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CHAPTER 1

ALGEBRAS AND MODULES

1.1. Generalities

1.1.1. Modules and Algebras.

Let $R$ be a commutative ring, with a unit element.

1.1.1.1. $R$–modules.

An $R$–module is an abelian group $X$ endowed with a ring morphism

$$R \to \text{End}(X)$$

$$\lambda \mapsto (x \mapsto \lambda x)$$

(we also set $x\lambda := \lambda x$).

A morphism $\varphi : X \to X'$ between two $R$–modules is a morphism of abelian groups which commutes with the actions of elements of $R$. The set of morphisms from $X$ to $X'$ is denoted by $\text{Hom}_R(X, X')$.

Convention 1. We let the elements of $\text{Hom}_R(X, X')$ act on the left of $X$, so that, for $\varphi \in \text{Hom}_R(X, X')$, $x \in X$ and $\lambda \in R$,

$$\varphi : x \mapsto \varphi(x) \quad \text{and} \quad \varphi(\lambda x) = \lambda \varphi(x),$$

and composing $\varphi$ followed by $\varphi'$ is denoted $\varphi' \cdot \varphi$.

1.1.1.2. $R$–algebras and $A$–modules.

An $R$–algebra is a ring $A$ with a unity element, endowed with a ring morphism $R \to ZA$. By abuse of notation, we denote by $\lambda a$ the product of the image of $\lambda \in R$ by $a \in A$. Thus $\lambda a = a\lambda$.

An $A$–module (or a left representation of $A$) is a pair $(X, \lambda)$ where

• $X$ is an $R$–module,

• $\lambda : A \to \text{End}_R(X)$ is a morphism of $R$–algebras.

The morphism $\lambda$ is called the structural morphism.

Remark 1.1.

When speaking of “modules”, one often omits the structural morphism (and only $X$ is called the module), by writing

$$ax := \lambda(a)(x) \quad \text{for} \ a \in A, x \in X.$$

When speaking of “representations”, one emphasizes more the structural morphism $\lambda$ (viewing then $X$ only as “the $R$–module of the representation”).
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Given two \(A\)-modules (left representations of \(A\)) \((X, \lambda)\) and \((X', \lambda')\), a morphism (resp. isomorphism)
\[
\varphi : (X, \lambda) \longrightarrow (X', \lambda')
\]
is an \(R\)-linear morphism (resp. isomorphism)
\[
\varphi : X \rightarrow X' \quad \text{such that} \quad \varphi \cdot \lambda(a) = \lambda'(a) \cdot \varphi.
\]

Expressing an isomorphism with successively the “module point of view” and the “representation point of view”, we get the following descriptions :

- Two \(A\)-modules \(X\) and \(X'\) are isomorphic if and only if there exists an \(R\)-linear isomorphism \(\varphi : X \rightarrow X'\) such that \(\varphi(ax) = a\varphi(x)\).
- Two representations \(\lambda : A \rightarrow \text{End}_R(X)\) and \(\lambda' : A \rightarrow \text{End}_R(X)\) (on the same \(R\)-module \(X\)) are isomorphic if and only if there is an \(R\)-linear automorphism \(\varphi\) of \(X\) such that
  \[
  \lambda'(a) = \varphi \cdot \lambda(a) \cdot \varphi^{-1} \quad (\forall a \in A).
  \]

**Convention 2.** For \(X\) and \(X'\) \(A\)-modules, we let the morphisms from \(X\) to \(X'\) act *on the right*, so that the commutation with the elements of \(A\) becomes just an associativity property : for \(\varphi : X \rightarrow X'\), \(a \in A\), \(x \in X\), we have
\[
(ax)\varphi = a(x\varphi).
\]
If \(X\) and \(X'\) are \(A\)-modules, then \(\text{Hom}_A(X, X')\) denotes the \(R\)-module of \(A\)-homomorphisms from \(X\) to \(X'\).

If \(X\) is an \(A\)-module, then \(E_A X := \text{End}_A(X)\) denotes the set of \(A\)-endomorphisms of \(X\).

1.1.1.3. Opposite algebra and modules–\(A\).

The opposite algebra \(A^{\text{op}}\) is by definition the \(R\)-module \(A\) where the multiplication is defined as \((a, a') \mapsto a'a\).

A module–\(A\) (or a right representation of \(A\)) is by definition an \(A^{\text{op}}\)-module.

Let \(Y\) be a module–\(A\). Letting the elements of \(A\) (which are the elements of \(A^{\text{op}}\)) act on the right of \(Y\), we get a structural morphism
\[
\rho : A \rightarrow \text{End}_R(Y)^{\text{op}}
\]
(where \(\text{End}_R(X)^{\text{op}}\) acts on the right of \(Y\)).

We then set
\[
y a := (y) \rho(a),
\]
thus justifying the name “module–\(A\)”. 
1.1. GENERALITIES

CONVENTION 3. For $Y$ and $Y'$ modules–$A$, we let the morphisms from $Y$ to $Y'$ act on the left, so that the commutation with the elements of $A$ becomes just an associativity property: for $\varphi : Y \rightarrow Y'$, $a \in A$, $y \in Y$, we have

$$\varphi(ya) = (\varphi y)a.$$ 

We denote by $\text{Hom}(Y, Y')_A$ the $R$–module of morphisms of modules–$A$ from $Y$ to $Y'$.

We set $EY_A := \text{End}(Y)_A$.

1.1.2. The language of $R$–linear categories and functors.

We briefly introduce and recall some basic notation and definitions about $R$–linear categories.

1.1.2.1. Categories.

**Definition 1.2.** An $R$–linear category consists of the following three mathematical entities:

- A class $\text{Ob}(\mathfrak{A})$, whose elements are called objects (for $X$ an object, we write $X \in \text{Ob}(\mathfrak{A})$, or even $X \in \mathfrak{A}$);
- for each pair of objects $X$ and $X'$, an $R$–module $\text{Mor}_\mathfrak{A}(X, X')$ (an element $f \in \text{Mor}_\mathfrak{A}(X, X')$ is then called a morphism with source $X$ and target $X'$ and denoted $f : X \rightarrow X'$);
- for each triple of objects $X$, $X'$, $X''$, an $R$–bilinear map called the composition:

$$\begin{align*}
\text{Mor}_\mathfrak{A}(X', X'') \times \text{Mor}_\mathfrak{A}(X, X') &\rightarrow \text{Mor}_\mathfrak{A}(X, X'') \\
(g, f) &\mapsto g.f, \\
\end{align*}$$

such that

1. $(h.g).f = h.(g.f)$,
2. whenever $X \in \mathfrak{A}$, there is an element $\text{Id}_X \in \text{Mor}_\mathfrak{A}(X, X)$ such that, for every morphism $f : X \rightarrow X'$, we have $f.\text{Id}_X = \text{Id}_{X'} . f$.

Let us give some definitions related to properties of morphisms. A morphism $f : X \rightarrow X'$ is

- a **monomorphism** (or monic) if $f.g_1 = f.g_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : X'' \rightarrow X$,
- an **epimorphism** (or epic) if $g_1.f = g_2.f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : X \rightarrow X''$,
- an **isomorphism** if there exists a morphism $g : X' \rightarrow X$ with $f.g = \text{Id}_{X'}$ and $g.f = \text{Id}_X$,
- an **endomorphism** if $X' = X$ ($\text{End}_\mathfrak{A}(X)$ denotes the $R$–algebra of endomorphisms of $X$),
- an **automorphism** if $f$ is both an endomorphism and an isomorphism. ($\text{Aut}(X)$ denotes the group of automorphisms of $X$).
The opposite category $\mathcal{A}^{\text{op}}$ of a category $\mathcal{A}$ is the category where

- $\text{Ob}(\mathcal{A}^{\text{op}}) := \text{Ob}(\mathcal{A})$,
- $\text{Mor}_{\mathcal{A}^{\text{op}}}(X, X') := \text{Mor}_{\mathcal{A}}(X', X)$,
- for $f \in \text{Mor}_{\mathcal{A}^{\text{op}}}(X, X')$ and $g \in \text{Mor}_{\mathcal{A}^{\text{op}}}(X', X'')$, $(g.f)_{\mathcal{A}^{\text{op}}} := (f.g)_{\mathcal{A}}$.

A full subcategory $\mathcal{A}'$ of a category $\mathcal{A}$ is a category where

- the objects of $\mathcal{A}'$ are some objects of $\mathcal{A}$,
- for $X$ and $X'$ objects of $\mathcal{A}'$, we have $\text{Mor}_{\mathcal{A}}(X, X') = \text{Mor}_{\mathcal{A}'}(X, X')$.

**Examples 1.3.**

We denote by $\text{Mod}_A$ (category of modules–$A$) the $R$–linear category whose objects are the modules–$A$ and where

$\text{Hom}_{\text{Mod}_A}(X, X') := \text{Hom}(X, X')_A$.

We denote by $\text{mod}_A$ the full subcategory of $\text{Mod}_A$ whose objects are the *finitely generated* modules–$A$.

We denote by $\text{AMod}_A$ (category of $A$–modules) the $R$–linear category whose objects are the $A$–modules and where

$\text{Hom}_{\text{AMod}_A}(X, X') := \text{Hom}_A(X, X')$.

We denote by $\text{Amod}_A$ the full subcategory of $\text{AMod}_A$ whose objects are the *finitely generated* $A$–modules.

We have

$(\text{AMod}_A)^{\text{op}} = \text{Mod}_{A^{\text{op}}}$.

The monomorphisms (resp. epimorphisms) of $\text{AMod}_A$ and $\text{Mod}_A$ are the injective (resp. surjective) homomorphisms.

**Remark 1.4.** An $R$–linear category $\mathcal{A}$ with a single object $X_0$ is defined by the $R$–algebra $A := \text{End}_{\mathcal{A}}(X_0)$.

