LIE GROUPS AND LIE ALGEBRAS 229B

RAPHAËL ROUQUIER

UCLA, Winter 2018

Contents

1. Lie algebras 1
  1.1. Generalities 1
  1.2. Nilpotent Lie algebras 2
  1.3. Solvable Lie algebras 2
  1.4. Semi-simple Lie algebras 3
  1.5. Cartan subalgebras 4
  2. Semi-simple Lie algebras 5
    2.1. Cartan subalgebras and roots 5
    2.2. Root systems 5

1. Lie algebras

1.1. Generalities.

1.1.1. Definitions. Let $k$ be a field.

A Lie algebra (over $k$) is a $k$-vector space $g$ endowed with a bilinear map $[-,-]: g \times g \to g$ such that $[a,b] = -[b,a]$ and

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for all $a, b, c \in g$.

Let $g$ be a Lie algebra and $h$ a $k$-subspace of $g$. We say that $h$ is

- a Lie subalgebra of $g$ if $[a, b] \in h$ for all $a, b \in h$.
- an ideal of $g$ if $[a, b] \in h$ for all $a \in g$ and $b \in h$.

Let $A$ be a $k$-algebra. This is a Lie algebra with $[a,b] = ab - ba$. This gives a functor from algebras to Lie algebras. It has a left adjoint, the universal enveloping algebra functor $g \mapsto U(g)$.

Example 1.1. Let $V$ be a vector space. The Lie algebra $\text{End}_k(V)$ is denoted by $\mathfrak{gl}(V)$. When $V = k^n$, we put $\mathfrak{gl}_n(k) = \mathfrak{gl}(k^n)$. We denote by $\mathfrak{sl}_n(k)$ the Lie subalgebra of $\mathfrak{gl}_n(k)$ of matrices with trace 0.

Example 1.2. Let $A$ be a $k$-vector space endowed with a bilinear map $A \times A \to A$, $(a, b) \mapsto a \cdot b$. A derivation of $A$ is a $k$-linear endomorphism $D$ of $A$ such that $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$. The set $\text{Der}(A)$ of derivations of $A$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

Date: February 20, 2018.
Proposition 1.3. Let $\mathfrak{g}$ be a Lie algebra. Given $x \in \mathfrak{g}$, define $\text{ad} x : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]$. This is a derivation of $\mathfrak{g}$. The corresponding map $\mathfrak{g} \to \text{Der}(\mathfrak{g})$ is a morphism of Lie algebras.

Given $\mathfrak{g}$ a Lie algebra and $V, V'$ two $k$-subspaces of $\mathfrak{g}$, we denote by $[V, V']$ the $k$-subspace of $\mathfrak{g}$ generated by elements $[v, v']$ with $v \in V$ and $v' \in V'$.

We denote by $\mathfrak{n}_\mathfrak{g}(V)$ the set of elements $x \in \mathfrak{g}$ such that $[x, v] \in V$ for all $v \in V$. This is a Lie subalgebra of $\mathfrak{g}$.

Lemma 1.4. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Then $[\mathfrak{g}, \mathfrak{h}]$ is an ideal of $\mathfrak{g}$.

We write $Z(\mathfrak{g}) = \{x \in \mathfrak{g} | \text{ad} x = 0\}$ for the center of $\mathfrak{g}$.

We say that $\mathfrak{g}$ is abelian if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, then $[x, y] = 0$ for all $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$, i.e., $\mathfrak{g}$ is the direct sum (and the direct product) of its ideals $\mathfrak{a}$ and $\mathfrak{b}$.

1.1.2. Representations. Given a $k$-vector space $V$, a representation of $\mathfrak{g}$ on $V$ is a morphism of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$.

Let $V = \mathfrak{g}$. The adjoint representation $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is defined by $\text{ad}(g) : g' \mapsto [g, g']$. Its kernel is $Z(\mathfrak{g})$.

