REPRESENTATIONS THEORY OF FINITE GROUPS 213B

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Plan of the course:
Basics of complex representations and characters of finite groups (should be reminders)
Categorical methods in representation theory
Representations and characters of symmetric groups (combinatorics of symmetric functions), general linear groups over finite fields, finite groups of Lie type (geometry needed)
Modular representation theory of finite groups.

1. Reminders: basic representation theory of finite groups

The first lectures are devoted to the basic representation theory of groups, in particular of finite groups over the field of complex numbers, and the theory of characters.

1.1. General representation theory of groups.

Let k be a field. All vector spaces will be finite-dimensional and over the field k.

1.1.1. *Definitions and first examples.* Let G be a group. A *representation* of G is a pair (V, ρ) where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a morphism of groups. We will often write "V" instead of "(V, ρ)" to denote the representation.

The dimension of V is called the *degree* or the *dimension* of the representation.

The *kernel* $\ker \rho$ of a representation is a normal subgroup of G, and the morphism ρ factors as the composition of the quotient morphism $G \to G/\ker\rho$ with a representation $\bar{\rho} : G/\ker\rho \to \mathrm{GL}(V)$.

The representation ρ is *faithful* if $\ker\rho = 1$.

Example 1.1. Let $G = \mathfrak{S}_n$ be the symmetric group on n letters. The permutation representation of G is defined as $(V = \mathbf{C}^n, \rho)$, where $\rho(w)(e_i) = e_{w(i)}$. It is a faithful representation.

When $V = \mathbf{C}^n$, we obtain a *matrix representation*: this is a morphism of groups $\rho : G \to \mathrm{GL}_n(\mathbf{C})$.

Let H be a subgroup of G and (V, ρ) a representation of G. The *restriction* $\mathrm{Res}_H^G(V, \rho)$ is the representation $(V, \rho_{|H})$ of H.

Some basic properties: $\rho(1) = 1$ and $\rho(g^{-1}) = \rho(g)^{-1}$ for all $g \in G$.

A *morphism of representations* $f : (V, \rho) \to (V', \rho')$ is a linear map $f \in \mathrm{Hom}_{\mathbf{C}}(V, V')$ such that $f \circ \rho(g) = \rho'(g) \circ f$ for all $g \in G$. We denote by $\mathrm{Hom}_{kG}(V, V')$ the space of such morphisms of representations.

Note that every representation is isomorphic to a matrix representation.

The *zero representation* corresponds to $V = 0$.

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A matrix representation of degree 1 is a morphism of groups $\rho : G \to \mathbf{C}^\times$.

Since $\mathbf{C}^\times$ is abelian, we have $[G,G] \subset \ker g$, where $[G,G]$ denotes the subgroup of G generated by the elements $ghg^{-1}h^{-1}$, for $g,h \in G$. As a consequence, ρ is the composition of the quotient morphism $G \to G/[G,G]$ with a morphism of groups $\rho' : G/[G,G] \to \mathbf{C}^\times$. This shows that G and $G/[G,G]$ have the same set of 1-dimensional matrix representations.

The *trivial representation* is the 1-dimensional representation given by $\rho : G \to \mathbf{C}^\times,\ g \mapsto 1$.

Remark 1.2. A key problem in the representation theory of finite groups is to relate certain representations of a group with those of certain subgroups or quotients. The discussion above of 1-dimensional representations is an example of that.

Example 1.3. Let $G = \mathbf{Z}/n\mathbf{Z}$ and assume $k = \mathbf{C}$. Given $r \in \mathbf{Z}$, we have a 1-dimensional matrix representation ρ_r of G given by given by $\rho_r(d) = \exp(2i\pi rd/n)$, for $d \in G$. Note that ρ_r depends only on the image of r in $\mathbf{Z}/n\mathbf{Z}$.

When k has characteristic p and n is a power of p, then all one-dimensional representations of $\mathbf{Z}/n\mathbf{Z}$ are trivial.

