g}$ is abelian, then $U(\mathfrak{g}) = S(\mathfrak{g})$ is a polynomial algebra.

Assume $\mathfrak{g}$ is semi-simple and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a faithful representation. There is a $\mathfrak{g}$-invariant symmetric bilinear form $\beta_\rho : \mathfrak{g} \times \mathfrak{g} \to k,\ (x, y) \mapsto \mathrm{Tr}_V(\rho(x)\rho(y))$. It is non-degenerate. We define $C_\rho = \sum e_i f_i \in U(\mathfrak{g})$, where $(e_i)_i$ is a basis of $\mathfrak{g}$ and (f_i) the dual basis with respect to β_ρ. We have $C_\rho \in Z(U(\mathfrak{g}))$. If ρ is simple, then C_ρ acts by $\frac{\dim \mathfrak{g}}{\dim V} \cdot \mathrm{id}_V$ on V.

When ρ is the adjoint representation, $C = C_\rho$ is the *Casimir* element.

1.4.4. *Complete reductibility.*

Theorem 1.19. *If $\mathfrak{g}$ is semi-simple, then all finite-dimensional $\mathfrak{g}$-modules are semi-simple.*

Theorem 1.20 (Levi)**.** *Every surjective map to a semi-simple Lie algebra splits.*

Definition 1.21. *$\mathfrak{g}$ is reductive if it is the direct sum of a semi-simple Lie algebra and an abelian Lie algebra.*

Theorem 1.22. *The following assertions are equivalent*

- *$\mathfrak{g}$ reductive*
- *The adjoint representation of $\mathfrak{g}$ is semisimple*
- *$\mathfrak{g}$ has a faithful semisimple representation*
- $\mathrm{Rad}(\mathfrak{g}) = Z(\mathfrak{g})$.

Theorem 1.23. *$\mathfrak{g}$ is semi-simple if and only if all its (finite-dimensional) representations are semi-simple.*

1.4.5. *Representations of $\mathfrak{sl}_2(\mathbf{C})$.* Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{C})$. We consider its basis $\{e, f, h\}$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g}$ has a presentation with generators e, f and h and relations

$$[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f.$$

Let V be a representation of $\mathfrak{g}$. Let $\lambda \in \mathbf{C}$. A non-zero vector $v \in V$ is a *weight vector of weight* λ if $h \cdot v = \lambda v$. It is a *highest weight vector of weight* λ if in addition $e \cdot v = 0$.

Lemma 1.24. *If V is finite-dimensional, then V has a highest weight vector.*

Proposition 1.25. *If V is finite-dimensional and v is a highest weight vector of weight λ, then $\lambda \in \mathbf{Z}_{\geq 0}$.*

Assume V is simple and finite-dimensional. Put $v_n = \frac{1}{n!} f^n(v)$. The set $\{v_n\}_{0 \leq n \leq \lambda}$ is a basis of V, $v_m = 0$ for $m < 0$ (by definition) and $m > \lambda$ and

$$e(v_n) = (\lambda - n + 1)v_{n-1},\ f(v_n) = (n+1)v_{n+1},\ h(v_n) = (\lambda - 2n)v_n.$$

Theorem 1.26. *Every finite-dimensional representation of $\mathfrak{g}$ is a direct sum of simple representations and $\mathfrak{g}$ has exactly one irreducible representation of dimension m for any $m > 0$.*

Exercise 1.5. Define V_d as the space of homogeneous polynomials of degree d in 2 variables x and y. Ie, $V_d = S^d V_1$. Show that $\mathfrak{g}$ acts on V_d as follows: e acts by $x\frac{\partial}{\partial y}$, f acts by $y\frac{\partial}{\partial x}$ and $h(x^a y^b) = (a-b)x^a y^b$.

Show that V_d is irreducible.

1.5. **Cartan subalgebras.** From now on, we will consider only the case $k = \mathbf{C}$.

Definition 1.27. *A* Cartan *subalgebra of $\mathfrak{g}$ is a nilpotent Lie subalgebra $\mathfrak{h}$ such that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.*

Proposition 1.28. *Let $\mathfrak{h} \subset \mathfrak{h}'$ be Cartan subalgebras of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{h}'$.*

Given $\mathfrak{g}$ and $\lambda \in \mathbf{C}$, let $\mathfrak{g}_x^\lambda$ be the λ-generalized eigenspace of $\operatorname{ad} x$. We have $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_x^\lambda$.

