1. Lie algebras

1.1. Generalities.

1.1.1. Definitions. Let \( k \) be a field.

A Lie algebra (over \( k \)) is a \( k \)-vector space \( g \) endowed with a bilinear map \([−, −] : g × g \to g\) such that 
\[
[a, b] = −[b, a]
\]
for all \( a, b \in g \).

Let \( g \) be a Lie algebra and \( h \) a \( k \)-subspace of \( g \). We say that \( h \) is

- a Lie subalgebra of \( g \) if \([a, b] \in h\) for all \( a, b \in h \).
- an ideal of \( g \) if \([a, b] \in h\) for all \( a \in g \) and \( b \in h \).

Let \( A \) be a \( k \)-algebra. This is a Lie algebra with \([a, b] = ab − ba\). This gives a functor from algebras to Lie algebras. It has a left adjoint, the universal enveloping algebra functor \( g \mapsto U(g) \).

Example 1.1. Let \( V \) be a vector space. The Lie algebra \( \text{End}_k(V) \) is denoted by \( \text{gl}(V) \). When \( V = k^n \), we put \( \text{gl}_n(k) = \text{gl}(k^n) \). We denote by \( \text{sl}_n(k) \) the Lie subalgebra of \( \text{gl}_n(k) \) of matrices with trace 0.

Example 1.2. Let \( A \) be a \( k \)-vector space endowed with a bilinear map \( a \cdot b : A \times A \to A \). A derivation of \( A \) is a \( k \)-linear endomorphism \( D \) of \( A \) such that \( D(a \cdot b) = D(a) \cdot b + a \cdot D(b) \). The set \( \text{Der}(A) \) of derivations of \( A \) is a Lie subalgebra of \( \text{gl}(A) \).

Proposition 1.3. Let \( g \) be a Lie algebra. Given \( x \in g \), define \( \text{ad} x : g \to g \) by \( y \mapsto [x, y] \). This is a derivation of \( g \). The corresponding map \( g \to \text{Der}(g) \) is a morphism of Lie algebras.

Given \( g \) a Lie algebra and \( V, V' \) two \( k \)-subspaces of \( g \), we denote by \([V, V']\) the \( k \)-subspace of \( g \) generated by elements \([v, v']\) with \( v \in V \) and \( v' \in V' \).

We denote by \( n_g(V) \) the set of elements \( x \in g \) such that \([x, v] \in V\) for all \( v \in V \). This is a Lie subalgebra of \( g \).

Lemma 1.4. Let \( h \) be an ideal of \( g \). Then \([g, h]\) is an ideal of \( g \).

We write \( Z(g) = \{x \in g | \text{ad} x = 0\} \) for the center of \( g \).

We say that \( g \) is abelian if \([x, y] = 0\) for all \( x, y \in g \).

If \( a \) and \( b \) are two ideals of \( g \) and \( g = a \oplus b \), then \([x, y] = 0\) for all \( x \in a \) and \( y \in b \), i.e., \( g \) is the direct sum (and the direct product) of its ideals \( a \) and \( b \).

Date: February 4, 2018.
1.1.2. **Representations.** Given a $k$-vector space $V$, a *representation* of $\mathfrak{g}$ on $V$ is a morphism of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$.

Let $V = \mathfrak{g}$. The *adjoint representation* $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is defined by $\text{ad}(g) : g' \mapsto [g, g']$. Its kernel is $Z(\mathfrak{g})$.

1.2. **Nilpotent Lie algebras.** From now on, all Lie algebras to be considered will be assumed to be finite-dimensional.

1.2.1. Let $\mathfrak{g}$ be a Lie algebra. The descending central series are the ideals defined by $C^1 \mathfrak{g} = \mathfrak{g}$ and $C^n \mathfrak{g} = [\mathfrak{g}, C^{n-1} \mathfrak{g}]$ for $n \geq 2$.

TFAE:

1. there is $n$ such that $C^n \mathfrak{g} = 0$
2. there is $n$ such that $(\text{ad} \ x_1) \cdots (\text{ad} \ x_n) = 0$ for all $x_1, \ldots, x_n \in \mathfrak{g}$.
3. there is a chain of ideals $0 = a_0 \subset \cdots \subset a_n = \mathfrak{g}$ such that $a_i/a_{i-1} \subset Z(\mathfrak{g}/a_{i-1})$ for all $i$ (iterated central extension of abelian Lie algebras).

A Lie algebra satisfying these equivalent conditions is called *nilpotent*.

**Exercise 1.1.** The Lie algebra $\mathfrak{g}$ of strictly upper triangular matrices in $\mathfrak{gl}_n$ is nilpotent. Determine the ideals $C^n \mathfrak{g}$.

1.2.2. Let $V$ be a finite-dimensional vector space over $k$. A full flag in $V$ is a sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $\dim V_i = i$.