Let G be a group defined by generators $g_1,\ldots,g_r$ and relations $R_1,\ldots,R_s$ (these are non-commutative polynomials in the variables $X_1^{\pm 1},\ldots,X_r^{\pm 1}$, and G is the quotient of the free group on $X_1,\ldots,X_r$ by the normal closure of the subgroup generated by $R_1,\ldots,R_s$). Let V be a vector space and $h_1,\ldots,h_r \in \mathrm{GL}(V)$. Assume that $R_i(h_1,\ldots,h_r) = 1$ for all $i = 1,\ldots,s$. Then, there is a unique morphism of groups $\rho : G \to \mathrm{GL}(V)$ such that $\rho(g_i) = h_i$. This provides a way of constructing and describing representations of groups defined by generators and relations.

Example 1.4. Let $G = \langle s,t | s^2 = t^2 = 1, (st)^2 = (ts)^2\rangle$. This is the dihedral group of order 8. There is a 2-dimensional representation of G given by

$$\rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \rho(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It can indeed easily be checked that the defining relations, given by $R_1 = X_1^2$, $R_2 = X_2^2$ and $R_3 = (X_1X_2)^2(X_2X_1)^{-2}$ are satisfied by those matrices. It can also be checked easily that this representation of G is faithful if the characteristic of k is not 2.

One can also construct four non-isomorphic one-dimensional representations ρ_1 (the trivial representation), ρ_ε, ρ_s and ρ_t of G. They are given by

$$\rho_\varepsilon(s) = \rho_\varepsilon(t) = -1,\ \rho_s(s) = -1,\ \rho_s(t) = 1,\ \rho_t(s) = 1,\ \rho_t(t) = -1.$$

1.1.2. *Subrepresentations, quotients, direct sums.* Let (V,ρ) be a representation of G and let V' be a subspace of V. If $\rho(g)(V') \subset V'$ for all $g \in G$, then V' defines a *subrepresentation* of V. The structure of representation is given by restricting the automorphisms $\rho(g)$ of V to V'. We also have a representation of G on the quotient space V/V'.

Note that given $f : (V,\rho) \to (V',\rho')$ a morphism of representations, then $\ker f$ is a subrepresentation of (V,ρ), $V/\ker f$ is a quotient representation of (V,ρ) isomorphic to the subrepresentation $\mathrm{im}\, f$ of (V',ρ'): the category of representations of G is an abelian category.

A non-zero representation (V,ρ) of G is said to be *irreducible* (or *simple*) if its only subrepresentations are V and 0.

We denote by $\mathrm{Irr}_k(G)$ a complete set of representatives of isomorphism classes of irreducible representations of G.

Remark 1.5. One of the most basic questions about representations of a group is to determine all simple representations up to isomorphism. This depends very much on the characteristic of k.

Exercise 1.1. Show that the representation of dimension 2 of Example 1.4 is simple if the characteristic of k is not 2. On the other hand, if k has characteristic 2, then $k(e_1 + e_2)$ is a subrepresentation.

The *direct sum* (V'', ρ'') of two representations (V, ρ) and (V', ρ') of G has underlying vector space $V'' = V \oplus V'$ and the action is given by $\rho''(g)(v, v') = (\rho(g)(v), \rho'(g)(v'))$ for $g \in G$, $v \in V$ and $v' \in V'$. We say that V' is a *complement* to the subrepresentation V of V''.

We say that a non-zero representation (V, ρ) is *indecomposable* if it is not isomorphic to the direct sum of two representations, none of which are zero.

A representation is *completely reducible* (or *semi-simple*) if it is isomorphic to a direct sum of irreducible representations. This is equivalent to the requirement that every subrepresentation has a complement.

From now on, the representations we consider are assumed to be finite-dimensional

Given a representation (V, ρ) and two subrepresentations V' and V'', if V' is simple and $V' \not\subset V''$, then $V' \cap V'' = 0$ and $V' \oplus V''$ is a subrepresentation of V. As a consequence, the sum of the irreducible subrepresentations of (V, ρ) is the largest completely reducible subrepresentation (it is non-zero if $V \neq 0$).

Proposition 1.6. *Let (V, ρ) be an irreducible representation of G and let H be a normal subgroup of G. Then $\mathrm{Res}_H^G(V, \rho)$ is a completely reducible representation of H.*

Proof. Let V' be the largest completely reducible subrepresentation of $\mathrm{Res}_H^G(V, \rho)$. Let L be a simple subrepresentation of V'. Let $g \in G$. Then $\rho(g)(L)$ is a subrepresentation of $\mathrm{Res}_H^G(V, \rho)$, since H is normal in G, and $\rho(g)(L)$ is a simple representation of H. It follows that $\rho(g)(L) \subset V'$, hence V' is G-stable. As (V, ρ) is simple, it follows that $V = V'$. □

We will see later that if G is a finite group, then all its representations are completely reducible if and only if the characteristic of k doesn't divide the order of G (Maschke's Theorem). In particular, if k has characteristic 0, then all representations are completely reducible.