1.1.2.2. **Functors.**

**Definition 1.5.** Let $\mathcal{A}$ and $\mathcal{B}$ be two ($R$–linear) categories. A (covariant) functor $F : \mathcal{A} \to \mathcal{B}$

- associates to each $X \in \text{Ob}(\mathcal{A})$ an object $F(X) \in \text{Ob}(\mathcal{B})$,
- for each pair $(X, X')$ of objects of $\mathcal{A}$ it defines a morphism of $R$–modules

$F : \text{Mor}_{\mathcal{A}}(X, X') \to \text{Mor}_{\mathcal{B}}(F(X), F(X'))$

such that

1. whenever $X \in \mathcal{A}$, $F(\text{Id}_X) = \text{Id}_{F(X)}$,
2. whenever $f : X \to X'$ and $g : X' \to X''$, then $F(g.f) = F(g).F(f)$.

A contravariant functor $F : \mathcal{A} \to \mathcal{B}$ is a (covariant) functor from $\mathcal{A}^{\text{op}}$ to $\mathcal{B}$.
The image of \( F \) is the full subcategory of \( \mathcal{B} \) with set of objects \( \{ F(X) \}_{X \in \text{Ob}(A)} \).

The essential image of \( F \) is the full subcategory of \( \mathcal{B} \) whose objects are the objects of \( \mathcal{A} \) isomorphic to objects of the image of \( F \).

We say that \( F \) is
- faithful if \( \text{Mor}_A(X, X') \to \text{Mor}_B(F(X), F(X')) \) is injective for all \( X, X' \in \mathcal{A} \),
- full if \( \text{Mor}_A(X, X') \to \text{Mor}_B(F(X), F(X')) \) is surjective for all \( X, X' \in \mathcal{A} \),
- fully faithful if it is full and faithful,
- essentially surjective if the essential image of \( F \) is \( \mathcal{B} \).

**Definition 1.6.** Let \( F, G : \mathcal{A} \to \mathcal{B} \) be two functors. A morphism \( \varepsilon : F \to G \)
- associates to each object \( X \) of \( \mathcal{A} \) a morphism \( \varepsilon_X : F(X) \to G(X) \)
- such that, whenever \( f : X \to X' \) is a morphism in \( \mathcal{A} \), the following diagram is commutative

\[
\begin{array}{ccc}
F(X) & \xrightarrow{\varepsilon_X} & G(X) \\
F(f) \downarrow & & \downarrow G(f) \\
F(X') & \xrightarrow{\varepsilon_{X'}} & G(X')
\end{array}
\]

We say that a morphism \( \varepsilon : F \to G \) is an isomorphism if, for all object \( X \) of \( \mathcal{A} \), \( \varepsilon_X : F(X) \to G(X) \) is an isomorphism. The functors \( F \) and \( G \) are then said to be isomorphic and we write \( F \cong G \).

**Example 1.7.** Whenever \( X \) is an \( \mathcal{A} \)-module, the \( \mathcal{R} \)-module \( \text{Hom}_A(A, X) \) is endowed with a natural structure of \( \mathcal{A} \)-module defined by

\[
(\varphi a)(b) := \varphi(ba) \quad \text{for} \ \varphi \in \text{Hom}_A(A, X) , \ a, b \in A.
\]

It defines a functor \( \text{Hom}_A(A, \cdot) \), which is isomorphic to the functor identity.

**Definition 1.8.** We say that a functor \( F : \mathcal{A} \to \mathcal{B} \) is an *equivalence of categories* if there exists a functor \( G : \mathcal{A} \to \mathcal{B} \) such that \( F.G \simeq \text{Id}_\mathcal{B} \) and \( G.F \simeq \text{Id}_\mathcal{A} \).

The following two propositions are left to the reader.

**Proposition 1.9.** A functor \( F : \mathcal{A} \to \mathcal{B} \) is an equivalence of categories if and only if it is fully faithful and essentially surjective.

**Proposition 1.10.** Assume that the functor \( F : \mathcal{A} \to \mathcal{B} \) is an equivalence of categories. Then

1. Whenever \( X, X' \in \mathcal{A} \), then \( F \) induces an isomorphism

\[
\text{Hom}_\mathcal{A}(X, X') \xrightarrow{\cong} \text{Hom}_\mathcal{B}(F(X), F(X')).
\]
(2) The image under \( F \) of a monomorphism (resp. an epimorphism) is a monomorphism (resp. an epimorphism).

1.1.3. Bimodules.

1.1.3.1. Generalities.

Let \( A \) and \( B \) be two \( R \)-algebras. We denote by \( A \otimes_R B \) the algebra defined on the tensor product by the multiplication \((a_1 \otimes b_1)(a_2 \otimes b_2) := a_1a_2 \otimes b_1b_2\).

**Notation 1.** In what follows, whenever the ring controlling the tensor product is not specified, it means that the tensor product is over \( R \).

An \((A,B)\)-bimodule, also called \( A \)-module–\( B \), is by definition an \((A \otimes_R B^{\text{op}})\)-module.

Let \( M \) be an \( A \)-module–\( B \). For \( a \in A \), \( b \in B^{\text{op}} \), \( m \in M \), we set \( amb := (a \otimes b)m \),

thus justifying the name “\( A \)-module–\( B \)”.

**Remark 1.11.** With the preceding notation, one has to consider that the elements of \( R \) act the same way on both sides of \( M \) : for \( \lambda \in R \) and \( m \in M \), we have \( \lambda m = m\lambda \).

Notice that an \( A \)-module–\( B \) is naturally a \( B^{\text{op}} \)-module–\( A^{\text{op}} \), i.e., a module–\((A^{\text{op}} \otimes_R B)\).

**Convention 4.** The question “where do the morphisms of bimodules act ?” is solved by the following convention : a morphism of \( A \)-modules–\( B \) is treated as a morphism of \((A \otimes_R B^{\text{op}})\)-modules, i.e., acts on the right.

We set

\[ \text{Hom}_A(M, M')_B := \text{Hom}_{A \otimes_R B^{\text{op}}}(M, M') \, . \]

Using the above conventions, many natural structures follow from associativity. We list just a few of them:

\[
\begin{align*}
X \in \text{AMod} \quad &\implies\quad X \in \text{AMod}_{EAX} \\
Y \in \text{Mod}_A \quad &\implies\quad Y \in \text{EY}_A \text{Mod}_A \\
M \in \text{AMod}_B \quad &\implies\quad \text{Hom}_A(M, N) \in \text{BMod}_C \\
N \in \text{AMod}_C \quad &\implies\quad \langle (m)(bfc) := (mb)fc \rangle \\
M \in \text{BMod}_A \quad &\implies\quad M \otimes_A N \in \text{BMod}_C \\
N \in \text{AMod}_C \quad &\implies\quad \text{Hom}_A(M, N) \in \text{BMod}_C \\
\end{align*}
\]
Let us set
\[
\begin{align*}
\lambda_A &: A \to \text{End}(A)_A, \quad a \mapsto (x \mapsto ax) \\
\rho_A &: A \to \text{End}_A(A), \quad a \mapsto (x \mapsto xa)
\end{align*}
\]
Then we have the following isomorphisms:
\[
\begin{align*}
\lambda_A &: A \xrightarrow{\sim} \text{End}(A)_A \\
\rho_A &: A \xrightarrow{\sim} \text{End}_A(A) \\
\lambda_A &: ZA \xrightarrow{\sim} \text{End}_A(A)_A.
\end{align*}
\]

1.1.4. Additive and abelian $R$–linear categories.

1.1.4.1. Additive $R$–linear categories, summands and idempotents.

An ($R$–linear) additive category is a category $\mathcal{A}$ with a zero object (i.e., $\text{Mor}_\mathcal{A}(0, X) = \text{Mor}_\mathcal{A}(X, 0) = 0$ for all $X$) and such that all pairs of objects $X, X' \in \mathcal{A}$ admit

(1) a product, i.e., an object $X \amalg X'$ endowed with morphisms
\[
\text{pr}_X : X \amalg X' \to X \quad \text{and} \quad \text{pr}_{X'} : X \amalg X' \to X'
\]
such that the map
\[
\text{Mor}_\mathcal{A}(Y, X \amalg X') \to \text{Mor}_\mathcal{A}(Y, X) \times \text{Mor}_\mathcal{A}(Y, X')
\]
\[
\varphi \mapsto (\text{pr}_X \cdot \varphi, \text{pr}_{X'} \cdot \varphi)
\]
is a bijection:

\[
\begin{tikzcd}
Y \arrow[swap]{dr}{\alpha} \arrow{r}{\alpha'} & X \amalg X' \arrow{d}{\text{pr}_{X'}} \arrow{r}{\text{pr}_X} & X' \arrow{d}{\text{pr}_{X'}} \\
& X &
\end{tikzcd}
\]

(2) a coproduct, i.e., an object $X \amalg X'$ endowed with morphisms
\[
i_X : X \to X \amalg X' \quad \text{and} \quad i_{X'} : X' \to X \amalg X'
\]
such that the map
\[
\text{Mor}_\mathcal{A}(X, Y) \times \text{Mor}_\mathcal{A}(X', Y) \to \text{Mor}_\mathcal{A}(X \amalg X', Y)
\]
\[
\varphi \mapsto (\varphi \cdot i_X, \varphi \cdot i_{X'})
\]
is a bijection:

\[
\begin{array}{ccc}
X & \xrightarrow{i_X} & X' \\
\downarrow & & \downarrow \alpha' \downarrow \\
\downarrow \alpha & & X \xleftarrow{i_X'} Y
\end{array}
\]

**Lemma 1.12.** Given products \((X \prod X', \text{pr}_X, \text{pr}_{X'})\) (resp. coproducts \((X \coprod X', i_X, i_{X'})\)), there exist coproducts \((X \coprod X', i_X, i_{X'})\) (resp. products \((X \prod X', \text{pr}_X, \text{pr}_{X'})\)), and we have

\[
i_X \cdot \text{pr}_X + i_{X'} \cdot \text{pr}_{X'} = \text{Id}_Y \\
\text{pr}_X \cdot i_X = \text{Id}_X, \text{pr}_{X'} \cdot i_{X'} = \text{Id}_{X'} \\
\text{pr}_X \cdot i_{X'} = 0, \text{pr}_{X'} \cdot i_X = 0.
\]

There are natural isomorphisms

\[X \prod X' \xrightarrow{\sim} X \coprod X'.\]

**Sketch of proof of 1.12.**

Indeed, assume for example the existence of coproduct. Then the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{0} & X' \\
\downarrow \text{Id}_X & & \downarrow \alpha' \downarrow \\
\downarrow i_X & & X \xleftarrow{i_{X'}} Y
\end{array}
\]

defines \(\text{pr}_X\), etc. \qed

**Definition 1.13.** For \(X\) and \(X'\) two objects of an additive category \(\mathfrak{A}\), we say that \(X'\) is a summand of \(X\) and we write \(X' \mid X\) if there exists an object \(X''\) and an isomorphism \(X \xrightarrow{\sim} X' \oplus X''\).

