1. Lie algebras

1.1. Generalities

1.1.1. Definitions. Let $k$ be a field.

A Lie algebra (over $k$) is a $k$-vector space $\mathfrak{g}$ endowed with a bilinear map $[-,-]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $[a,b] = -[b,a]$ and

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$$

for all $a,b,c \in \mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a $k$-subspace of $\mathfrak{g}$. We say that $\mathfrak{h}$ is

- a Lie subalgebra of $\mathfrak{g}$ if $[a,b] \in \mathfrak{h}$ for all $a,b \in \mathfrak{h}$.
- an ideal of $\mathfrak{g}$ if $[a,b] \in \mathfrak{h}$ for all $a \in \mathfrak{g}$ and $b \in \mathfrak{h}$.

Let $A$ be a $k$-algebra. This is a Lie algebra with $[a,b] = ab - ba$. This gives a functor from algebras to Lie algebras. It has a left adjoint, the universal enveloping algebra functor $\mathfrak{g} \mapsto U(\mathfrak{g})$.

Example 1.1. Let $V$ be a vector space. The Lie algebra $\text{End}_k(V)$ is denoted by $\mathfrak{gl}(V)$. When $V = k^n$, we put $\mathfrak{gl}_n(k) = \mathfrak{gl}(k^n)$. We denote by $\mathfrak{sl}_n(k)$ the Lie subalgebra of $\mathfrak{gl}_n(k)$ of matrices with trace 0.

Example 1.2. Let $A$ be a $k$-vector space endowed with a bilinear map $A \times A \to A$, $(a,b) \mapsto a \cdot b$. A derivation of $A$ is a $k$-linear endomorphism $D$ of $A$ such that $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$. The set $\text{Der}(A)$ of derivations of $A$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

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Proposition 1.3. Let $\mathfrak{g}$ be a Lie algebra. Given $x \in \mathfrak{g}$, define $\text{ad} x : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]$. This is a derivation of $\mathfrak{g}$. The corresponding map $\mathfrak{g} \to \text{Der}(\mathfrak{g})$ is a morphism of Lie algebras.

Given $\mathfrak{g}$ a Lie algebra and $V, V'$ two $k$-subspaces of $\mathfrak{g}$, we denote by $[V, V']$ the $k$-subspace of $\mathfrak{g}$ generated by elements $[v, v']$ with $v \in V$ and $v' \in V'$.

We denote by $\mathfrak{n}_s(V)$ the set of elements $x \in \mathfrak{g}$ such that $[x, v] \in V$ for all $v \in V$. This is a Lie subalgebra of $\mathfrak{g}$.

Lemma 1.4. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Then $[\mathfrak{g}, \mathfrak{h}]$ is an ideal of $\mathfrak{g}$.

We write $Z(\mathfrak{g}) = \{x \in \mathfrak{g} | \text{ad} x = 0\}$ for the center of $\mathfrak{g}$.

We say that $\mathfrak{g}$ is abelian if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, then $[x, y] = 0$ for all $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$, i.e., $\mathfrak{g}$ is the direct sum (and the direct product) of its ideals $\mathfrak{a}$ and $\mathfrak{b}$.

1.1.2. Representations. Given a $k$-vector space $V$, a representation of $\mathfrak{g}$ on $V$ is a morphism of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$.

Let $V = \mathfrak{g}$. The adjoint representation $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is defined by $\text{ad}(g) : g' \mapsto [g, g']$. Its kernel is $Z(\mathfrak{g})$.

1.2. Nilpotent Lie algebras. From now on, all Lie algebras to be considered will be assumed to be finite-dimensional.

1.2.1. Let $\mathfrak{g}$ be a Lie algebra. The descending central series are the ideals defined by $C^1 \mathfrak{g} = \mathfrak{g}$ and $C^n \mathfrak{g} = [\mathfrak{g}, C^{n-1} \mathfrak{g}]$ for $n \geq 2$.

TFAE:

1. there is $n$ such that $C^n \mathfrak{g} = 0$
2. there is $n$ such that $(\text{ad} x_1) \cdots (\text{ad} x_n) = 0$ for all $x_1, \ldots, x_n \in \mathfrak{g}$.
3. there is a chain of ideals $0 = \mathfrak{a}_0 \subset \cdots \subset \mathfrak{a}_n = \mathfrak{g}$ such that $\mathfrak{a}_i / \mathfrak{a}_{i-1} \subset Z(\mathfrak{g}/\mathfrak{a}_{i-1})$ for all $i$ (iterated central extension of abelian Lie algebras).