Lemma 1.29. *Given $\lambda, \mu \in \mathbf{C}$, we have $[\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu] \subset \mathfrak{g}_x^{\lambda+\mu}$. In particular, $\mathfrak{g}_x^0$ is a Lie subalgebra of $\mathfrak{g}$ containing x.*

We have $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{g}_x^0) = \mathfrak{g}_x^0$.

Definition 1.30. *Define the* rank *of $\mathfrak{g}$ as $\operatorname{rank}(\mathfrak{g}) = \max\{\dim \mathfrak{g}_x^0 | x \in \mathfrak{g}\}$.*

An element $x \in \mathfrak{g}$ is regular *if $\dim \mathfrak{g}_x^0 = \operatorname{rank}(\mathfrak{g})$.*

Theorem 1.31. *If x is regular, then $\mathfrak{g}_x^0$ is a Cartan subalgebra of $\mathfrak{g}$, with dimension the rank of $\mathfrak{g}$.*

Conversely, given $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, there is a regular element x of $\mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{g}_x^0$.

Theorem 1.32. *Let G be the subgroup of $\operatorname{Aut}(\mathfrak{g})$ generated by $\{\exp(\operatorname{ad}(y))\}_{y \in \mathfrak{g}}$. Given $\mathfrak{h}$ and $\mathfrak{h}'$ two Cartan subalgebras of $\mathfrak{g}$, there is $g \in G$ such that $\mathfrak{h}' = g(\mathfrak{h})$.*

2. Semi-simple Lie algebras

From now on, $\mathfrak{g}$ will be a semisimple Lie alegbra.

2.1. Cartan subalgebras and roots.

2.1.1. *Jordan decomposition.*

Definition 2.1. *An element $x \in \mathfrak{g}$ is* semi-simple *if $\operatorname{ad} x$ is diagonalizable.*

An element $x \in \mathfrak{g}$ is nilpotent *if $\operatorname{ad} x$ is nilpotent.*

Theorem 2.2 (Jordan-Chevalley decomposition)**.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $x \in \mathfrak{g}$. There exists unique elements $x_s, x_n \in \mathfrak{g}$ such that $x = x_s + x_n$, x_s is semisimple, x_n is nilpotent and $[x_s, x_n] = 0$.*

2.1.2. *Properties of Cartan subalgebras.* Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Proposition 2.3.
- *$\mathfrak{h}$ is abelian*
- *All elements of $\mathfrak{h}$ are semisimple*
- *The Killing form on $\mathfrak{g}$ restricts to a non-degenerate form on $\mathfrak{h}$.*

Note as a consequence that all regular elements of $\mathfrak{g}$ are semisimple.

Given $\alpha \in \mathfrak{h}^*$, we put $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [y, x] = \alpha(y)x \ \forall y \in \mathfrak{h}\}$. We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. We have $\mathfrak{g}_0 = \mathfrak{h}$.

Definition 2.4. *The set of* roots of $\mathfrak{g}$ *is* $R = \{\alpha \in \mathfrak{h}^* - \{0\} | \ \mathfrak{g}_\alpha \neq 0\}$.

We have $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. The Killing form is non-degenerate on $\mathfrak{h}$ and the subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are dual with respect to the Killing form.

Let V be the $\mathbf{R}$-subspace of $\mathfrak{h}^*$ spanned by R.

Theorem 2.5. (V, R) *is a root system.*

2.2. **Root systems.**

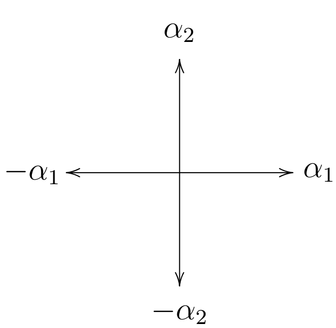
2.2.1. *Definition.*

Definition 2.6. *A root system is the data of an Euclidean space V and a finite subset R of $V - \{0\}$ with the following properties:*