**Definition 1.14.** For \(X\) an object of an additive category \(\mathfrak{A}\), we say that \(X\) is indecomposable if there is no isomorphism between \(X\) and an object \(X' \oplus X''\) where both \(X'\) and \(X''\) are nonzero.

An additive category \(\mathfrak{A}\) is called complete if, for any set-indexed family \((X_\alpha)_{\alpha \in I}\) of objects in \(\mathfrak{A}\), there is a product

\[
\prod_{\alpha \in I} X_\alpha, (p_\alpha)_{\alpha \in I}.
\]

It is called cocomplete if for each family \((X_\alpha)_{\alpha \in I}\) of objects in \(\mathfrak{A}\), there exists a coproduct

\[
\coprod_{\alpha \in I} X_\alpha, (i_\alpha)_{\alpha \in I}.
\]
(which represents $\prod_{\alpha \in I} \text{Hom}_A(X_{\alpha}, \bullet)$ — see below).

**Examples 1.15.** The categories $\mathcal{A}\text{Mod}$ and $\text{Mod}_A$ are complete and cocomplete.

The categories $\mathcal{A}\text{mod}$ and $\text{mod}_A$ are neither complete nor cocomplete.

1.1.4.2. Representable and corepresentable functors.

A contravariant functor $F$ defined on $\mathcal{A}$ is **representable** if there is an object $X \in \mathcal{A}$ and an isomorphism of functors:

$$\text{Mor}_A(\bullet, X) \xrightarrow{\sim} F,$$

and note that such a property determines the object $X$ uniquely up to a unique isomorphism.

**Example 1.16.** The product $X \amalg X'$ represents the product functor

$$\text{Mor}_A(\bullet, X) \times \text{Mor}_A(\bullet, X').$$

Dually, a covariant functor $F$ defined on $\mathcal{A}$ is corepresentable if there is an object $X \in \mathcal{A}$ and an isomorphism of functors:

$$\text{Mor}_A(X, \bullet) \xrightarrow{\sim} F,$$

and note that such a property determines the object $X$ uniquely up to a unique isomorphism.

**Example 1.17.** The coproduct $X \amalg X'$ is defined as “the” object (unique up to a unique isomorphism) which represents the product functor

$$\text{Mor}_A(X, \bullet) \times \text{Mor}_A(X', \bullet).$$

1.1.4.3. Summands and idempotents.

Here we come back to the particular case of the additive category $\mathcal{A}\text{Mod}$.

**Definition 1.18.**

An idempotent in $A$ is an element $i$ with $i^2 = i$.

Two idempotents $i_1$ and $i_2$ are orthogonal if $i_1i_2 = i_2i_1 = 0$.

An idempotent $i$ is primitive if $i \neq 0$ and $i$ cannot be expressed as a sum of two non-zero orthogonal idempotents.

Let $X$ be an $A$–module.

- If $X = X' \oplus X''$, the composition $X \rightarrow X' \hookrightarrow X$ (of the projection from $X$ onto $X'$ by the injection of $X'$ into $X$) is an idempotent of $E_AX$. Conversely, an idempotent $i$ of $E_AX$ determines a decomposition $X = X.i \oplus X.(1 - i)$.

- More generally a decomposition $X = \bigoplus_{\alpha} X_{\alpha}$ corresponds to a family $(i_{\alpha})_\alpha$ of mutually orthogonal idempotents in $E_AX$ such that $1 = \sum_{\alpha} i_{\alpha}$.
• The idempotent \( i \) of \( E_A X \) is primitive if and only if its image \( X . i \) (a summand of \( X \)) is indecomposable.
• The summands of the \( A \)-module \( A \) correspond to the idempotents \( i \) of \( A \). The \( A \)-module \( A i \) is indecomposable if and only if \( i \) is primitive.
• The central idempotents of \( A \) (idempotents of \( Z A \)) correspond to the summands of \( A \) as an \( A \)-module–\( A \), or, in other words, to the twosided ideals of \( A \) which are summands of \( A \).

If \( i \) is an idempotent of \( A \), the \( R \)-submodule \( i A i \) of \( A \), endowed with the composition laws of \( A \), is an algebra with unit element \( i \). The proof of the following lemma is left to the reader.

**Lemma 1.19.**

1. Assume that \( A \) is isomorphic to a direct product of a family of algebras \((A_i)_{i \in I} : A \xrightarrow{\sim} \prod_{i \in I} A_i .$$

Let us denote by \( e_i \) the element of \( A \) whose image in \( A_i \) is the identity and whose image in \( A_j \) for \( j \neq i \) is 0. Then

- the family \((e_i)_{i \in I}\) is a family of mutually orthogonal central idempotents in \( A \),
- \( 1 = \sum_{i \in I} e_i \),

\[ 1.1.4.4. \text{Central idempotents and products of algebras.} \]

The following proposition is much longer to state than to prove. Its proof is left to the reader.

**Proposition 1.20.**

Let \( I \) be a finite set.

1. Assume that \( A \) is isomorphic to a direct product of a family of algebras \((A_i)_{i \in I} :$$
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• If \( a_i := A e_i \), then \( a_i \) is a twosided ideal of \( A \), and we have

\[
A = \bigoplus_{i \in I} a_i .
\]

(2) Assume that \( A \) is a direct sum of a family \((a_i)_{i \in I}\) of twosided ideals:

\[
A = \bigoplus_{i \in I} a_i .
\]

Then

• there is a family \((e_i)_{i \in I}\) of mutually orthogonal central idempotents in \( A \) such that, for each \( i \in I \), \( a_i = A e_i \),
• if we endow \( a_i \) with the structure of algebra induced by the laws of \( A \) (with \( e_i \) as the identity element), then the projections

\[
\pi_i : A \twoheadrightarrow a_i = A e_i , \quad a \mapsto ae_i
\]

induce an algebra isomorphism

\[
A \xrightarrow{\sim} \prod_{i \in I} A e_i .
\]

(3) If this is the case, then the functors

\[
\begin{align*}
\mathcal{A} \text{Mod} & \to \mathcal{A}_i \text{mod} , \quad X \mapsto e_i X \\
\mathcal{A}_i \text{Mod} & \to \mathcal{A} \text{Mod} , \quad X_i \mapsto \text{Res}_{\pi_i} X_i
\end{align*}
\]

induce an equivalence

\[
\mathcal{A} \text{Mod} \simeq \bigoplus_{i \in I} \mathcal{A}_i \text{Mod} .
\]

1.1.4.5. Abelian categories.

Let \( \varphi : X \to X' \) be a morphism in \( \mathcal{A} \).

• The kernel \( \ker(\varphi) \) of \( \varphi \) is the object (if it exists) which represents the functor

\[
\ker (\varphi_\bullet : \text{Mor}_\mathcal{A}(\bullet , X) \to \text{Mor}_\mathcal{A}(\bullet , X')) .
\]

• The cokernel \( \text{coker}(\varphi) \) of \( \varphi \) is the object (if it exists) which represents the functor

\[
\ker (\varphi^* : \text{Mor}_\mathcal{A}(X', \bullet) \to \text{Mor}_\mathcal{A}(X , \bullet)) .
\]

• The image \( \text{im}(\varphi) \) is the kernel (if it exists) of the morphism \( X' \to \text{coker}(\varphi) \).
• The coimage \( \text{coim}(\varphi) \) is the cokernel (if it exists) of the morphism \( \ker(\varphi) \to X \).
Assume that these four objects exist. Then there is a unique morphism $\bar{\varphi}$ such that the following diagram is commutative:

![Diagram](image)

An abelian category is an additive category $\mathcal{A}$ such that

(Ab1) every morphism admits a kernel and a cokernel,

(Ab2) for each morphism $\varphi$, the corresponding morphism $\bar{\varphi}$ is an isomorphism.

**Examples 1.21.** The categories $\mathcal{A} \text{Mod}$ and $\text{Mod}_A$ are abelian.

The categories $\mathcal{A} \text{mod}$ and $\text{mod}_A$ are abelian if and only if $A$ is noetherian.

In an abelian category, the following properties are immediate consequences of the definitions.

- A monomorphism is a morphism whose kernel is zero.
- An epimorphism is a morphism whose cokernel is zero.

1.1.4.6. **Short exact sequences and split short exact sequences.**

The proof of the following proposition is left to the reader. It is written in the category $\mathcal{A} \text{Mod}$, so that the morphisms act on the right.

**Proposition 1.22.**

Let

\[(1.1) \quad 0 \to X' \xrightarrow{\alpha'} X \xrightarrow{\alpha''} X'' \to 0\]

be a short exact sequence in $\mathcal{A} \text{Mod}$.

The following assertions are equivalent.

(i) The exists a morphism $\beta' : X \to X'$ such that $\beta' \alpha' = \text{Id}_{X'}$.

(ii) There exists a morphism $\beta'' : X'' \to X$ such that $\beta'' \alpha'' = \text{Id}_{X''}$.

(iii) There exist morphisms $\beta' : X \to X'$ and $\beta'' : X'' \to X$ such that $\alpha' \beta' + \alpha'' \beta'' = \text{Id}_X$.