1.2. Nilpotent Lie algebras. From now on, all Lie algebras to be considered will be assumed to be finite-dimensional.

1.2.1. Let $\mathfrak{g}$ be a Lie algebra. The descending central series are the ideals defined by $C^1 \mathfrak{g} = \mathfrak{g}$ and $C^n \mathfrak{g} = [\mathfrak{g}, C^{n-1} \mathfrak{g}]$ for $n \geq 2$.

TFAE:

1. there is $n$ such that $C^n \mathfrak{g} = 0$
2. there is $n$ such that $(\text{ad} x_1) \cdots (\text{ad} x_n) = 0$ for all $x_1, \ldots, x_n \in \mathfrak{g}$.
3. there is a chain of ideals $0 = \mathfrak{a}_0 \subset \cdots \subset \mathfrak{a}_n = \mathfrak{g}$ such that $\mathfrak{a}_i/\mathfrak{a}_{i-1} \subset Z(\mathfrak{g}/\mathfrak{a}_{i-1})$ for all $i$ (iterated central extension of abelian Lie algebras).

A Lie algebra satisfying these equivalent conditions is called nilpotent.

Exercise 1.1. The Lie algebra $\mathfrak{g}$ of strictly upper triangular matrices in $\mathfrak{gl}_n$ is nilpotent. Determine the ideals $C^i \mathfrak{g}$.

1.2.2. Let $V$ be a finite-dimensional vector space over $k$. A full flag in $V$ is a sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $\dim V_i = i$.

Theorem 1.5 (Engel). Consider $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation such that $\rho(x)$ is nilpotent for all $x \in \mathfrak{g}$. Then, there is a full flag $V_\ast$ in $V$ such that $\rho(x)(V_i) \subset V_{i-1}$ for all $x \in \mathfrak{g}$ and all $i$.

Corollary 1.6. $\mathfrak{g}$ is nilpotent iff $\text{ad}(x)$ is nilpotent for all $x \in \mathfrak{g}$.

1.3. Solvable Lie algebras. The derived series of $\mathfrak{g}$ are the ideals defined by $D^1 \mathfrak{g} = \mathfrak{g}$ and $D^n \mathfrak{g} = [D^{n-1} \mathfrak{g}, D^{n-1} \mathfrak{g}]$ for $n \geq 2$.

TFAE:

- There is $n$ such that $D^n \mathfrak{g} = 0$
- $\mathfrak{g}$ is a successive extension of abelian Lie algebras.

A Lie algebra satisfying these equivalent conditions is called solvable.
Exercise 1.2. The Lie algebra \( \mathfrak{g} \) of upper triangular matrices in \( \mathfrak{gl}_n \) is solvable. Determine the ideals \( D^i \mathfrak{g} \).

Theorem 1.7 (Lie). Assume \( k \) is algebraically closed and has characteristic 0. Let \( \rho : \mathfrak{g} \to \mathfrak{gl}(V) \) be a representation of \( \mathfrak{g} \) with \( V \) a finite-dimensional vector space. If \( \mathfrak{g} \) is solvable, there is a full flag \( V_i \) of \( V \) such that \( \rho(x)(V_i) \subseteq V_i \) for all \( x \in \mathfrak{g} \) and all \( i \).

Corollary 1.8. \( \mathfrak{g} \) solvable, \( k \) arbitrary (\( \text{char} 0 \)). Then, \([\mathfrak{g}, \mathfrak{g}]\) is nilpotent.

1.4. Semi-simple Lie algebras.

1.4.1. Bilinear forms. Let \( \rho : \mathfrak{g} \to \mathfrak{gl}(V) \) be a representation. A bilinear form \( \beta : V \times V \to k \) is \( \mathfrak{g} \)-invariant if \( \beta(\rho(x)v_1, v_2) = -\beta(v_1, \rho(x)v_2) \) for all \( x \in \mathfrak{g} \) and \( v_1, v_2 \in V \).