Exercise 1.2. Show that $k(e_1 + \cdots + e_n)$ is a subrepresentation V' of the representation (V, ρ) of $\mathfrak{S}_n$ of Example 1.1. Show that the representation (V, ρ) is indecomposable if and only if the characteristic of k divides n, if and only if $k(e_1 + \cdots + e_n)$ admits no complement.

1.1.3. *Permutation representations.* Let Ω be a finite set. An action of G on Ω is a morphism of groups $\phi : G \to \mathfrak{S}(\Omega)$, where $\mathfrak{S}(\Omega)$ is the group of permutations of Ω (isomorphic to $\mathfrak{S}_n$ if $n = |\Omega|$). The associated *permutation representation* is given by $V = \bigoplus_{\omega \in \Omega} \mathbf{C}\omega$ and $\rho(g)(\omega) = \phi(g)(\omega)$.

Example 1.7. The symmetric group $\mathfrak{S}_n$ acts on $\Omega = \{1, \ldots, n\}$ and the representation of Example 1.1 is the associated permutation representation.

Note that a non-trivial permutation representation is never simple, since $k(\sum_{\omega \in \Omega} \omega)$ is a subrepresentation.

Given H a subroup of finite index of G (*i.e.*, $|G/H| < \infty$), then G acts by left multiplication on G/H, and we obtain a corresponding permutation representation. Note that the corresponding representation is faithful if and only if G acts faithfully on G/H, which is equivalent to the property that H contains no non-trivial normal subgroup of G.

When G is finite, the *regular representation* of G is the permutation representation associated with the action of G on itself by left multiplication. It is a faithful representation.

1.1.4. *Duals and tensor products.* Given (V, ρ) a representation of G, we have a *dual representation* (V^*, ρ^*), where $\rho^*(g) = {}^t(\rho(g^{-1}))$.

The tensor product (V'', ρ'') of two representations (V, ρ) and (V', ρ') of G is given by $V'' = V \otimes_k V'$ and $\rho''(g) = \rho(g) \otimes \rho'(g)$.

Note that the tensor product and the duals equip the set of isomorphism classes of 1-dimensional representations of G with a group structure.

1.1.5. *Characters.* Let (V, ρ) be a representation of G. The *character* χ_V of V is the map

$$\chi_V : G \to k, \ g \mapsto \mathrm{Tr}_V(\rho(g)).$$

A *class function* on G with values in k is a map $f : G \to k$ such that $f(gg'g^{-1}) = f(g')$ for all $g, g' \in G$. In other terms, f is constant on conjugacy classes. We denote by $\mathrm{CF}(G, k)$ the space of class functions $G \to k$. Given C a conjugacy class of G, we denote by δ_C the class function given by $\delta_C(g) = 1$ if $g \in C$ and $\delta_C(g) = 0$ if $g \not\in C$. Then $(\delta_C)_C$, where C runs over conjugacy classes of G, is a basis of $\mathrm{CF}(G, k)$, when G is a finite group.

Note that the character of a representation is a class function. We have the following properties:

$$\chi_{V \oplus V'} = \chi_V + \chi_{V'}, \ \chi_{V \otimes V'} = \chi_V \cdot \chi_{V'}, \ \chi_{V^*}(g) = \chi_V(g^{-1}), \ \chi_{\mathrm{Res}^G_H V} = (\chi_V)_{|H}, \chi_{k\Omega}(g) = |\Omega^g|.$$

1.2. **Complex representations of finite groups.** In this section, we consider only finite groups and all vector spaces will be finite-dimensional and over the field $k = \mathbf{C}$.

1.2.1. *Characters.* We introduce a hermitian structure on $\mathrm{CF}(G, \mathbf{C})$:

$$\langle f, f' \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{f'(g)}.$$

We recall without proof some fundamental results (we will comment on the proofs later when we consider modular representations and the failure of these results).