(iv) There exists an isomorphism $\sigma : X \xrightarrow{\sim} X' \oplus X''$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
0 & \xrightarrow{} & X' & \xrightarrow{\alpha'} & X & \xrightarrow{\beta'} & X'' & \xrightarrow{} & 0 \\
& & \downarrow{\sigma} & & \downarrow{\sigma} & & \downarrow{\sigma} & & \\
0 & \xrightarrow{} & X' & \xrightarrow{} & X' \oplus X'' & \xrightarrow{} & X' & \xrightarrow{} & 0
\end{array}
\]

Moreover, if the above properties hold, then the sequence

\[(1.2) \quad 0 \to X'' \xrightarrow{\beta''} X \xrightarrow{\beta'} X' \to 0\]

is also exact.
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If the above properties hold, we say that the sequence $1.1$ (resp. the sequence $1.2$) is split.

1.1.4.7. Exact functors.

**Definition 1.23.** A functor $F : _{A}\text{Mod} \to _{B}\text{Mod}$ is **exact** if whenever $X' \to X \to X''$ is exact in $_{A}\text{Mod}$, then $F(X') \to F(X) \to F(X'')$ is exact in $_{B}\text{Mod}$.

The functor $F$ is exact if and only if the image under $F$ of any short exact sequence in $_{A}\text{Mod}$ is a short exact sequence in $_{B}\text{Mod}$.

**Definition 1.24.**

- A (covariant) functor $F : _{A}\text{Mod} \to _{B}\text{Mod}$ is **left exact** if whenever $0 \to X' \to X$ is exact in $_{A}\text{Mod}$, then $0 \to F(X') \to F(X)$ is exact in $_{B}\text{Mod}$.
- A (covariant) functor $F : _{A}\text{Mod} \to _{B}\text{Mod}$ is **right exact** if whenever $X \to X'' \to 0$ is exact in $_{A}\text{Mod}$, then $F(X) \to F(X'') \to 0$ is exact in $_{B}\text{Mod}$.
- A contravariant functor $F : _{A}\text{Mod} \to _{B}\text{Mod}$ is left exact (resp. right exact) if the corresponding functor $(_{A}\text{Mod})^{\text{op}} \to _{B}\text{Mod}$ is left exact (resp. right exact).

Thus a functor is exact if and only if it is both left and right exact.

**Example 1.25 (Homomorphisms and tensor products as functors).** Let $M$ be an $A$–module–$B$. Then

- The functor $\text{Hom}_{A}(M, \cdot) : _{A}\text{Mod} \to _{B}\text{Mod}$ is covariant and left exact, i.e., if $0 \to X' \to X \to X''$ is exact, the sequence $0 \to \text{Hom}_{A}(M, X') \to \text{Hom}_{A}(M, X) \to \text{Hom}_{A}(M, X'')$ is exact.
- The functor $\text{Hom}_{A}(\cdot, M) : _{A}\text{Mod} \to \text{Mod}_{B}$ is contravariant and left exact, i.e., if $X' \to X \to X'' \to 0$ is exact, the sequence $0 \to \text{Hom}_{A}(X'', M) \to \text{Hom}_{A}(X, M) \to \text{Hom}_{A}(X', M)$ is exact.
- The functor $M \otimes_{B} \cdot : _{B}\text{Mod} \to _{A}\text{Mod}$ is covariant and right exact, i.e., if $Y' \to Y \to Y'' \to 0$ is exact, the sequence $M \otimes_{B} Y' \to M \otimes_{B} Y \to M \otimes_{B} Y'' \to 0$ is exact.
- The functor $\cdot \otimes_{A} M : \text{Mod}_{A} \to \text{Mod}_{B}$ is covariant and right exact, i.e., if $X' \to X \to X'' \to 0$ is exact, the sequence $X' \otimes_{A} M \to X \otimes_{A} M \to X'' \otimes_{A} M \to 0$ is exact.
1.2. Projective modules

1.2.1. Projective modules.
The following proposition is well-known and its (easy) proof is left to the reader – moreover, it will be proved below in the case of finitely generated modules.

**Proposition 1.26.** Let $M$ be an $A$–module. The following properties are equivalent:

(i) The functor $\text{Hom}_{A}(M, \bullet) : \mathcal{A}\text{Mod} \to \mathcal{E}_{A}\mathcal{M}\text{Mod}$ is exact.

(ii) Whenever $X$ and $Y$ are $A$–modules, $\varphi : X \to Y$ is an epimorphism, and $\psi : M \to Y$ is a morphism, there is a morphism $\hat{\psi} : M \to X$ such that $\hat{\psi}\varphi = \psi : X \to Y$.

(iii) $M$ is isomorphic to a direct summand of a free $A$–module.

The module $M$ is said to be projective if the preceding equivalent properties are satisfied.

1.2.2. Finitely generated projective modules.

**Lemma 1.27.** Let $X$, $Y$ and $M$ be $A$–modules.

(1) The image of
$$\text{Hom}_{A}(X, M) \otimes_{R} \text{Hom}_{A}(M, Y) \longrightarrow \text{Hom}_{A}(X, Y)$$

consists of those morphisms $X \to Y$, which factorize through $M^{n}$, for some natural integer $n$.

(2) If $M$ is an $A$–module–$B$, the preceding map factorizes through a map
$$\text{Hom}_{A}(X, M) \otimes_{B} \text{Hom}_{A}(M, Y) \longrightarrow \text{Hom}_{A}(X, Y)$$

**Proof.** Let
$$x = \sum_{i=1}^{n} \alpha_{i} \otimes \beta_{i} \in \text{Hom}_{A}(X, M) \otimes_{R} \text{Hom}_{A}(M, Y).$$

The image of $x$ in $\text{Hom}_{A}(X, Y)$ is $\sum_{i=1}^{n} \alpha_{i}\beta_{i}$. The maps $\alpha_{i}$ ($1 \leq i \leq n$), respectively $\beta_{i}$ ($1 \leq i \leq n$), describe a unique map $\alpha : X \to M^{n}$, respectively $\beta : M^{n} \to Y$. Their composition $\alpha\beta$ is equal to $\sum_{i=1}^{n} \alpha_{i}\beta_{i}$, which proves the assertion (1).

The proof of (2) is left to the reader. □  □
The $A$–dual of an $A$–module $X$ is the module–$A$ defined by
$$X^\vee := \text{Hom}_A(X, A).$$
We define the map $\tau_{X,Y}$ as the composition
$$\tau_{X,Y} : X^\vee \otimes_A Y \rightarrow \text{Hom}_A(X, Y).$$
We also set
$$\tau_X := \tau_{X,X}.$$
Applying 1.27 to the particular case where $M = A$, we see that

**Lemma 1.28.** The image of $\tau_{X,Y}$ consists of those morphisms which factorize through $A^n$, for some $n$.

**Definition 1.29.** The elements of the image of $\tau_{X,Y}$ are called the projective maps from $X$ to $Y$. We denote the set of all projective maps from $X$ to $Y$ by $\text{Hom}^\text{pr}_A(X, Y)$.

By 1.28, we see that $\text{Hom}^\text{pr}_A(X, Y)$ is a ‘twosided ideal’ in $\text{Hom}_A(X, Y)$, i.e., if $f \in \text{Hom}^\text{pr}_A(X, Y)$, $g \in \text{Hom}_A(Y, Z)$ and $h \in \text{Hom}_A(W, X)$, then $fg \in \text{Hom}^\text{pr}_A(X, Z)$ and $hf \in \text{Hom}^\text{pr}_A(W, Y)$.

The following omnibus theorem is classical.

**Theorem—Definition 1.30.** Let $M$ be an $A$–module. The following assertions are equivalent.

(i) $M$ is finitely generated, and whenever $\varphi$ is a surjective morphism from the $A$–module $X$ onto the $A$–module $Y$ and $\psi$ is a morphism of $M$ to $Y$, then there exists a morphism $\rho$ of $M$ to $X$ such that $\rho \varphi = \psi$.

(ii) $M$ is finitely generated, and the functor $\text{Hom}_A(M, \cdot) : A\text{Mod} \rightarrow \text{End}_A(M)$ is an exact functor.

(iii) $M$ is finitely generated, and any surjection with image $M$ is split.

(iv) $M$ is a direct summand of a free module, i.e., $M \mid A^n$, for some integer $n$.

(v) The map $\tau_M : M^\vee \otimes_A M \rightarrow \text{Hom}_A(M, M)$ is onto.

(vi) The map $\tau_{X,M} : X^\vee \otimes_A M \rightarrow \text{Hom}_A(X, M)$ is an isomorphism for all $A$–modules $X$.

(vii) The map $\tau_{M,X} : M^\vee \otimes_A X \rightarrow \text{Hom}_A(M, X)$ is an isomorphism for all $A$–modules $X$.

(viii) The map $\tau_M$ is an isomorphism.

**Proof.** Short proof of 1.30

(i) $\Rightarrow$ (ii). (i) implies that the functor $\text{Hom}_A(M, \cdot)$ is right exact. Since it is always left exact, it is exact.

(ii) $\Rightarrow$ (iii). One applies the functor $\text{Hom}_A(M, \cdot)$ and uses a preimage of $1_M$ to define a splitting.

(iii) $\Rightarrow$ (iv). Because $M$ is finitely generated over $A$, it is an image, hence a summand, of $A^n$ for some $n$. 

(iv) $\Rightarrow$ (v). Since $M \mid A^n$, we know that $\text{Id}_M$ is in the image of $\tau_M$. Furthermore, $\tau_M$ is a map in $\text{End}_A(M)\text{Mod}_{\text{End}_A(M)}$ and consequently it is onto.

(v) $\Rightarrow$ (vi). We exhibit the inverse of $\tau_{X,M}$. By (v) there exists an element $\sum_{i=1}^n n_i \otimes m_i$ such that $\tau_M(\sum_{i=1}^n n_i \otimes m_i) = 1_M$. We define the map

$$\psi : \text{Hom}_A(X, M) \to X^\vee \otimes_A M$$

by $\alpha \mapsto \sum_{i=1}^n \alpha n_i \otimes m_i$. This map $\psi$ satisfies $\psi \circ \tau_{X,M} = \text{Id}_{\text{Hom}_A(X,M)}$ and $\tau_{X,M} \circ \psi = \text{Id}_{X^\vee \otimes_A M}$.