**Theorem 1.5** (Engel). Consider $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation such that $\rho(x)$ is nilpotent for all $x \in \mathfrak{g}$. Then, there is a full flag $V_\bullet$ in $V$ such that $\rho(x)(V_i) \subset V_{i-1}$ for all $x \in \mathfrak{g}$ and all $i$.

**Corollary 1.6.** $\mathfrak{g}$ is nilpotent iff $\text{ad}(x)$ is nilpotent for all $x \in \mathfrak{g}$.

1.3. **Solvable Lie algebras.** The derived series of $\mathfrak{g}$ are the ideals defined by $D^1 \mathfrak{g} = \mathfrak{g}$ and $D^n \mathfrak{g} = [D^{n-1} \mathfrak{g}, D^{n-1} \mathfrak{g}]$ for $n \geq 2$.

TFAE:

- There is $n$ such that $D^n \mathfrak{g} = 0$
- $\mathfrak{g}$ is a successive extension of abelian Lie algebras.

A Lie algebra satisfying these equivalent conditions is called *solvable*.

**Exercise 1.2.** The Lie algebra $\mathfrak{g}$ of upper triangular matrices in $\mathfrak{gl}_n$ is solvable. Determine the ideals $D^n \mathfrak{g}$.

**Theorem 1.7** (Lie). Assume $k$ is algebraically closed and has characteristic $0$. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ with $V$ a finite-dimensional vector space. If $\mathfrak{g}$ is solvable, there is a full flag $V_\bullet$ of $V$ such that $\rho(x)(V_i) \subset V_i$ for all $x \in \mathfrak{g}$ and all $i$.

**Corollary 1.8.** $\mathfrak{g}$ solvable, $k$ arbitrary (char 0). Then, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

1.4. **Semi-simple Lie algebras.**
1.4.1. Bilinear forms. Let $\rho : g \rightarrow gl(V)$ be a representation. A bilinear form $\beta : V \times V \rightarrow k$ is $g$-invariant if $\beta(\rho(x)v_1, v_2) = -\beta(v_1, \rho(x)v_2)$ for all $x \in g$ and $v_1, v_2 \in V$.

Remark 1.9. Assume $k = \mathbb{C}$. Let $G$ be a complex Lie group with Lie algebra $g$ and let $\psi : G \rightarrow GL(V)$ be a representation of $G$ whose associated Lie algebra representation is $\rho$. The bilinear form $\beta$ is $G$-invariant if and only if $\beta(\psi(g)v_1, \psi(g)v_2) = \beta(v_1, v_2)$ for all $g \in G$ and $v_1, v_2 \in V$. Equivalently: $\beta(\psi(g)v_1, v_2) = \beta(v_1, \psi(g^{-1})v_2)$ for all $g, v_1, v_2$. This equality implies the $g$-equivariance of $\beta$.

Fix a representation and a $g$-invariant bilinear form. Given $L \subset V$, let $L^\perp = \{v \in V | \beta(l, v) = 0 \ \forall l \in L\}$.

Consider the adjoint representation $ad : g \rightarrow gl(g)$. A bilinear form $\alpha : g \times g \rightarrow k$ is $g$-invariant (for the adjoint representation) if and only if $\alpha([x, y], z) = \alpha(x, [y, z])$ for all $x, y, z \in g$.

The bilinear form given by $\beta(x, y) = Tr_g(ad x ad y)$ is called the Killing form. It is $g$-invariant. If $a$ is an ideal of $g$, then $a^\perp$ is also an ideal. Note also that the restriction of the Killing form of $g$ to $a$ is the Killing form of $a$.

Exercise 1.3. Show that if $g$ is nilpotent, then $\beta = 0$.

1.4.2. Radical and semi-simple Lie algebras. Note that given $a_1$ and $a_2$ two solvable ideals of $g$, then $a_1 + a_2$ is a solvable ideal.

Definition 1.10. The radical $rad(g)$ is the largest solvable ideal of $g$.

Definition 1.11. $g$ is semi-simple if $rad(g) = 0$.

Note that $g$ is semi-simple if and only if it has no non-zero abelian ideal.

Theorem 1.12. $g$ is semisimple iff the Killing form is non degenerate.

Theorem 1.13 (Cartan). Let $V$ be a vector space and $g$ be a Lie subalgebra of $gl(V)$. Then $g$ is solvable if and only if $Tr_V(xy) = 0$ for all $x \in g$ and $y \in [g, g]$.

Exercise 1.4. Let $g = gl_n(\mathbb{C})$. Show that the Killing form is $\beta(x, y) = 2ntr(xy) - 2tr(x)tr(y)$. Deduce that $sl_n(\mathbb{C})$ is semi-simple for $n \geq 2$.

Proposition 1.14. $rad(g) = [g, g]^\perp$.

Proposition 1.15. Let $g$ be a semisimple Lie algebra and $a$ an ideal. Then, $a^\perp$ is an ideal and $g = a \oplus a^\perp$.

