Homological algebra

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Contents

CHAPTER 1

Homological algebra

1. Categories

1.1. Definitions.

1.1.1. *Categories.*

Definition 1.1. *A category $\mathcal{C}$ is the data of*

- *a set* $\mathrm{Ob}(\mathcal{C})$
- *a small set* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ *for every* $X,Y \in \mathrm{Ob}(\mathcal{C})$
- *a map* $\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}(X,Z),\ (f,g) \mapsto g \circ f$ *for every* $X,Y,Z \in \mathrm{Ob}(\mathcal{C})$

such that

- $\circ$ *is associative*
- *for every* $X \in \mathrm{Ob}(\mathcal{C})$, *there is an element* $\mathbf{1}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ *such that given* $X,Y \in \mathrm{Ob}(\mathcal{C})$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, *then* $\mathbf{1}_Y \circ f \circ \mathbf{1}_X = f$.

The elements of $\mathrm{Ob}(\mathcal{C})$ are the *objects* of $\mathcal{C}$ and the elements of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are called *morphisms* or *arrows* between X and Y. We denote by $\mathrm{Ar}(\mathcal{C})$ the set of arrows of $\mathcal{C}$. We sometimes write "$X \in \mathcal{C}$" to mean "$X \in \mathrm{Ob}(\mathcal{C})$".

Note that $\mathbf{1}_X$ is uniquely determined by the conditions of the definition.

A *small category* is a category $\mathcal{C}$ such that $\mathcal{C}$ is a small set.

Remark 1.2. The data of a small category with one object is the same as the data of a monoid with a unit. In general, one should think of a category as a monoid with "several objects": let $\mathcal{C}$ be a category. Recall that a magma is a set together with a partially defined associative multiplication law. Then, composition defines a structure of magma on $M = \coprod_{X,Y\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X,Y)$: given $X \xrightarrow{f} Y$ and $Z \xrightarrow{g} T$, we say that $g \cdot f$ is defined when $Y = Z$ and we put then $g \cdot f = g \circ f$.

An object $X \in \mathcal{C}$ is *initial* if $\mathrm{Hom}_{\mathcal{C}}(X,M)$ has cardinality 1 for every $M \in \mathcal{C}$. An object $X \in \mathcal{C}$ is *final* if $\mathrm{Hom}_{\mathcal{C}}(M,X)$ has cardinality 1 for every $M \in \mathcal{C}$. An initial (resp. terminal) object is unique up to a unique isomorphism, it it exists.

Note that X is final in $\mathcal{C}$ if and only if it is terminal in $\mathcal{C}^{\mathrm{opp}}$.

An object $X \in \mathcal{C}$ is a *zero object* if it is initial and final.

A map $X \xrightarrow{f} Y$ is a

- *epimorphism* (write $X \overset{f}{\twoheadrightarrow} Y$) if given $g,h : Y \to Z$ such that $gf = hf$, then $g = h$
- *monomorphism* (write $X \overset{f}{\hookrightarrow} Y$) if given $g,h : W \to X$ such that $fg = fh$, then $g = h$

- *isomorphism* (write $X \xrightarrow[\sim]{f} Y$) if there exists $g : Y \to X$ such that $f \circ g = \mathbf{1}_Y$ and $g \circ f = \mathbf{1}_X$.

Example 1.3. The first and foremost example of category is the category Sets. Its objects are the small sets and the morphisms are the maps between sets. The monomorphisms are the injections, the epimorphisms the surjections, and the isomorphisms the bijections. The empty set $\emptyset$ is an initial object, while a set with one element is a final object.

We denote by $\mathcal{C}^{\mathrm{opp}}$ the *opposite category* to $\mathcal{C}$. We have $\mathrm{Ob}(\mathcal{C}^{\mathrm{opp}}) = \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and given $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(Y, Z)$, then the composition $g \circ^{\mathrm{opp}} f$ of f and g in $\mathcal{C}^{\mathrm{opp}}$ is $f \circ g$ (taken in $\mathcal{C}$).