(v) $\Rightarrow$ (vii). Using the same element $\sum_{i=1}^n n_i \otimes m_i$ as above, one can give an explicit formula of the inverse of $\tau_{M,X}$, namely

$$\text{Hom}_A(M, X) \to M^\vee \otimes_A X$$

$$\alpha \mapsto \sum_{i=1}^n n_i \otimes m_i \alpha.$$ 

The implications (vi) $\Rightarrow$ (v) and (vii) $\Rightarrow$ (v) are trivial because $\tau_M = \tau_{M,M}$.

(vii) $\Rightarrow$ (i). Since $M^\vee \otimes_A \bullet$ is a right exact functor, the map $\varphi$ in (i) induces a surjection

$$M^\vee \otimes_A X \xrightarrow{\varphi^*} M^\vee \otimes_A Y.$$ 

But $M^\vee \otimes_A X$ and $M^\vee \otimes_A Y$ are respectively isomorphic to $\text{Hom}_A(M, X)$ and $\text{Hom}_A(M, Y)$, and so $\varphi$ induces a surjection

$$\text{Hom}_A(M, X) \xrightarrow{\varphi^*} \text{Hom}_A(M, Y).$$

Now, any preimage of $\psi$ satisfies the condition on $\rho$ in (i).

Moreover, since the identity of $M$ is a projective map, $M$ is a summand of some $A^n$, hence is finitely generated.

(vii) $\Rightarrow$ (viii) is trivial, as well as (viii) $\Rightarrow$ (v). $\square \quad \square$

We denote the full subcategory of $A\text{mod}$ consisting of all the projective $A$–modules by $A\text{proj}$. If $M$ is an $(A,B)$-bimodule which is projective as an $A$–module, then we write $M \in A\text{mod}_B \cap A\text{proj}$, by abuse of notation.

Similarly, we denote by $\text{proj}_A$ the category of finitely generated projective right $A$–modules (“projective modules–$A$”).

Notice also that the $R$–module of projective maps $\text{Hom}_A^\text{pr}(X, Y)$ may be defined as the set of those morphisms from $X$ to $Y$ which factorize through a projective $A$–module.

1.2.3. Projective modules and duality.

We recall that for an $A$–module $X$, we denote by $X^\vee$ its $A$–dual, a module-$A$. Now if $Y$ is a module–$A$, we denote by $Y^\vee$ its dual–$A$, an $A$–module.
If \( \varphi : X \to X' \) is a morphism in \( \mathcal{A} \text{Mod} \), then the map

\[ \varphi^\vee : X'^\vee \to X^\vee, \quad (y' : X' \to A) \mapsto (\varphi \cdot y' : X \to A) \]

is a morphism in \( \text{Mod}_A \). Hence we have a contravariant functor

\[ \mathcal{A} \text{Mod} \to \text{Mod}_A, \quad X \to X^\vee, \]

as well as a contravariant functor

\[ \text{Mod}_A \to \mathcal{A} \text{Mod}, \quad Y \to Y^\vee. \]

We have a natural morphism of \( A \)-modules

\[ X \to (X^\vee)^\vee, \quad x \mapsto (y \mapsto xy). \]

The next proposition follows easily from the fact that finitely generated projective modules are nothing but summands of free modules with finite rank.

**Proposition 1.31.**

(a) Whenever \( X \) is a finitely generated projective \( A \)-module (resp. \( Y \) is a finitely generated projective module–\( A \)), then \( X^\vee \) is a finitely generated projective module–\( A \) (resp. \( Y^\vee \) is a finitely generated projective \( A \)-module).

(b) If \( X \in \mathcal{A} \text{proj} \), the natural morphism \( X \mapsto (X^\vee)^\vee \) is an isomorphism and the functors \( X \mapsto X^\vee \) and \( Y \mapsto Y^\vee \) induce inverse isomorphisms between \( \mathcal{A} \text{proj} \) and \( \text{proj}_A \).

### 1.3. Complete reducibility, Isotypic components

Throughout this paragraph, we denote by \( k \) a (commutative) field.

We call division algebra over \( k \) (or division \( k \)-algebra) a \( k \)-algebra, finite dimensional as a \( k \)-vector space, which is a (non necessarily commutative) field (in other words, a field whose center contains \( k \)).

Let \( A \) a \( k \)-algebra which is finite dimensional as a \( k \)-vector space.

We have an equivalence of categories between \( \mathcal{A} \text{mod} \) and \( \text{mod}_A \) defined by the \( k \)-duality functor

\[ X \mapsto X^* = \text{Hom}(X, k), \quad (\varphi : X_1 \to X_2) \mapsto (\varphi^* : X^*_2 \to X^*_1). \]

Whenever \( X \) is a finitely generated \( A \)-module or module–\( A \), and \( X_1 \) is a submodule of \( X \), we denote by \( X^*_1 \) its orthogonal in \( X^* \) : it is a submodule of \( X^* \), and \( X^*_1 \perp = X_1 \).

#### 1.3.1. Irreducible modules : generalities.

For the convenience of the reader, we repeat the definition given in chapter 1.

**Definition 1.32.** Let \( S \) be an \( A \)-module. We say that \( S \) is irreducible (or “simple”) if \( S \neq 0 \) and if the only submodules of \( S \) are 0 and \( S \).
One may of course give an analogous definition for modules–$A$.

The following first properties of irreducible modules are easy to check.

(I1) Let $S$ and $T$ be irreducible $A$–modules. Then, either $S$ and $T$ are isomorphic, and $\text{Hom}_A(S, T)$ consists of $0$ and of isomorphisms, or $S$ and $T$ are not isomorphic, and $\text{Hom}_A(S, T) = 0$. In particular, the algebra $E_A S := \text{Hom}_A(S, S)$ is a division $k$–algebra.

(I2) Let $S$ be irreducible. Whenever $s$ is a non–zero element of $S$, we have $S = As$. Thus every irreducible module is isomorphic to a quotient of $A A$, a finite dimensional $k$–vector space and a finitely generated $A$–module,

(I3) A submodule $S$ of an $A$–module $X$ is irreducible if and only if it is a minimal nonzero submodule. Every $A$–module has an irreducible submodule.

Indeed, since $A$ is finite dimensional over $k$, every cyclic $A$–module has finite dimension over $k$, hence every $A$–module has submodules of finite dimension. It follows that every $A$–module has submodules of minimal nonzero dimension : such submodules are minimal submodules.

(I4) A quotient of an $A$–module $X$ is irreducible if and only if it is the quotient of $X$ by a maximal submodule. In particular every $A$–module with finite dimension over $k$ has an irreducible quotient.

(I5) Whenever $X$ is an $A$–module with finite dimension over $k$, there is a finite sequence

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

of submodules of $X$ such that every quotient $X_i/X_{i-1}$ is an irreducible $A$–module.

(I6) If $\varphi : X \twoheadrightarrow S$ is a surjective morphism from $X$ onto an irreducible module $S$, and if $0 = X_0 \subset \cdots \subset X_n = X$ is a composition series of $X$, there exists a unique $i \geq 1$ such that $\varphi$ maps $X_i$ onto $S$ and $X_{i-1}$ onto $0$. In particular we have $S \cong X_i/X_{i-1}$.

(I7) Every irreducible module is a composition factor of $A$ (viewed as an $A$–module), and there is only a finite number of isomorphism classes of irreducible $A$–modules.

(I8) The $k$–duality functor sends an irreducible $A$–module on an irreducible module–$A$.

We denote by $\text{Irr}(\text{mod-}A)$ (or simply by $\text{Irr}(A)$) the (finite) set of isomorphism classes of irreducible $A$–modules.

1.3.2. Completely reducible modules.
1.3. COMPLETE REDUCIBILITY, ISOTYPIC COMPONENTS

1.3.2.1. Definition and characterization.

Proposition–Definition 1.33.
Let $X$ be an $A$–module.

(1) The following properties are equivalent:
   (i) $X$ is a direct sum of irreducible submodules.
   (ii) $X$ is a sum of irreducible submodules.
   (iii) Every submodule $X'$ of $X$ is a summand in $X$, i.e., there exists a submodule $X''$ such that $X = X' \oplus X''$.

(2) Definition: A module $X$ which satisfies the above equivalent assertions is called completely reducible, or semi-simple.

Proof of Proposition 1.33.
It relies on the following lemma.

Lemma 1.34.
Let $X$ be an $A$–module. Assume that there exists a family $(S_i)_{i \in I}$ of irreducible submodules of $X$ such that $X = \sum_{i \in I} S_i$. Let $X'$ be a submodule of $X$. Then there exists a subset $J$ of $I$ such that
$$X = X' \oplus \left( \sum_{j \in J} S_j \right).$$

Proof of Lemma 1.34.
We can choose (by Zorn lemma) $J$ maximal such that
$$\left( \sum_{j \in J} S_j \right) \cap X' = 0.$$
Let us then prove that $X = X' + \left( \sum_{j \in J} S_j \right)$. For that, it suffices to prove that for all $i \in I - J$ we have $S_i \subseteq X' + \left( \sum_{j \in J} S_j \right)$. But if this is not the case, we have $S_i \cap (X' + \left( \sum_{j \in J} S_j \right)) = 0$ (since $S_i$ is irreducible), from which we deduce
$$\left( \sum_{j \in J \cup \{i\}} S_j \right) \cap X' = 0,$$
a contradiction.

We now prove proposition 1.33.
(i) $\Rightarrow$ (ii) : clear.
(ii) $\Rightarrow$ (iii) : results from lemma 1.34.
(iii) $\Rightarrow$ (i) : First, let us remark that if a module satisfies (iii), then any submodule satisfies (iii) as well.