Definition 1.16. A Lie algebra is simple if it is non abelian and it has no non-zero proper ideal.

Proposition 1.17. Let $g$ be a semi-simple Lie algebras. Then, there are ideals $a_1, \ldots, a_n$ of $g$ such that $g = a_1 \times \cdots \times a_n$ and $a_1, \ldots, a_n$ are simple Lie algebras. This decomposition is unique up to ordering.

Proposition 1.18. If $g$ is semisimple, then $[g, g] = g$. 
1.4.3. Enveloping algebras and Casimir. The enveloping algebra of $\mathfrak{g}$ is the $k$-algebra defined by $U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])_{x, y \in \mathfrak{g}}$. The functor $U$ is left adjoint to the canonical functor from algebras to Lie algebras.

If $\mathfrak{g}$ is abelian, then $U(\mathfrak{g}) = S(\mathfrak{g})$ is a polynomial algebra.

Assume $\mathfrak{g}$ is semi-simple and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a faithful representation. There is a $\mathfrak{g}$-invariant symmetric bilinear form $\beta_\rho : \mathfrak{g} \times \mathfrak{g} \to k$, $(x, y) \mapsto \text{Tr}_V(\rho(x)\rho(y))$. It is non-degenerate. We define $C_\rho = \sum e_i f_i \in U(\mathfrak{g})$, where $(e_i)$ is a basis of $\mathfrak{g}$ and $(f_i)$ the dual basis with respect to $\beta_\rho$. We have $C_\rho \in Z(U(\mathfrak{g}))$. If $\rho$ is simple, then $C_\rho$ acts by $\dim \mathfrak{g} \cdot \dim V \cdot \text{id}_V$ on $V$.

When $\rho$ is the adjoint representation, $C = C_\rho$ is the Casimir element.

1.4.4. Complete reductibility.

Theorem 1.19. If $\mathfrak{g}$ is semi-simple, then all finite-dimensional $\mathfrak{g}$-modules are semi-simple.

Theorem 1.20 (Levi). Every surjective map to a semi-simple Lie algebra splits.

Definition 1.21. $\mathfrak{g}$ is reductive if it is the direct sum of a semi-simple Lie algebra and an abelian Lie algebra.

Theorem 1.22. The following assertions are equivalent
- $\mathfrak{g}$ reductive
- The adjoint representation of $\mathfrak{g}$ is semisimple
- $\mathfrak{g}$ has a faithful semisimple representation
- $\text{Rad}(\mathfrak{g}) = Z(\mathfrak{g})$.

Theorem 1.23. $\mathfrak{g}$ is semi-simple if and only if all its (finite-dimensional) representations are semi-simple.

1.4.5. Representations of $\mathfrak{sl}_2(\mathbb{C})$. Define $V_d$ as $(d + 1)$-dimensional representations on homogeneous polynomials of degree $d$ in 2 variables $x$ and $y$. Ie, $V_d = S^d V_1$.

Fact: $e$ acts by $x \frac{\partial}{\partial y}$, $f$ acts by $y \frac{\partial}{\partial x}$. So, $h(x^a y^b) = (a - b)x^a y^b$.

1.5. Cartan subalgebras.

1.5.1. From now on, we will consider only the case $k = \mathbb{C}$.

Definition 1.24. A Cartan subalgebra of $\mathfrak{g}$ is a nilpotent Lie subalgebra $\mathfrak{h}$ such that $n_\mathfrak{g}(\mathfrak{h}) = \mathfrak{h}$.

Given $\mathfrak{g}$ and $\lambda \in \mathbb{C}$, let $\mathfrak{g}_x^\lambda$ be the $\lambda$-generalized eigenspace of $\text{ad} x$. We have $\mathfrak{g} = \bigoplus \lambda \mathfrak{g}_x^\lambda$.

Lemma 1.25. Given $\lambda, \mu \in \mathbb{C}$, we have $[\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu] \subseteq \mathfrak{g}_x^{\lambda + \mu}$. In particular, $\mathfrak{g}_x^0$ is a Lie subalgebra of $\mathfrak{g}$ containing $x$.

We have $n_\mathfrak{g}(\mathfrak{g}_x^0) = \mathfrak{g}_x^0$.

Definition 1.26. Define the rank of $\mathfrak{g}$ as $\text{rank}(\mathfrak{g}) = \max \{\dim \mathfrak{g}_x^0 | x \in \mathfrak{g}\}$.

An element $x \in \mathfrak{g}$ is regular if $\dim \mathfrak{g}_x^0 = \text{rank}(\mathfrak{g})$.

Theorem 1.27. If $x$ is regular, then $\mathfrak{g}_x^0$ is a Cartan subalgebra of $\mathfrak{g}$, with dimension the rank of $\mathfrak{g}$.

Conversely, given $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, there is a regular element $x$ of $\mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{g}_x^0$. 