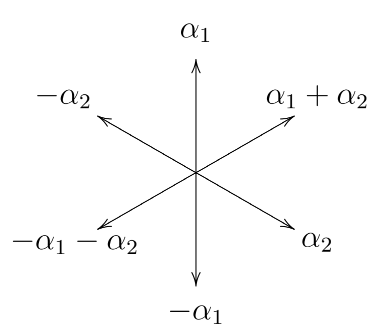
- *R generates V*
- *Given $\alpha \in R$, the symmetry $s_\alpha : v \mapsto v - 2\frac{\langle \alpha, v\rangle}{\langle \alpha, \alpha\rangle}\alpha$ leaves R invariant*
- *Given $\alpha, \beta \in R$, we have $s_\alpha(\beta) - \beta \in \mathbf{Z}\alpha$*
- *$R \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$.*

Examples (rank 2). Take $V = \mathbf{R}^2$.

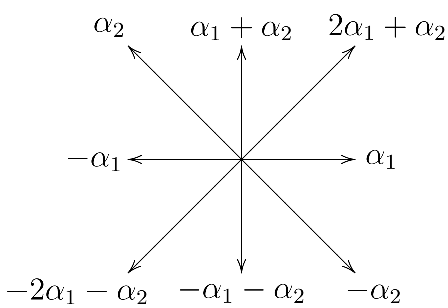
Type $A_1 \times A_1$



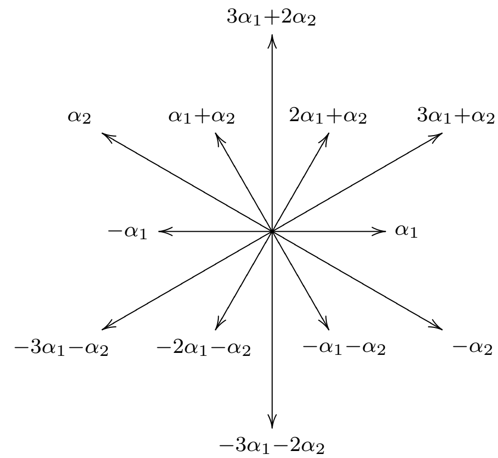
Type A_2



Type $B_2 = C_2$



Type G_2



Two root systems (V, R) and (V', R') are *isomorphic* if there is an isomorphism of vector spaces $\phi : V \xrightarrow{\sim} V'$ with $\phi(R) = R'$ (ϕ needs not respect the Euclidean structure).

Assume $V = V_1 \oplus V_2$, $R = R_1 \coprod R_2$ and R_i is a root system in V_i, the subspace of V generated by R_i, for $i = 1, 2$. We say that (V, R) is the *direct sum* of the root systems (V_1, R_1) and (V_2, R_2). A root system is *irreducible* if it is non-empty and it is not the direct sum of two non-empty root systems.

The *Weyl group* of the root system W is the subgroup of $\mathrm{GL}(V)$ generated by the reflections s_α for $\alpha \in R$. It is a finite group (it is a subgroup of the symmetric group on R).

2.2.2. *Bases.*

Definition 2.7. *A* basis *of R is a subset Δ of R that is a basis of V with the following property. Define $R^+ = R \cap (\bigoplus_{\alpha\in\Delta} \mathbf{Z}_{\geq 0}\alpha)$ (the* positive *roots) and $R^- = R \cap (\bigoplus_{\alpha\in\Delta} \mathbf{Z}_{\leq 0}\alpha)$ (the* negative *roots). The additional requirement is that $R = R^+ \coprod R^-$.*

Note that given R^+, the basis Δ can be recovered as the set of indecomposable elements of R^+ (those elements of R^+ that are not the sum of two elements of R^+).

Theorem 2.8. *Let $\zeta \in V^*$ such that $\zeta(\alpha) \neq 0$ for all $\alpha \in R$. Then $R^+_\zeta := \{\alpha \in R | \zeta(\alpha) > 0\}$ is the set of positive roots for a basis Δ_ζ of R.*

Given Δ a basis of R, we have $\Delta = \Delta_\zeta$ for any $\zeta \in V^*$ such that $\zeta(\alpha) > 0$ for all $\alpha \in \Delta$.

Theorem 2.9. *Let Δ be a basis of R. The group W is generated by $\{s_\alpha\}_{\alpha\in\Delta}$.*

If Δ' is a base of R, then there is $w \in W$ such that $\Delta' = w(\Delta)$.