Let us choose a family $(S_i)_{i \in I}$ maximal subject to the condition $\sum_{i \in I} S_i = \oplus_{i \in I} S_i$. We shall prove that $X = \oplus_{i \in I} S_i$. By hypothesis, we know that there exists a submodule $X'$ such that $X = (\oplus_{i \in I} S_i) \oplus X'$. We must prove that $X' = 0$. If this is not the case, $X'$ contains an irreducible module, a contradiction with the maximality of the family $(S_i)_{i \in I}$. 

$\square$
Proposition 1.35. Let \( X \) be a completely reducible \( A \)-module. An \( A \)-module \( X' \) is isomorphic to a submodule of \( X \) if and only if \( X' \) is isomorphic to a quotient of \( X \).

Proof. This is a consequence of the characterization (iii) (see 1.33 above) of completely reducible modules.

1.3.2.2. Isotypic components.

Definition 1.36. Let \( X \) be an \( A \)-module, and let \( S \) be an irreducible \( A \)-module.

1. We call \( S \)-isotypic component of \( X \) and we denote by \( \text{Iso}(S, X) \) the sum of all submodules of \( X \) isomorphic to \( S \).

2. We say that \( X \) is \( S \)-isotypic if \( X = \text{Iso}(S, X) \).

Notice that an \( S \)-isotypic module is completely reducible. The following property result from the fact that a completely reducible module has same submodules and quotients.

Lemma 1.37. If \( X \) is \( S \)-isotypic, all submodules and all quotients of \( X \) are also \( S \)-isotypic.

Let us prove a few properties of the \( S \)-isotypic component.

Proposition 1.38.

1. \( \text{Iso}(S, X) \) is an \( A \)-submodule \( - E_A X \) of \( X \).

2. The morphism

\[
S \otimes_{E_A} \text{Hom}_A(S, X) \longrightarrow X , \; s \otimes \varphi \mapsto s \varphi
\]

defines an isomorphism of \( A \)-modules \( - E_A X \) between \( S \otimes_{E_A} \text{Hom}_A(S, X) \) and \( \text{Iso}(S, X) \).

Proof. (1) results from the fact that the image of an irreducible submodule of \( X \) by an \( A \)-homomorphism is either 0 or isomorphic to this submodule.

(2) For the same reason as above, whenever \( \varphi \in \text{Hom}_A(S, X) \), the image of \( \varphi \) is contained in \( \text{Iso}(S, X) \). It follows that the image of the morphism described in the assertion (2) is contained in \( \text{Iso}(S, X) \).

The assertion can be proven using Proposition 1.35.

The proof of the next proposition is left to the reader.

Proposition 1.39.

Let \( X \) be a completely reducible \( A \)-module. Let \( \text{Irr}(A) \) denote a complete set of representatives of isomorphism classes of irreducible \( A \)-modules.

1. We have

\[
X = \bigoplus_{S \in \text{Irr}(A)} \text{Iso}(S, X).
\]
1.4. Jacobson radical, semisimple algebras

(2) The morphisms (for \( S \in \text{Irr}(A) \))

\[ \mu_S(x) : S \otimes_{E_A} \text{Hom}_A(S, X) \rightarrow X \]

induce an isomorphism of \( A \)-modules–

\[ \bigoplus_{S \in \text{Irr}(A)} S \otimes_{E_A} \text{Hom}_A(S, X) \sim X. \]

The proof of the following proposition is straightforward.

**Proposition 1.40.**

Let \( X \) be a completely reducible \( A \)-module. We set \( \text{Irr}(X) := \{ S \in \text{Irr}(A) \mid \text{Iso}(S, X) \neq 0 \} \).

(1) Let \( Y \) be a submodule of \( X \). Then we have

\[ \text{Iso}(S, Y) = \text{Iso}(S, X) \cap Y, \]

and so

\[ Y = \bigoplus_{S \in \text{Irr}(X)} \text{Iso}(S, X) \cap Y. \]

(2) Assume that for all \( S \in \text{Irr}(X) \), \( \text{Iso}(S, X) \) is irreducible (we then say that \( X \) is “multiplicity free”). Then the map

\[ I \mapsto \bigoplus_{S \in I} \text{Iso}(S, X) \]

is a bijection from the set of subsets of \( \text{Irr}(X) \) onto the set of submodules of \( Y \).

1.4. Jacobson radical, semisimple algebras

1.4.1. Definition of the Jacobson radical.

An element \( r \in A \) is said “left quasi-nilpotent” (resp. “right quasi-nilpotent”) if \( 1 - r \) is left invertible (resp. right invertible), i.e., if there exists \( u \in A \) such that \( u(1 - r) = 1 \) (resp. \( (1 - r)u = 1 \)). A left and right quasi-nilpotent element is said quasi-nilpotent. Notice that a nilpotent element is (left and right) quasi-nilpotent.

**Proposition–Definition 1.41.**

Let \( A \) be an \( R \)-algebra. The following subsets of \( A \) are well defined and are all equal:

(i) The intersection of all maximal proper left ideals of \( A \).

(i’) The intersection of all maximal proper right ideals of \( A \).

(ii) The set of elements \( r \in A \) such that \( ar \) is left quasi-nilpotent for all \( a \in A \).

(ii’) The set of elements \( r \in A \) such that \( ra \) is right quasi-nilpotent for all \( a \in A \).
The largest left ideal consisting of left quasi-nilpotent elements.

(iii') The largest right ideal consisting of right quasi-nilpotent elements.

(iv) The largest twosided ideal consisting of quasi-nilpotent elements.

The above conditions define a twosided ideal of $A$ which is called the (Jacobson) radical of $A$ and is denoted by $\text{Rad}(A)$.

**Proof.** Let us first prove the equality of the sets defined in (i), (ii) and (iii). We denote temporarily by $\text{Rad}_A(A)$ and $\text{Rad}_A(A)$ the sets defined respectively by the conditions (i) and (ii).

- If $r \in \text{Rad}_A(A)$, for all $a \in A$ we have $ar \in \text{Rad}_A(A)$, and $1 - ar$ belongs to no proper left ideal of $A$, which shows that $1 - ar$ is left invertible. Thus $\text{Rad}_A(A) \subseteq \text{Rad}_A(A)$.

- If $r \in \text{Rad}_A(A)$, then (by definition of $\text{Rad}_A(A)$) the left ideal generated by $r$ consists of left quasi-nilpotent elements. Let us show that any left ideal $n$ consisting of left quasi-nilpotent elements is contained in $\text{Rad}_A(A)$: this will show that $\text{Rad}_A(A) \subseteq \text{Rad}_A(A)$, that there is an ideal as defined in (iii) and that it coincides with $\text{Rad}_A(A)$.

  If $n \not\subseteq \text{Rad}_A(A)$, there exists a maximal left ideal $a$ such that $n \not\subseteq a$, whence $n + a = A$. So there exists $n \in n$ and $a \in a$ with $1 = n + a$, which shows that $a$ is left invertible, a contradiction.

**Lemma 1.42.**

(1) Any right ideal consisting of right quasi-nilpotent elements consists in fact of (left and right) quasi-nilpotent elements.

(2) For $a, a' \in A$, the element $aa'$ is quasi-nilpotent if and only if the element $a'a$ is quasi-nilpotent.

**Proof.** (1) Let $n$ be such a right ideal. For $n \in n$, let us denote by $1 - n'$ a right inverse of $1 - n$. From the equality $(1 - n)(1 - n') = 1$, we deduce that $n' = n n' - n$, hence $n' \in n$. It follows that $1 - n'$ has a right inverse. Thus $1 - n'$ has a left inverse and a right inverse, hence is invertible and so is its inverse $1 - n$.

(2) An elementary computation shows that if $(1 - a'a)c = 1$, then $(1 - aa')(1 + ac a') = 1$. □

The ideal $\text{Rad}(A)_A$ (by the equivalence of (i') and (iii')) consists of right quasi-nilpotent elements, hence by the preceding lemma 1.42 it consists of elements $r$ such that $ar$ is quasi-nilpotent for all $a \in A$. By the equality of the sets defined by (i) and (iii), it follows that $\text{Rad}(A)_A \subseteq \text{Rad}(A)$. By symmetry, we deduce that $\text{Rad}(A)_A = \text{Rad}(A)_A$. This shows that the set defined by the above conditions is a twosided ideal. The condition (iv) is now obviously fulfilled. □

Let $A^\times$ denote the group of unit of an algebra $A$. Notice that the set

$$1 + \text{Rad}(A) := \{ 1 + r \mid r \in \text{Rad}(A) \}$$
is a normal subgroup of $A^\times$.

**Proposition 1.43.**

We have the following short exact sequence

$$1 \to 1 + \text{Rad}(A) \to A^\times \to (A/\text{Rad}(A))^\times \to 1.$$ 

**Proof.** It suffices to prove the surjectivity of $A \times 1 + \text{Rad}(A) \to A/\text{Rad}(A)$. Let $a \in A$ such that its image in $A/\text{Rad}(A)$ is invertible. So there is $a' \in A$ such that $aa' \in 1 + \text{Rad}(A)$, which shows that $aa'$ is invertible and thus $a$ is invertible. □

### 1.4.2. Nakayama lemma and more proprieties.

#### 1.4.2.1. Nakayama’s lemma.

**Proposition 1.44 (Nakayama lemma).**

Let $X$ be a finitely generated $A$–module.

1. If $\text{Rad}(A)X = X$, then $X = 0$.
2. Let $E$ be a subset of $X$ whose image in $X/\text{Rad}(A)X$ is a generating set. Then $E$ is a generating set for $X$.
3. Let $\varphi : Y \to X$ be a morphism of $A$–modules. Then $\varphi$ is onto if and only if the composition of $\varphi$ by the canonical surjection $X \to X/\text{Rad}(A)X$ is onto.

**Proof.** (1) Let $m$ be the smallest integer such that $X$ can be generated by $m$ elements. Assume $X \neq 0$, hence $m \geq 1$, and let $(x_1, x_2, \ldots, x_m)$ be a set of generators of $X$. If $\text{Rad}(A)X = X$, then in particular $x_m \in \text{Rad}(A)X$ and there exist elements $r_1, r_2, \ldots, r_m \in \text{Rad}(A)$ such that $x_m = r_1x_1 + r_2x_2 + \cdots + r_mx_m$. It follows that $(1 - r_m)x_m = r_1x_1 + \cdots + r_{m-1}x_{m-1}$, and since $1 - r_m$ is invertible, $x_m$ is a linear combination of $x_1, \ldots, x_{m-1}$, hence $X$ is generated by $x_1, \ldots, x_{m-1}$, a contradiction.