Remark 1.9. Assume \( k = \mathbb{C} \). Let \( G \) be a complex Lie group with Lie algebra \( \mathfrak{g} \) and let \( \psi : G \to \text{GL}(V) \) be a representation of \( G \) whose associated Lie algebra representation is \( \rho \). The bilinear form \( \beta \) is \( G \)-invariant if and only if \( \beta(\psi(g)v_1, \psi(g)v_2) = \beta(v_1, v_2) \) for all \( g \in G \) and \( v_1, v_2 \in V \). Equivalently: \( \beta(\psi(g)v_1, v_2) = \beta(v_1, \psi(g^{-1})v_2) \) for all \( g, v_1, v_2 \). This equality implies the \( \mathfrak{g} \)-equivariance of \( \beta \).

Fix a representation and a \( \mathfrak{g} \)-invariant bilinear form. Given \( L \subseteq V \), let \( L^\perp = \{ v \in V | \beta(l, v) = 0 \ \forall l \in L \} \).

Consider the adjoint representation \( \text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \). A bilinear form \( \alpha : \mathfrak{g} \times \mathfrak{g} \to k \) is \( \mathfrak{g} \)-invariant (for the adjoint representation) if and only if \( \alpha([x, y], z) = \alpha(x, [y, z]) \) for all \( x, y, z \in \mathfrak{g} \).

The bilinear form given by \( \beta(x, y) = \text{Tr}_\mathbb{C}(\text{ad} x \text{ ad} y) \) is called the Killing form. It is \( \mathfrak{g} \)-invariant. If \( \mathfrak{a} \) is an ideal of \( \mathfrak{g} \), then \( \mathfrak{a}^\perp \) is also an ideal. Note also that the restriction of the Killing form of \( \mathfrak{g} \) to \( \mathfrak{a} \) is the Killing form of \( \mathfrak{a} \).

Exercise 1.3. Show that if \( \mathfrak{g} \) is nilpotent, then \( \beta = 0 \).

1.4.2. Radical and semi-simple Lie algebras. Note that given \( \mathfrak{a}_1 \) and \( \mathfrak{a}_2 \) two solvable ideals of \( \mathfrak{g} \), then \( \mathfrak{a}_1 + \mathfrak{a}_2 \) is a solvable ideal.

Definition 1.10. The radical \( \text{rad}(\mathfrak{g}) \) is the largest solvable ideal of \( \mathfrak{g} \).

Definition 1.11. \( \mathfrak{g} \) is semi-simple if \( \text{rad}(\mathfrak{g}) = 0 \).

Note that \( \mathfrak{g} \) is semi-simple if and only if it has no non-zero abelian ideal.

Theorem 1.12. \( \mathfrak{g} \) is semisimple iff the Killing form is non degenerate.

Theorem 1.13 (Cartan). Let \( V \) be a vector space and \( \mathfrak{g} \) be a Lie subalgebra of \( \mathfrak{gl}(V) \). Then \( \mathfrak{g} \) is solvable if and only if \( \text{Tr}_V(xy) = 0 \) for all \( x \in \mathfrak{g} \) and \( y \in [\mathfrak{g}, \mathfrak{g}] \).

Exercise 1.4. Let \( \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \). Show that the Killing form is \( \beta(x, y) = 2n\text{tr}(xy) - 2\text{tr}(x)\text{tr}(y) \). Deduce that \( \mathfrak{sl}_n(\mathbb{C}) \) is semi-simple for \( n \geq 2 \).

Proposition 1.14. \( \text{rad}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^\perp \).

Proposition 1.15. Let \( \mathfrak{g} \) be a semisimple Lie algebra and \( \mathfrak{a} \) an ideal. Then, \( \mathfrak{a}^\perp \) is an ideal and \( \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp \).