A *subcategory* $\mathcal{C}'$ of $\mathcal{C}$ is a category $\mathcal{C}'$ with $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$, with $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for $X, Y \in \mathrm{Ob}(\mathcal{C}')$, with $\mathbf{1}_{X \in \mathcal{C}'} = \mathbf{1}_{X \in \mathcal{C}}$ for all $X \in \mathrm{Ob}(\mathcal{C}')$, and where the composition in $\mathcal{C}'$ coincides with the one in $\mathcal{C}$.

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is a *full subcategory* if $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}(\mathcal{C}')$. So, a full subcategory of $\mathcal{C}$ is determined by its set of objects and conversely, given any $I \subset \mathrm{Ob}(\mathcal{C})$, there is a unique full subcategory $\mathcal{C}'$ of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}') = I$. A *strictly full subcategory* $\mathcal{C}'$ of $\mathcal{C}$ is a full subcategory such that every object of $\mathcal{C}$ isomorphic to an object of $\mathcal{C}'$ is an object of $\mathcal{C}'$.

Remark 1.4. When describing graphically a category, we will usually omit the identity arrows.

1.1.2. *Functors.* Let $\mathcal{C}'$ be a category. A *functor* $F : \mathcal{C} \to \mathcal{C}'$ is the data of

- a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$
- a map $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$

such that $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$ and $F_{X,Z}(g \circ f) = F_{Y,Z}(g) \circ F_{X,Y}(f)$ for all $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$.

We will often write "F" for "$F_{X,Y}$" and "FX" for "$F(X)$".

Let $G : \mathcal{C} \to \mathcal{C}'$ be a functor. A *morphism of functors* $\alpha : F \to G$ is the data of morphisms $\alpha(X) \in \mathrm{Hom}_{\mathcal{C}'}(FX, GX)$ for $X \in \mathcal{C}$ making the following diagram commutative, for all $X \xrightarrow{f} Y$ in $\mathcal{C}$:

$$\begin{array}{ccc} FX & \xrightarrow{F(f)} & FY \\ {\scriptstyle \alpha(X)}\downarrow & & \downarrow{\scriptstyle \alpha(Y)} \\ GX & \xrightarrow[G(f)]{} & GY \end{array}$$

We denote by $\mathrm{Id}_{\mathcal{C}}$ the *identity functor* of $\mathcal{C}$ and by $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ the *category of functors* from $\mathcal{C}$ to $\mathcal{C}'$.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor.

The *image* of F is the full subcategory of $\mathcal{C}'$ with set of objects $\{F(X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$. The *essential image* of F is the full subcategory $\mathrm{im}\, F$ of $\mathcal{C}'$ with objects those objects of $\mathcal{C}$ isomorphic to objects of the image of F.

We say that F is

- *faithful* if $\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}'}(FX, FY)$ is injective for all $X, Y \in \mathcal{C}$

- *full* if $F(X,Y) : \mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}'}(FX,FY)$ is surjective for all $X,Y \in \mathcal{C}$
- *fully faithful* if it is full and faithful
- *essentially surjective* if the essential image of F is $\mathcal{C}'$
- an *equivalence of categories* if there is a functor $G : \mathcal{C}' \to \mathcal{C}$ such that $F \circ G \simeq \mathrm{Id}_{\mathcal{C}'}$ and $G \circ F \simeq \mathrm{Id}_{\mathcal{C}}$.
- an *isomorphism of categories* if there is a functor $G : \mathcal{C}' \to \mathcal{C}$ such that $F \circ G = \mathrm{Id}_{\mathcal{C}'}$ and $G \circ F = \mathrm{Id}_{\mathcal{C}}$.

We leave the following proposition as an exercice.

PROPOSITION 1.5. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. Then, F is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

A functor $F : \mathcal{C} \to \mathcal{C}'$ gives rise to a functor $F^{\mathrm{opp}} : \mathcal{C}^{\mathrm{opp}} \to \mathcal{C}'^{\mathrm{opp}}$ and this gives a canonical isomorphism of categories $\mathrm{Fun}(\mathcal{C},\mathcal{C}')^{\mathrm{opp}} \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}},\mathcal{C}'^{\mathrm{opp}})$.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $\mathcal{C}^{\mathrm{opp}} \to \mathcal{C}'$.