A Lie algebra satisfying these equivalent conditions is called nilpotent.

Exercise 1.1. The Lie algebra $\mathfrak{g}$ of strictly upper triangular matrices in $\mathfrak{gl}_n$ is nilpotent. Determine the ideals $C^i \mathfrak{g}$.

1.2.2. Let $V$ be a finite-dimensional vector space over $k$. A full flag in $V$ is a sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $\dim V_i = i$.

Theorem 1.5 (Engel). Consider $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation such that $\rho(x)$ is nilpotent for all $x \in \mathfrak{g}$. Then, there is a full flag $V_\bullet$ in $V$ such that $\rho(x)(V_i) \subset V_{i-1}$ for all $x \in \mathfrak{g}$ and all $i$.

Corollary 1.6. $\mathfrak{g}$ is nilpotent iff $\text{ad}(x)$ is nilpotent for all $x \in \mathfrak{g}$.

1.3. Solvable Lie algebras. The derived series of $\mathfrak{g}$ are the ideals defined by $D^1 \mathfrak{g} = \mathfrak{g}$ and $D^n \mathfrak{g} = [D^{n-1} \mathfrak{g}, D^{n-1} \mathfrak{g}]$ for $n \geq 2$.

TFAE:

- There is $n$ such that $D^n \mathfrak{g} = 0$
- $\mathfrak{g}$ is a successive extension of abelian Lie algebras.

A Lie algebra satisfying these equivalent conditions is called solvable.
Exercise 1.2. The Lie algebra $\mathfrak{g}$ of upper triangular matrices in $\mathfrak{gl}_n$ is solvable. Determine the ideals $D^i\mathfrak{g}$.

Theorem 1.7 (Lie). Assume $k$ is algebraically closed and has characteristic 0. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ with $V$ a finite-dimensional vector space. If $\mathfrak{g}$ is solvable, there is a full flag $V_i$ of $V$ such that $\rho(x)(V_i) \subset V_{i}$ for all $x \in \mathfrak{g}$ and all $i$.

Corollary 1.8. $\mathfrak{g}$ solvable, $k$ arbitrary (char 0). Then, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

1.4. Semi-simple Lie algebras.

1.4.1. Bilinear forms. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. A bilinear form $\beta : V \times V \to k$ is $\mathfrak{g}$-invariant if $\beta(\rho(x)v_1, v_2) = -\beta(v_1, \rho(x)v_2)$ for all $x \in \mathfrak{g}$ and $v_1, v_2 \in V$.

Remark 1.9. Assume $k = \mathbb{C}$. Let $G$ be a complex Lie group with Lie algebra $\mathfrak{g}$ and let $\psi : G \to \text{GL}(V)$ be a representation of $G$ whose associated Lie algebra representation is $\rho$. The bilinear form $\beta$ is $G$-invariant if and only if $\beta(\psi(g)v_1, \psi(g)v_2) = \beta(v_1, v_2)$ for all $g \in G$ and $v_1, v_2 \in V$. Equivalently: $\beta(\psi(g)v_1, v_2) = \beta(v_1, \psi(g^{-1})v_2)$ for all $g, v_1, v_2$. This equality implies the $\mathfrak{g}$-equivariance of $\beta$.

Fix a representation and a $\mathfrak{g}$-invariant bilinear form. Given $L \subset V$, let $L^\perp = \{v \in V | \beta(l, v) = 0 \forall l \in L\}$.

Consider the adjoint representation $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. A bilinear form $\alpha : \mathfrak{g} \times \mathfrak{g} \to k$ is $\mathfrak{g}$-invariant (for the adjoint representation) if and only if $\alpha([x, y], z) = \alpha(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$.

The bilinear form given by $\beta(x, y) = \text{Tr}(\text{ad } x \text{ ad } y)$ is called the Killing form. It is $\mathfrak{g}$-invariant. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{a}^\perp$ is also an ideal. Note also that the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{a}$ is the Killing form of $\mathfrak{a}$.

Exercise 1.3. Show that if $\mathfrak{g}$ is nilpotent, then $\beta = 0$.