Definition 1.16. A Lie algebra is simple if it is non abelian and it has no non-zero proper ideal.
Proposition 1.17. Let \( \mathfrak{g} \) be a semi-simple Lie algebras. Then, there are ideals \( a_1, \ldots, a_n \) of \( \mathfrak{g} \) such that \( \mathfrak{g} = a_1 \times \cdots \times a_n \) and \( a_1, \ldots, a_n \) are simple Lie algebras. This decomposition is unique up to ordering.

Proposition 1.18. If \( \mathfrak{g} \) is semisimple, then \( [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \).

1.4.3. Enveloping algebras and Casimir. The enveloping algebra of \( \mathfrak{g} \) is the \( k \)-algebra defined by \( U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])_{x,y \in \mathfrak{g}} \). The functor \( U \) is left adjoint to the canonical functor from algebras to Lie algebras.

If \( \mathfrak{g} \) is abelian, then \( U(\mathfrak{g}) = S(\mathfrak{g}) \) is a polynomial algebra.

Assume \( \mathfrak{g} \) is semi-simple and \( \rho : \mathfrak{g} \to \mathfrak{gl}(V) \) is a faithful representation. There is a \( \mathfrak{g} \)-invariant symmetric bilinear form \( \beta_\rho : \mathfrak{g} \times \mathfrak{g} \to k \), \( (x, y) \mapsto \text{Tr}_V(\rho(x)\rho(y)) \). It is non-degenerate. We define \( C_\rho = \sum e_i f_i \in U(\mathfrak{g}) \), where \( (e_i)_i \) is a basis of \( \mathfrak{g} \) and \( (f_i)_i \) the dual basis with respect to \( \beta_\rho \). We have \( C_\rho \in Z(U(\mathfrak{g})) \). If \( \rho \) is simple, then \( C_\rho \) acts by \( \frac{\dim \mathfrak{g}}{\dim V} \cdot \text{id}_V \) on \( V \).

When \( \rho \) is the adjoint representation, \( C = C_\rho \) is the Casimir element.

1.4.4. Complete reductibility.

Theorem 1.19. If \( \mathfrak{g} \) is semi-simple, then all finite-dimensional \( \mathfrak{g} \)-modules are semi-simple.

Theorem 1.20 (Levi). Every surjective map to a semi-simple Lie algebra splits.

Definition 1.21. \( \mathfrak{g} \) is reductive if it is the direct sum of a semi-simple Lie algebra and an abelian Lie algebra.

Theorem 1.22. The following assertions are equivalent

- \( \mathfrak{g} \) reductive
- The adjoint representation of \( \mathfrak{g} \) is semisimple
- \( \mathfrak{g} \) has a faithful semisimple representation
- \( \text{Rad}(\mathfrak{g}) = Z(\mathfrak{g}) \).

Theorem 1.23. \( \mathfrak{g} \) is semi-simple if and only if all its (finite-dimensional) representations are semi-simple.

1.4.5. Representations of \( \mathfrak{sl}_2(C) \). Define \( V_d \) as \((d + 1)\)-dimensional representations on homogeneous polynomials of degree \( d \) in 2 variables \( x \) and \( y \). I.e, \( V_d = S^d V_1 \).

Fact: \( e \) acts by \( x \frac{\partial}{\partial y} \), \( f \) acts by \( y \frac{\partial}{\partial x} \). So, \( h(x^a y^b) = (a - b)x^a y^b \).

1.5. Cartan subalgebras. From now on, we will consider only the case \( k = C \).

Definition 1.24. A Cartan subalgebra of \( \mathfrak{g} \) is a nilpotent Lie subalgebra \( \mathfrak{h} \) such that \( n_\mathfrak{g}(\mathfrak{h}) = \mathfrak{h} \).

Proposition 1.25. Let \( \mathfrak{h} \subset \mathfrak{h}' \) be Cartan subalgebras of \( \mathfrak{g} \). Then \( \mathfrak{h} = \mathfrak{h}' \).