If $\alpha \in R$, then there is $w \in W$ such that $w(\alpha) \in \Delta$.

Given $\alpha, \beta \in R$, we put $n(\alpha, \beta) = 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$.

Fix a basis Δ of R. The *Cartan matrix* of R is $(n(\alpha, \beta))_{\alpha,\beta\in\Delta}$. Up to permutation of rows and columns, it does not depend on the choice of the basis. We have $n(\alpha, \alpha) = 2$, while $n(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$.

Theorem 2.10. *Let (V_1, R_1) and (V_2, R_2) be two root systems, with bases Δ_1 and Δ_2 and Cartan matrices C_1 and C_2. The root systems (V_1, R_1) and (V_2, R_2) are isomorphic if and only if there is a bijection $\phi : \Delta_1 \xrightarrow{\sim} \Delta_2$ such that $(C_2)_{(\phi(\alpha),\phi(\beta))} = C_1$.*

2.3. **Structure of semi-simple Lie algebras.**

2.3.1. *Root spaces.* We give the steps of the proof of Theorem 2.5.

- R generates $\mathfrak{h}^*$ as a complex vector space.

Given $\alpha \in R$, let $h_\alpha \in \mathfrak{h}$ such that $\alpha = \langle h_\alpha, - \rangle$. Let $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

- Given $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, we have $[x, y] = \langle x, y \rangle h_\alpha$. So $\mathfrak{h}_\alpha = \mathbf{C} h_\alpha \neq 0$.

We denote by $H_\alpha \in \mathfrak{h}_\alpha$ the unique element such that $\alpha(H_\alpha) = 2$. Fix $X_\alpha \in \mathfrak{g}_\alpha$ non-zero.

- There is an element $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Y_\alpha] = H_\alpha$. There is an isomorphism of Lie algebras from $\mathfrak{sl}_2(\mathbf{C})$ to the Lie subalgebra $\mathfrak{s}_\alpha = \mathbf{C}X_\alpha \oplus \mathbf{C}H_\alpha \oplus \mathbf{C}Y_\alpha$ of $\mathfrak{g}$ given by $e \mapsto X_\alpha$, $h \mapsto H_\alpha$ and $f \mapsto Y_\alpha$.
- $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 1$ and Y_α as above is unique.
- Given $\beta \in R \setminus \{\pm\alpha\}$, the subspace $L = \bigoplus_{m \in \mathbf{Z}} \mathfrak{g}^{\alpha + m\beta}$ of $\mathfrak{g}$ is an irreducible representation of $\mathfrak{s}_\alpha$.
- We have $s_\alpha(\beta) \in R$.
- $2\alpha \notin R$.

2.3.2. *Generators.* Given a basis Δ of R, define $\mathfrak{n}_+ = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$, $\mathfrak{n}^- = \bigoplus_{\alpha \in R_-} \mathfrak{g}_\alpha$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$.

Proposition 2.11. *We have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (triangular decomposition). The subspaces $\mathfrak{n}_+$ and $\mathfrak{n}^-$ are nilpotent subalgebras of $\mathfrak{g}$, $\mathfrak{b}$ is a solvable subalgebra of $\mathfrak{g}$ and $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}_+$.*

Theorem 2.12. *The Lie algebra $\mathfrak{g}$ is generated by the elements X_α, Y_α and H_α for $\alpha \in \Delta$. It has a presentation with those generators and the following relations*

$$[H_\alpha, H_\beta] = 0,\ [H_\alpha, X_\beta] = n(\alpha,\beta) X_\beta,\ [H_\alpha, Y_\beta] = -n(\alpha,\beta) Y_\beta,\ [X_\alpha, Y_\beta] = \delta_{\alpha,\beta} H_\alpha$$
$$\mathrm{ad}^{1-n(\alpha,\beta)}(X_\alpha)(X_\beta) = 0,\ \mathrm{ad}^{1-n(\alpha,\beta)}(Y_\alpha)(Y_\beta) = 0 \textit{ for } \alpha \neq \beta.$$

Theorem 2.13. *The construction above gives a bijection from isomorphism classes of complex semisimple Lie algebras to isomorphism classes of root systems.*