1.4.2. Radical and semi-simple Lie algebras. Note that given $\mathfrak{a}_1$ and $\mathfrak{a}_2$ two solvable ideals of $\mathfrak{g}$, then $\mathfrak{a}_1 + \mathfrak{a}_2$ is a solvable ideal.

Definition 1.10. The radical $\text{rad}(\mathfrak{g})$ is the largest solvable ideal of $\mathfrak{g}$.

Definition 1.11. $\mathfrak{g}$ is semi-simple if $\text{rad}(\mathfrak{g}) = 0$.

Note that $\mathfrak{g}$ is semi-simple if and only if it has no non-zero abelian ideal.

Theorem 1.12. $\mathfrak{g}$ is semisimple iff the Killing form is non degenerate.

Theorem 1.13 (Cartan). Let $V$ be a vector space and $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{g}$ is solvable if and only if $\text{Tr}_V(xy) = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$.

Exercise 1.4. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Show that the Killing form is $\beta(x, y) = 2n\text{tr}(xy) - 2\text{tr}(x)\text{tr}(y)$. Deduce that $\mathfrak{sl}_n(\mathbb{C})$ is semi-simple for $n \geq 2$.

Proposition 1.14. $\text{rad}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^\perp$.

Proposition 1.15. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{a}$ an ideal. Then, $\mathfrak{a}^\perp$ is an ideal and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$.

Definition 1.16. A Lie algebra is simple if it is non abelian and it has no non-zero proper ideal.
Proposition 1.17. Let \( g \) be a semi-simple Lie algebras. Then, there are ideals \( a_1, \ldots, a_n \) of \( g \) such that \( g = a_1 \times \cdots \times a_n \) and \( a_1, \ldots, a_n \) are simple Lie algebras. This decomposition is unique up to ordering.

Proposition 1.18. If \( g \) is semisimple, then \([g, g] = g\).

1.4.3. Enveloping algebras and Casimir. The enveloping algebra of \( g \) is the \( k \)-algebra defined by \( U(g) = T(g)/(x \otimes y - y \otimes x - [x, y])_{x,y \in g} \). The functor \( U \) is left adjoint to the canonical functor from algebras to Lie algebras.

If \( g \) is abelian, then \( U(g) = S(g) \) is a polynomial algebra.

Assume \( g \) is semi-simple and \( \rho : g \to gl(V) \) is a faithful representation. There is a \( g \)-invariant symmetric bilinear form \( \beta_{\rho} : g \times g \to k \), \((x, y) \mapsto Tr_{V}(\rho(x)\rho(y))) \). It is non-degenerate. We define \( C_{\rho} = \sum e_i f_i \in U(g) \), where \((e_i)_i \) is a basis of \( g \) and \((f_i) \) the dual basis with respect to \( \beta_{\rho} \). We have \( C_{\rho} \in Z(U(g)) \). If \( \rho \) is simple, then \( C_{\rho} \) acts by \( \frac{\dim g}{\dim V} \cdot \text{id}_V \) on \( V \)

When \( \rho \) is the adjoint representation, \( C = C_{\rho} \) is the Casimir element.

1.4.4. Complete reductibility.

Theorem 1.19. If \( g \) is semi-simple, then all finite-dimensional \( g \)-modules are semi-simple.

Theorem 1.20 (Levi). Every surjective map to a semi-simple Lie algebra splits.

Definition 1.21. \( g \) is reductive if it is the direct sum of a semi-simple Lie algebra and an abelian Lie algebra.

Theorem 1.22. The following assertions are equivalent

- \( g \) reductive
- The adjoint representation of \( g \) is semisimple
- \( g \) has a faithful semisimple representation
- \( \text{Rad}(g) = Z(g) \).

Theorem 1.23. \( g \) is semi-simple if and only if all its (finite-dimensional) representations are semi-simple.

1.4.5. Representations of \( \mathfrak{sl}_2(C) \). Define \( V_d \) as \((d + 1)\)-dimensional representations on homogeneous polynomials of degree \( d \) in 2 variables \( x \) and \( y \). Ie, \( V_d = S^d V_1 \).

Fact: \( e \) acts by \( x \frac{\partial}{\partial y} \), \( f \) acts by \( y \frac{\partial}{\partial x} \). So, \( h(x^ay^b) = (a - b)x^ay^b \).