Given \( \mathfrak{g} \) and \( \lambda \in C \), let \( \mathfrak{g}_x^\lambda \) be the \( \lambda \)-generalized eigenspace of \( \text{ad} \, x \). We have \( \mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_x^\lambda \).

Lemma 1.26. Given \( \lambda, \mu \in C \), we have \( [\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu] \subset \mathfrak{g}_x^{\lambda + \mu} \). In particular, \( \mathfrak{g}_x^0 \) is a Lie subalgebra of \( \mathfrak{g} \) containing \( x \).

We have \( n_\mathfrak{g}(\mathfrak{g}_x^0) = \mathfrak{g}_x^0 \).

Definition 1.27. Define the rank of \( \mathfrak{g} \) as \( \text{rank}(\mathfrak{g}) = \max \{ \dim \mathfrak{g}_x^0 \mid x \in \mathfrak{g} \} \).

An element \( x \in \mathfrak{g} \) is regular if \( \dim \mathfrak{g}_x^0 = \text{rank}(\mathfrak{g}) \).
Theorem 1.28. If \( x \) is regular, then \( g_x^0 \) is a Cartan subalgebra of \( g \), with dimension the rank of \( g \).
Conversely, given \( h \) a Cartan subalgebra of \( g \), there is a regular element \( x \) of \( g \) such that \( h = g_x^0 \).

Theorem 1.29. Let \( G \) be the subgroup of \( \text{Aut}(g) \) generated by \( \{ \exp(\text{ad}(y)) \}_{y \in g} \). Given \( h \) and \( h' \) two Cartan subalgebras of \( g \), there is \( g \in G \) such that \( h' = g(h) \).

2. Semi-simple Lie algebras

From now on, \( g \) will be a semisimple Lie algebra.

2.1. Cartan subalgebras and roots.

2.1.1. Jordan decomposition.

Definition 2.1. An element \( x \in g \) is semi-simple if \( \text{ad} \, x \) is diagonalizable.
An element \( x \in g \) is nilpotent if \( \text{ad} \, x \) is nilpotent.

Theorem 2.2 (Jordan-Chevalley decomposition). Let \( g \) be a complex semisimple Lie algebra and let \( x \in g \). There exists unique elements \( x_s, x_n \in g \) such that \( x = x_s + x_n \), \( x_s \) is semisimple, \( x_n \) is nilpotent and \([x_s, x_n] = 0\).

2.1.2. Properties of Cartan subalgebras. Fix a Cartan subalgebra \( h \) of \( g \).

Proposition 2.3. \( h \) is abelian.
• All elements of \( h \) are semi-simple
• The Killing form on \( g \) restricts to a non-degenerate form on \( h \).

Note as a consequence that all regular elements of \( g \) are semisimple.
Given \( \alpha \in h^* \), we put \( g_\alpha = \{ x \in g \mid [y, x] = \alpha(y)x \forall y \in h \} \). We have \([g_\alpha, g_\beta] \subset g_{\alpha+\beta} \). We have \( g_0 = h \).

Definition 2.4. The set of roots of \( g \) is \( R = \{ \alpha \in h^* - \{0\} \mid g_\alpha \neq 0 \} \).

We have \( g = h \oplus \bigoplus_{\alpha \in R} g_\alpha \). The Killing form is non-degenerate on \( h \) and the subspaces \( g_\alpha \) and \( g_{-\alpha} \) are dual with respect to the Killing form.

Let \( V \) be the \( R \)-subspace of \( h^* \) spanned by \( R \).

Theorem 2.5. \((V, R)\) is a root system.

2.2. Root systems.

2.2.1. Definition.

Definition 2.6. A root system is the data of an Euclidean space \( V \) and a finite subset \( R \) of \( V - \{0\} \) with the following properties:
• \( R \) generates \( V \)
• Given \( \alpha \in R \), the symmetry \( s_\alpha : v \mapsto v - 2\frac{\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle} \alpha \) leaves \( R \) invariant
• Given \( \alpha, \beta \in R \), we have \( s_\alpha(\beta) - \beta \in \mathbb{Z}\alpha \)
• \( R \cap R\alpha = \{ \alpha, -\alpha \} \).
Examples (rank 2). Take $V = \mathbb{R}^2$.