(2) Let us denote by $X'$ the submodule of $X$ generated by $E$. By hypothesis, we have $X' + \text{Rad}(A)X = X$, which can be reinterpreted as $\text{Rad}(A)(X/X') = X/X'$. By the first assertion, it follows that $X/X' = 0$, hence $X = X'$.

(3) : Call $E$ the image of $\varphi$ and apply (2). □

#### 1.4.2.2. Radical and Irreducible modules.

**Definition 1.45.** An $A$–module is said to be irreducible if it has exactly two submodules, i.e., if $S \neq 0$ and the only nonzero submodule of $S$ is $S$.

**Lemma 1.46.**

Up to isomorphism, the irreducible $A$–modules are the modules $A/m$ for $m$ a maximal left ideal of $A$. 
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Proof. Notice that if $S$ is irreducible, then whenever $s \in S$, $s \neq 0$, we have $As = S$. So an irreducible $A$–module is an image of $A$, hence of the shape $A/\mathfrak{m}$ where $\mathfrak{m}$ is a left ideal of $A$. The ideal $\mathfrak{m}$ is maximal if and only if the module $A\mathfrak{m}$ has exactly two submodules. □

Proposition 1.47.

$\text{Rad}(A)$ is the annihilator of all irreducible $A$–modules, i.e., the following assertions are equivalent for $a \in A$:

(i) $a \in \text{Rad}(A)$,
(ii) Whenever $S$ is an irreducible $A$–module, we have $aS = 0$.

Proof. Let us first prove that the annihilator of all irreducible $A$–modules is contained in $\text{Rad}(A)$. Indeed, if $a$ annihilates $A\mathfrak{m}$, then $aA \subset \mathfrak{m}$, hence in particular $a \in \mathfrak{m}$. Thus if $a$ annihilates $A/\mathfrak{m}$ for all maximal left ideal $\mathfrak{m}$, we see that $a$ belongs to $\bigcap \mathfrak{m}$, i.e., belongs to $\text{Rad}(A)$.

Conversely, if $S$ is irreducible, $S$ is finitely generated and by Nakayama’s lemma we have $\text{Rad}(A)S \neq S$, hence $\text{Rad}(A)S = 0$. □

Let $i$ be an idempotent of $A$. Recall that $iAi$ inherits a structure of algebra whose unity element is $i$.

Proposition 1.48.

We have $\text{Rad}(iAi) = i\text{Rad}(A)i = iAi \cap \text{Rad}(A)$.

Proof. It is clear that $i\text{Rad}(A)i = iAi \cap \text{Rad}(A)$.

In order to prove that $\text{Rad}(iAi) \subset i\text{Rad}(A)i$, it suffices to prove that $\text{Rad}(iAi) \subset \text{Rad}(A)$. Let us prove that $\text{Rad}(iAi)$ annihilates all irreducible $A$–modules.

Let $S$ be an irreducible $A$–module. If $iS = 0$, then $\text{Rad}(iAi)S \neq 0$. Assume $iS \neq 0$. Let us check that $iS$ is an irreducible $iAi$–module. Indeed, assume that $S'$ is a nonzero $iAi$–submodule of $iS$; by the irreducibility of $S$, we have $AS' = iS$, hence $iAiS' = S$. Since $iS$ is irreducible, it is annihilated by $\text{Rad}(iAi)$, hence $S$ is annihilated by $\text{Rad}(iAi)$.

Let us now prove that $i\text{Rad}(A)i \subset \text{Rad}(iAi)$. Let $r \in \text{Rad}(A)$. Hence $iri \in \text{Rad}(A)$ and there exists $t \in A$ such that $(1-iri)(1-b) = 1$, i.e., $1-b-iri+iribi = 0$. Multiplying on both sides by $i$ yields

$$i - ibi - iri + iribi = 0,$$

which shows that $(i-iri)$ is (right) invertible in $iAi$. Since $i\text{Rad}(A)i$ is a twosided ideal in $iAi$, this shows indeed that $i\text{Rad}(A)i \subset \text{Rad}(iAi)$. □

1.4.3. Jacobson radical and semi–simple algebras.

Throughout this section, we assume that $A$ is a finite dimension $k$–algebra where $k$ is a field.
1.4. JACOBSON RADICAL, SEMISIMPLE ALGEBRAS

1.4.3.1. Yet another characterization of $\text{Rad}(A)$.

We have the following supplementary characterization of the radical.

**Proposition 1.49.**

Let $k$ be a (commutative) field, and let $A$ be a $k$–algebra, finite dimensional over $k$. Then $\text{Rad}(A)$ is the largest two-sided nilpotent ideal of $A$.

**Proof.** It suffices to prove that $\text{Rad}(A)$ is nilpotent, so it suffices to prove that, whenever $n \geq 0$ is an integer, then

$$\text{Rad}(A)^n \neq 0 \implies \text{Rad}(A)^{n+1} \subset \text{Rad}(A)^n.$$ 

That implication is an immediate consequence of Nakayama’s lemma. \qed

1.4.3.2. Socle and radical of a module.

**Definitions 1.50.** Whenever $X$ is a finitely generated $A$–module.

1. We denote by $\text{Rad}_A(X)$ (or simply $\text{Rad}(X)$) and we call radical of $X$ the intersection of all the maximal proper submodules of $X$.

2. We denote by $\text{Soc}_A(X)$ (or simply $\text{Soc}(X)$) and we call socle of $X$ the sum of all the minimal nonzero submodules of $X$.

Similarly, whenever $Y$ is a module–$A$, we define its radical $\text{Rad}(Y)_A$ and its socle $\text{Soc}(Y)_A$.

Note that a nonzero submodule of $X$ is minimal if and only if it is irreducible, and that similarly a proper submodule of $X$ is maximal if and only if its quotient is irreducible.

**Proposition 1.51.**

Let $X$ be a finitely generated $A$–module.

1. **Socle, radical and duality:**
   - (a) the orthogonal of $\text{Soc}_A(X)$ in $X^*$ is $\text{Rad}(X^*)_A$,
   - (b) the orthogonal of $\text{Rad}_A(X)$ in $X^*$ is $\text{Soc}(X^*)_A$.

2. **Socle, radical and complete reducibility:**
   - (a) $\text{Rad}(X)$ is the smallest submodule of $X$ providing a semisimple quotient,
   - (b) $\text{Soc}(X)$ is the largest completely reducible submodule of $X$.

3. **Socle and radical as bimodules:**

   If $M$ is a $A$–module–$B$, $\text{Soc}_A(M)$ and $\text{Rad}_A(M)$ are $A$–submodules–$B$ of $M$.

**Proof.** The assertions (1) are obvious.

In order to prove (2), it follows that it suffices to prove that $\text{Soc}(X)$ is the largest completely reducible submodule of $X$, which is immediate.
Since the image of an irreducible module by an $A$–morphism is either 0 or an irreducible module, we see that, whenever $M$ is an $A$–module–$B$, $\text{Soc}_A(M)$ is an $A$–submodule–$B$. It follows by duality that $\text{Rad}(M^*)_A$ is a $B$–submodule–$A$ of $M^*$. Interchanging right and left, and $M$ and $M^*$, shows then that $\text{Rad}_A(M)$ is an $A$–submodule–$B$ of $M$. □

The following proposition is an immediate consequence of the definitions.

**Proposition 1.52.**

*The Jacobson radical* $\text{Rad}(A)$ *coincides with the radical* $\text{Rad}_A(A)$ *of* $A$ *as an* $A$–*module and with the radical* $\text{Rad}(A)_A$ *of* $A$ *as a module–* $A$.

\(\dagger\) **Attention** (\(\dagger\)) It is not true in general that $\text{Soc}_A(A) = \text{Soc}(A)_A$ (see the example after 1.54). We shall see, nevertheless, that this equality holds for symmetric algebras.

1.4.3.3. *Semi–simple algebras : Definition.*

Let us first notice that, since $\text{Rad}(A) = \text{Rad}_A(A)$, the $A$–module $A/\text{Rad}(A)$ is completely reducible (Similarly, the module–$A A/\text{Rad}(A)$ is completely reducible).

**Proposition 1.53.**

(1) Whenever $X$ is an $A$–module, we have $\text{Rad}(X) = \text{Rad}(A)X$.

(2) $X$ is completely reducible if and only if $\text{Rad}(A)X = 0$.

(3) We have $\text{Rad}(A/\text{Rad}(A)) = 0$.

**Proof.** (1) Since $X/\text{Rad}(X)$ is completely reducible and since $\text{Rad}(A)$ annihilates all irreducible modules, we have $\text{Rad}(A)X \subseteq \text{Rad}(X)$. Reciprocally, $X/\text{Rad}(A)X$ is a homomorphic image of the $A$–module $(A/\text{Rad}(A))^{(I)}$ (for some set $I$), hence is completely reducible. This shows that $\text{Rad}(X) \subseteq \text{Rad}(A)X$.

(2) results immediately from (1) and from the definition of $\text{Rad}(X)$.

(3) By what precedes applied to the $A$–module $A$, we have $\text{Rad}(A/\text{Rad}(A)) = \text{Rad}(A)(A/\text{Rad}(A)) = 0$. □

**Corollary 1.54.**

*Let* $X$ *be a finitely generated* $A$–*module. *We have

$$\text{Soc}(X) = \{ x \in X \mid (\text{Rad}(A)x = 0) \}.$$  

**Proof.** Since, by definition, $\text{Soc}(X)$ is completely reducible, we have $\text{Rad}(A)\text{Soc}(X) = 0$. Conversely, suppose that $x \in X$ is such that $\text{Rad}(A)x = 0$. Then the $A$–module $Ax$ is annihilated by $\text{Rad}(A)$ hence is completely reducible, hence is contained in $\text{Soc}(X)$. □
Example 1.55. Let us consider the $k$–algebra $A := \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$. Then it is easy to check that $\text{Rad}(A) = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$, from which it follows (by 1.54, (2)) that $\text{Soc}_A(A) = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$ and $\text{Soc}(A)_A = \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$.