REMARK 1.6. In defining a morphism of functors, it is common to specify only the effect on objects when the effect on morphisms is "clear" in the context.

1.1.3. *Adjoint functors.* Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}$ be two functors.

DEFINITION 1.7. *An* adjunction (F, G, α) *is the data of an isomomorphism of functors from* $\mathcal{C}^{\mathrm{opp}} \times \mathcal{C}'$ *to* Sets

$$\alpha : \mathrm{Hom}_{\mathcal{C}'}(F(?), ?') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(?, G(?')).$$

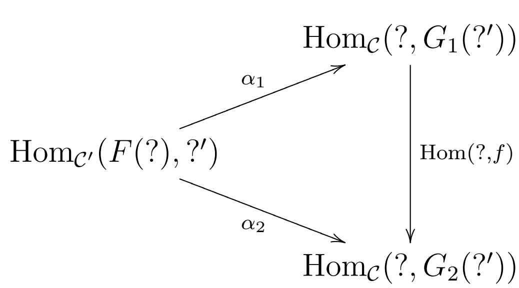
If (F, G) is an *adjoint pair*, we say that F is *left adjoint* to G and that G is *right adjoint* to F. We write $G = F^{\vee}$ and $F = {}^{\vee}G$.

LEMMA 1.8. *If (F, G, α) is an adjunction, then $(G^{\mathrm{opp}}, F^{\mathrm{opp}}, \alpha')$ is an adjunction, where α' is the composition*

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(G^{\mathrm{opp}}(?'), ?) = \mathrm{Hom}_{\mathcal{C}}(?, G(?')) \xrightarrow{\alpha^{-1}} \mathrm{Hom}_{\mathcal{C}'}(F(?), ?') = \mathrm{Hom}_{\mathcal{C}'^{\mathrm{opp}}}(?', F^{\mathrm{opp}}(?)).$$

The following proposition shows that adjoints are unique up to a unique isomorphism.

PROPOSITION 1.9. *If (F, G_1, α_1) and (F, G_2, α_2) are adjoint pairs, then there is a unique isomorphism of functors $f : G_1 \xrightarrow{\sim} G_2$ such that the following diagram commutes*



1.2. Representability.

1.2.1. *Yoneda's Lemma.* Let $\mathcal{C}^\wedge = \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets})$.

There is a canonical functor

$$\mathrm{can} : \mathcal{C} \to \mathcal{C}^\wedge, \ M \mapsto \mathrm{Hom}_{\mathcal{C}}(-, M).$$

Let $M \in \mathcal{C}$ and $F \in \mathcal{C}^\wedge$. We have canonical maps

$$\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{Hom}_{\mathcal{C}}(-, M), F) \to F(M), \quad f \mapsto f(M)(\mathrm{id}_M)$$

$$\text{and } F(M) \to \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{Hom}_{\mathcal{C}}(-, M), F), \quad x \mapsto (N \mapsto (\phi \mapsto F(\phi)(x)))\,.$$

The following theorem, whose proof is trivial, is a fundamental result in the theory of categories.

Theorem 1.10 (Yoneda's Lemma). *The maps above define inverse bijections, functorial in $M \in \mathcal{C}$ and $F \in \mathcal{C}^\wedge$*

$$\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{Hom}_{\mathcal{C}}(-, M), F) \underset{\sim}{\overset{\sim}{\rightleftarrows}} F(M)$$

In particular, the canonical functor $\mathcal{C} \to \mathcal{C}^\wedge$ is fully faithful.

We say that the object M of $\mathcal{C}$ *represents* the functor $F : \mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets}$ if $\mathrm{Hom}_{\mathcal{C}}(-, M) \xrightarrow{\sim} F$. Such an object M, endowed with the isomorphism $\mathrm{Hom}_{\mathcal{C}}(-, M) \xrightarrow{\sim} F$, is unique up to a unique isomorphism.