Proposition 1.8. *All representations of G are completely reducible.*

Given V and V' two representations of G, we have $\dim \mathrm{Hom}_{\mathbf{C}G}(V, V') = \langle \chi_V, \chi_{V'} \rangle$.

We have $V \simeq V'$ if and only if $\chi_V = \chi_{V'}$.

An *irreducible character* of G is a character of the form χ_V, with $V \in \mathrm{Irr}_{\mathbf{C}}(G)$.

Let (V, ρ) be a representation of G. The endomorphisms $\rho(g)$ are diagonalizable, since $\rho(g)^n = 1$ for some n. Their eigenvalues are roots of unity, so $\chi_V(g)$ is a sum of roots of unity and $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

Proposition 1.9. *The set $\{\chi_V\}_{V \in \mathrm{Irr}_{\mathbf{C}}(G)}$ is an orthonormal basis of* $\mathrm{CF}(G, \mathbf{C})$. *The character of the regular representation of G is $\sum_{V \in \mathrm{Irr}_{\mathbf{C}}(G)} \chi_V(1) \chi_V$. In particular, $|G| = \sum_{V \in \mathrm{Irr}_{\mathbf{C}}(G)} \chi_V(1)^2$.*

We deduce that $|\operatorname{Irr}_{\mathbf{C}}(G)|$ is the number of conjugacy classes of G. Note nevertheless that there is no natural bijection in general between conjugacy classes and irreducible characters of G.

Example 1.10. Let G be the dihedral group of order 8. We have constructed in Example 1.4 four non-isomorphic representations of degree 1 and an irreducible representation of degree 2. Since $4 \cdot 1^2 + 2^2 = 8$, we deduce that these are all irreducible representations of G, up to isomorphism.

1.2.2. *Orthogonality relations.*

Proposition 1.11 (Schur's Lemma)**.** *Let V and V' be two irreducible representations of G. We have $\operatorname{Hom}_G(V, V') = 0$ if V is not isomorphic to V' and $\operatorname{End}_G(V) = \mathbf{C}\,\mathrm{id}_V$.*

Proof. Let $f : V \to V'$ be a non-zero morphism of representations. Since $\operatorname{im} f$ is a subrepresentation of V', it follow that $\operatorname{im} f = V'$. Similarly, $\ker f$ is a subrepresentation of V, hence $\ker f = 0$. So, f is an isomorphism. So, $\operatorname{End}_G(V)$ is a finite-dimensional division algebra over $\mathbf{C}$, hence $\operatorname{End}_G(V) = \mathbf{C}$. □

Proposition 1.12. *Let (V, ρ) and (V', ρ'') be two irreducible representations of G with characters χ and χ'. We have $\langle \chi, \chi' \rangle = 0$ if $V \not\simeq V'$ and $\langle \chi, \chi \rangle = 1$.*

Proof. Let $\phi \in \operatorname{Hom}_{\mathbf{C}}(V', V)$ and let $f = \frac{1}{|G|} \sum_g \rho(g^{-1}) f \rho'(g)$. We have $f \in \operatorname{Hom}_G(V', V)$. It follows from Schur's Lemma that $f = 0$ if $V \not\simeq V'$. Assume $V' = V$ and $\rho' = \rho$. By Schur's Lemma, we have $f = \lambda \cdot \mathrm{id}$ for some $\lambda \in \mathbf{C}$ and $\operatorname{Tr}(f) = \operatorname{Tr}(\phi) = \lambda\chi(1)$.

Fix bases of V and V' and let $\rho_{ik}(g)$ and $\rho'_{jl}(g)$ denote the matrix coefficients.

Let $\phi = e_{ji}$. We have $f_{ji} = \frac{1}{|G|} \sum_g \rho_{ii}(g^{-1}) \rho'_{jj}(g) = 0$ if $V \not\simeq V'$. So, $\langle \chi, \chi' \rangle = \sum_{g,i,j} \rho_{ii}(g^{-1}) \rho'_{jj}(g) = 0$ if $V \not\simeq V'$.

Assume $V = V'$. We have $\frac{1}{|G|} \sum_g \rho_{ii}(g^{-1}) \rho_{jj}(g) = \delta_{ij} \chi(1)^{-1}$, hence $\langle \chi, \chi \rangle = 1$. □