Proposition–Definition 1.56.

(1) The following assertions are equivalent:

(i) $\text{Rad}(A) = 0$.

(ii) All $A$–modules are completely reducible.

(iii) All $A$–modules are projective.

(iii’) All modules–$A$ are projective.

An algebra satisfying the above properties is called semi–simple.

(2) $\text{Rad}(A)$ is the smallest twosided ideal of $A$ whose quotient is a semi–simple algebra.

Remark 1.57. We shall see (cf. 1.63 below) that an algebra $A$ is semi–simple if and only if $A$ is a completely reducible $A$–module–$A$.

1.4.3.4. Group algebra over a field.

Proposition 1.58.

Let $k$ be a field and let $G$ be a finite group. The following assertions are equivalent:

(i) The algebra $kG$ is semisimple.

(ii) The characteristic of $k$ does not divide the order $|G|$ of $G$.

Proof. (i)$\Rightarrow$(ii): Let us set $SG := \sum_{g \in G} g \in kG$. We have $SG \in ZkG$ and $(SG)^2 = |G|SG$.

Thus if the characteristic of $k$ divides $|G|$, we see that $SG$ is a nonzero nilpotent element of $ZkG$, which proves that $\text{Rad}(kG) \neq 0$ and $kG$ is not semisimple.

(ii)$\Rightarrow$(i): We prove that every $kG$–module is completely reducible.

Let $X$ be a $kG$–module, and let $Y$ be a submodule. There is a $k$–linear projector $\pi : X \rightarrow Y$. Let us set $\pi_G := \frac{1}{|G|} \sum_{g \in G} g \pi g^{-1}$.

We check that $\pi_G$ is a $kG$–linear projector from $X$ onto $Y$, showing that there is a $kG$–submodule $Y'$ such that $X = Y \oplus Y'$.

Indeed, it is clear that $\pi_G$ is $kG$–linear. Let us prove that $\pi_G^2 = \pi_G$. We have $\pi_G^2 = \frac{1}{|G|^2} \sum_{g,h \in G} g \pi g^{-1} h \pi h^{-1}$. Since $g^{-1}h$ stabilizes
the image $Y$ of $\pi$, we see that $\pi g^{-1} h \pi = g^{-1} h \pi$, which implies $\pi^2 = \pi G$. 

1.5. Structure of semi–simple algebras

1.5.1. Simple algebras.

**Definition 1.59.** We say that the algebra $A$ is simple if $A \neq 0$ and if its only twosided ideals are 0 and $A$. In other words, $A$ is simple if and only if $A$ is an irreducible $A$–module–$A$.

A division algebra is obviously simple.

Any matrix algebra $\text{Mat}_n(K)$ of $n \times n$ matrices with coefficients in a division $k$–algebra $K$ is a simple algebra. The following result shows that any simple $k$–algebra is of this type.

**Theorem 1.60.**

1. Let $A$ be a simple algebra. Then $A$ is semi–simple, and there is only one isomorphism class of irreducible $A$–modules.

2. Let $A$ be a semi–simple algebra with only one isomorphism class of irreducible $A$–modules. Then $A$ is simple.

More precisely, let $S$ be an irreducible $A$–module.

(a) The natural morphism

$$S^\vee \otimes_A S \longrightarrow E_A S$$

is an isomorphism of $E_A S$–modules–$E_A S$, and the natural morphism

$$\mu_S(A) : S \otimes_{E_A S} S^\vee \longrightarrow A$$

is an isomorphism of $A$–modules–$A$.

(b) The pair of bimodules $(S, S^\vee)$ is a Morita pair for $A$ and $E_A S$. More precisely, the structural morphism

$$A \rightarrow \text{End}(S)_{E_A S}$$

is an isomorphism.

In particular we have an algebras isomorphism

$$A \sim \text{Mat}_m((E_A S)^{op})$$

where $m$ denotes the dimension of $S$ as a right vector space over $E_A S$, and $A$ is simple.

Thus the simple algebras are just the algebras isomorphic to matrix algebras over division $k$–algebra which are finite extensions of $k$. 


1.5. STRUCTURE OF SEMI–SIMPLE ALGEBRAS

**Proof.** (1) Let $A$ be simple. If $S$ is an irreducible $A$–submodule of $A$, we have $A = \text{Iso}(S, A)$ since $\text{Iso}(S, A)$ is a twosided ideal in $A$. This shows that $A$ is semi–simple and that it has only one isomorphism class of irreducible modules.

(2) By assumption we have $A = \text{Iso}(S, A)$.

(a) By 1.38, we know that $\mu_S(A) : S \otimes_{E_A S} S^\vee \longrightarrow A$ is an isomorphism.

Moreover, we know that $A$ is isomorphic to the direct sum of (a finite number of) submodules isomorphic to $S$, which shows that $S$ is a (finitely generated) projective $A$–module. By the characterization of finitely generated projective modules, we know that the natural morphism

$$S^\vee \otimes_A S \longrightarrow E_A S, \quad \varphi \otimes s \mapsto (x \mapsto (x\varphi)s)$$

is an isomorphism.

(b) is an immediate consequence of the properties of a Morita equivalence and of the fact that $E_A S$ is a division algebra. \hfill $\Box$

1.5.2. Semi–simple algebras.

Let $A$ be semi–simple.

Let us denote by $\text{Irr}(A)$ a complete set of representatives of isomorphism classes of irreducible $A$–modules.

Since the $A$–module $A$ is completely reducible, it is the direct sum of its isotypic components:

$$A = \bigoplus_{S \in \text{Irr}(A)} \text{Iso}(S, A).$$

Since the above decomposition is actually a decomposition of $A$ into a direct sum of $A$–submodules–$A$ (twosided ideals of $A$), and since the algebra of endomorphisms of $A$ (as an $A$–module–$A$) is the center $Z_A$ of $A$, this decomposition corresponds to a decomposition of 1 into a sum of mutually orthogonal idempotents in $Z_A$:

$$1 = \sum_{S \in \text{Irr}(A)} e_S,$$

where $e_S e_T = \delta_{S,T} e_S$, and $\text{Iso}(S, A) = Ae_S$.

The twosided ideal $\text{Iso}(S, A)$ inherits a structure of algebra, where the composition laws are those of $A$, and where the unit element is $e_S$. We then have an isomorphism of algebras

$$A \xrightarrow{\sim} \prod_{S \in \text{Irr}(A)} \text{Iso}(S, A), \quad a \mapsto (ae_S)_{S \in \text{Irr}(A)}.$$

For every $S \in \text{Irr}(A)$, the algebra $\text{Iso}(S, A)$ is simple, since it is semi–simple and it has only one irreducible module up to isomorphism (cf. theorem 1.60).

Thus we have proved the following theorem.
**Theorem 1.61.**

Let $A$ be a finite dimensional $k$–algebra. The following assertions are equivalent:

(i) $A$ is semisimple.

(ii) $A$ is isomorphic to a (finite) direct product of simple algebras.

More precisely, let $A$ be semi–simple.

1. The map

$$
\bigoplus_{S \in \text{Irr}(A)} S \otimes_{E_{A S}} S^\vee \to A , \quad \sum_{S \in \text{Irr}(A)} s \otimes f \mapsto \sum_{S \in \text{Irr}(A)} s f
$$

is an isomorphism of $A$–modules–$A$, which (for every $S \in \text{Irr}(A)$) sends the factor $S \otimes_{E_{A S}} S^\vee$ onto the twosided ideal $\text{Iso}(S, A)$ of $A$.

2. Each algebra $S \otimes_{E_{A S}} S^\vee$ is simple and isomorphic to $\text{End}(S)_{E_{A S}}$, and the structural morphisms

$$
\lambda_S : A \to \text{End}(S)_{E_{A S}}
$$

define an algebra isomorphism

$$
A \sim \prod_{S \in \text{Irr}(A)} \text{End}(S)_{E_{A S}} ,
$$

hence an algebra isomorphism

$$
A \sim \prod_{S \in \text{Irr}(A)} \text{Mat}_{m_S}(E_{A S})^{\text{op}}
$$

(where $m_S$ denotes the dimension of $S$ as a right vector space over $E_{A S}$).

**Proposition 1.62.**

Let $A = \bigoplus_{S \in \text{Irr}(A)} A e_S$ be a semisimple algebra. The map

$$
I \mapsto \tau_I := \bigoplus_{S \in I} A e_S
$$

is a bijection from $\text{Irr}(A)$

- onto the set of all twosided ideals of $A$,
- and onto the set of all quotient algebras of $A$.

**Proof.** Viewed as an $A \otimes A^{\text{op}}$–module, $A$ is multiplicity free (see 1.40), and its isotypic components are the $A e_S$ for $S \in \text{Irr}(A)$. Thus the result is an application of 1.40.

**Proposition 1.63.**

The following assertions are equivalent.

(i) The algebra $A$ is semi–simple.

(ii) Viewed as an $A$–module–$A$, $A$ is completely reducible.
Proof. The structure theorem of semi–simple algebras shows that if $A$ is semi–simple, it is a direct product of simple algebras, hence a direct sum of minimal twosided ideals, i.e., a direct sum of irreducible $A$–modules–$A$. Thus $A$ is a completely reducible $A$–module–$A$.

Conversely, if $A$ is a completely reducible $A$–module–$A$, then $A$ is a direct sum of minimal twosided ideals. As above in the proof of theorem 1.61, we see that each of these minimal twosided ideals is endowed with a structure of simple algebra, hence is isomorphic to a bimodule of the shape $S \otimes_{E_A} S^\vee$ for some irreducible $A$–module $S$, hence is completely reducible as an $A$–module. This shows that $A$ is completely reducible as an $A$–module, hence that $A$ is semi–simple. \qed