**Type $A_1 \times A_1$**

**Type $A_2$**

**Type $B_2 = C_2$**

**Type $G_2$**

Two root systems $(V, R)$ and $(V', R')$ are *isomorphic* if there is an isomorphism of vector spaces $\phi : V \cong V'$ with $\phi(R) = R'$ ($\phi$ needs not respect the Euclidean structure).

Assume $V = V_1 \oplus V_2$, $R = R_1 \bigsqcup R_2$ and $R_i$ is a root system in $V_i$, the subspace of $V$ generated by $R_i$, for $i = 1, 2$. We say that $(V, R)$ is the *direct sum* of the root systems $(V_1, R_1)$ and $(V_2, R_2)$. A root system is *irreducible* if it is non-empty and it is not the direct sum of two non-empty root systems.

The *Weyl group* of the root system $W$ is the subgroup of $\text{GL}(V)$ generated by the reflections $s_\alpha$ for $\alpha \in R$. It is a finite group (it is a subgroup of the symmetric group on $R$).

2.2.2. Bases.

**Definition 2.7.** A basis of $R$ is a subset $\Delta$ of $R$ that is a basis of $V$ with the following property. Define $R^+ = R \cap (\bigoplus_{\alpha \in \Delta} \mathbb{Z}_{\geq 0} \alpha)$ (the positive roots) and $R^- = R \cap (\bigoplus_{\alpha \in \Delta} \mathbb{Z}_{\leq 0} \alpha)$ (the negative roots). The additional requirement is that $R = R^+ \bigsqcup R^-$.

Note that given $R^+$, the basis $\Delta$ can be recovered as the set of indecomposable elements of $R^+$ (those elements of $R^+$ that are not the sum of two elements of $R^+$).
Theorem 2.8. Let \( \zeta \in V^* \) such that \( \zeta(\alpha) \neq 0 \) for all \( \alpha \in R \). Then \( R^+_\zeta := \{ \alpha \in R | \zeta(\alpha) > 0 \} \) is the set of positive roots for a basis \( \Delta_\zeta \) of \( R \).

Given a basis of \( R \), we have \( \Delta = \Delta_\zeta \) for any \( \zeta \in V^* \) such that \( \zeta(\alpha) > 0 \) for all \( \alpha \in \Delta \).

Theorem 2.9. Let \( \Delta \) be a basis of \( R \). The group \( W \) is generated by \( \{ s_\alpha \}_{\alpha \in \Delta} \).

If \( \Delta' \) is a base of \( R \), then there is \( w \in W \) such that \( \Delta' = w(\Delta) \).

If \( \alpha \in R \), then there is \( w \in W \) such that \( w(\alpha) \in \Delta \).

Given \( \alpha, \beta \in R \), we put \( n(\alpha, \beta) = 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \).

Fix a basis \( \Delta \) of \( R \). The Cartan matrix of \( R \) is \( (n(\alpha, \beta))_{\alpha, \beta \in \Delta} \). Up to permutation of rows and columns, it does not depend on the choice of the basis. We have \( n(\alpha, \alpha) = 2 \), while \( n(\alpha, \beta) \leq 0 \) for \( \alpha \neq \beta \).

Theorem 2.10. Let \( (V_1, R_1) \) and \( (V_2, R_2) \) be two root systems, with bases \( \Delta_1 \) and \( \Delta_2 \) and Cartan matrices \( C_1 \) and \( C_2 \). The root systems \( (V_1, R_1) \) and \( (V_2, R_2) \) are isomorphic if and only if there is a bijection \( \phi : \Delta_1 \rightarrow \Delta_2 \) such that \( (C_2)_{(\phi(\alpha), \phi(\beta))} = C_1 \).