1.5. Cartan subalgebras. From now on, we will consider only the case \( k = C \).

Definition 1.24. A Cartan subalgebra of \( g \) is a nilpotent Lie subalgebra \( h \) such that \( n_{g}(h) = h \).

Proposition 1.25. Let \( h \subset h' \) be Cartan subalgebras of \( g \). Then \( h = h' \).

Given \( g \) and \( \lambda \in C \), let \( g^\lambda \) be the \( \lambda \)-generalized eigenspace of \( \text{ad} x \). We have \( g = \bigoplus_{\lambda} g^\lambda \).

Lemma 1.26. Given \( \lambda, \mu \in C \), we have \([g^\lambda_x, g^\mu_y] \subset g^{\lambda + \mu} \). In particular, \( g^0_x \) is a Lie subalgebra of \( g \) containing \( x \).

We have \( n_{g}(g^0_x) = g^0_x \).

Definition 1.27. Define the rank of \( g \) as \( \text{rank}(g) = \max\{\dim g^0_x | x \in g\} \).

An element \( x \in g \) is regular if \( \dim g^0_x = \text{rank}(g) \).
Theorem 1.28. If $x$ is regular, then $g^0_x$ is a Cartan subalgebra of $g$, with dimension the rank of $g$.

Conversely, given $h$ a Cartan subalgebra of $g$, there is a regular element $x$ of $g$ such that $h = g^0_x$.

Theorem 1.29. Let $G$ be the subgroup of Aut($g$) generated by $\{\exp(\text{ad}(y))\}_{y \in g}$. Given $h$ and $h'$ two Cartan subalgebras of $g$, there is $g \in G$ such that $h' = g(h)$.

2. SEMI-SIMPLE LIE ALGEBRAS

From now on, $g$ will be a semisimple Lie algebra.

2.1. Cartan subalgebras and roots.

2.1.1. Jordan decomposition.

Definition 2.1. An element $x \in g$ is semi-simple if $\text{ad } x$ is diagonalizable.

An element $x \in g$ is nilpotent if $\text{ad } x$ is nilpotent.

Theorem 2.2 (Jordan-Chevalley decomposition). Let $g$ be a complex semisimple Lie algebra and let $x \in g$. There exists unique elements $x_s, x_n \in g$ such that $x = x_s + x_n$, $x_s$ is semisimple, $x_n$ is nilpotent and $[x_s, x_n] = 0$.

2.1.2. Properties of Cartan subalgebras. Fix a Cartan subalgebra $h$ of $g$.

Proposition 2.3. 
- $h$ is abelian
- All elements of $h$ are semi-simple
- The Killing form on $g$ restricts to a non-degenerate form on $h$.

Note as a consequence that all regular elements of $g$ are semisimple.

Given $\alpha \in h^*$, we put $g_\alpha = \{ x \in g | [y,x] = \alpha(y)x \ \forall y \in h \}$. We have $[g_\alpha, g_\beta] \subset g_{\alpha + \beta}$. We have $g_0 = h$.

Definition 2.4. The set of roots of $g$ is $R = \{ \alpha \in h^* - \{0\} | g_\alpha \neq 0 \}$.

We have $g = h \oplus \bigoplus_{\alpha \in R} g_\alpha$. The Killing form is non-degenerate on $h$ and the subspaces $g_\alpha$ and $g_{-\alpha}$ are dual with respect to the Killing form.

Let $V$ be the $R$-subspace of $h^*$ spanned by $R$.

Theorem 2.5. $(V, R)$ is a root system.

2.2. Root systems.

2.2.1. Definition.

Definition 2.6. A root system is the data of an Euclidean space $V$ and a finite subset $R$ of $V - \{0\}$ with the following properties:

- $R$ generates $V$
- Given $\alpha \in R$, the symmetry $s_\alpha : v \mapsto v - 2\frac{\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle} \alpha$ leaves $R$ invariant
- Given $\alpha, \beta \in R$, we have $s_\alpha(\beta) - \beta \in \mathbb{Z}\alpha$
- $R \cap R\alpha = \{ \alpha, -\alpha \}$. 

Examples (rank 2). Take $V = \mathbb{R}^2$.

**Type $A_1 \times A_1$**

**Type $A_2$**

**Type $B_2 = C_2$**

**Type $G_2$**