1.2.2. Consider now $\mathcal{C}^\vee = \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}}, \mathrm{Sets}^{\mathrm{opp}})$. There is a canonical isomorphism $\mathcal{C}^\vee \xrightarrow{\sim} \left((\mathcal{C}^{\mathrm{opp}})^\wedge\right)^{\mathrm{opp}}$. We deduce from §1.2.1 that the canonical functor

$$\mathcal{C} \to \mathcal{C}^\vee, \ M \mapsto \mathrm{Hom}_{\mathcal{C}}(M, -)$$

is fully faithful and that $\mathrm{Hom}_{\mathcal{C}^\vee}(G, \mathrm{Hom}_{\mathcal{C}}(M, -)) \xrightarrow{\sim} G(M)$.

Let $G : \mathcal{C} \to \mathrm{Sets}$ be a functor. We say that M *corepresents* G if $\mathrm{Hom}_{\mathcal{C}}(M, -) \xrightarrow{\sim} G$, where G is viewed as a functor $\mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets}^{\mathrm{opp}}$. Such an object M (endowed with the isomorphism $\mathrm{Hom}_{\mathcal{C}}(M, -) \xrightarrow{\sim} G$) is unique up to a unique isomorphism.

1.3. Limits. The limit (resp. the colimit) of a functor with value in $\mathcal{C}$ is, when it exists, an object of $\mathcal{C}$ mapping to a well-defined object of $C^\wedge$ (resp. $\mathcal{C}^\vee$), which is constructed using limits in the category of sets. This object will then be unique up to unique isomorphism.

We fix is a small category I.

1.3.1. *Limits in* Sets. Let $F : I^{\mathrm{opp}} \to \mathrm{Sets}$ be a functor. We define the limit of F, denoted by $\lim F$, as the subset of $\prod_{i \in I} F(i)$ given by those families $(x_i)_{i \in I}$ such that for every $j \in I$ and $\phi : i \to j$, then $F(\phi)(x_j) = x_i$.

1.3.2. *Limits and colimits.* Let $F : I^{\mathrm{opp}} \to \mathcal{C}$ be a functor. A *limit* of F is, if it exists, an object of $\mathcal{C}$ denoted by $\lim F$ that represents the functor

$$\mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets}, \ M \mapsto \lim \mathrm{Hom}_{\mathcal{C}}(M, F)$$

where $\mathrm{Hom}_{\mathcal{C}}(M, F) : I^{\mathrm{opp}} \to \mathrm{Sets}$ is the functor $i \mapsto \mathrm{Hom}_{\mathcal{C}}(M, F(i))$. So, there is an isomorphism $\mathrm{Hom}_{\mathcal{C}}(-, \lim F) \xrightarrow{\sim} \lim \mathrm{Hom}_{\mathcal{C}}(-, F)$.

Similarly, given $G : I \to \mathcal{C}$ a functor, a *colimit* of G is an object of $\mathcal{C}$, if it exists, denoted by $\mathrm{colim}\, G$ that corepresents the functor

$$\mathcal{C} \to \mathrm{Sets}, \ M \mapsto \lim \mathrm{Hom}_{\mathcal{C}}(G, M)$$

where $\mathrm{Hom}_{\mathcal{C}}(G, M) : I^{\mathrm{opp}} \to \mathrm{Sets}$ is the functor $i \mapsto \mathrm{Hom}_{\mathcal{C}}(G(i), M)$. So, there is an isomorphism $\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim} G, -) \xrightarrow{\sim} \lim \mathrm{Hom}_{\mathcal{C}}(G, -)$.

We say that the limits and colimits above are *indexed* by the category I. We denote them also by $\lim_{i\in I} F(i)$ and $\operatorname{colim}_{i\in I} G(i)$.

Remark 1.11. A limit (resp. a colimit) is sometimes also called a projective limit (resp. an inductive or a direct limit).

Remark 1.12. Let $F : I^{\mathrm{opp}} \to \mathcal{C}$ be a functor. The colimit of F (indexed by the category I^{opp}) should not be confused with the limit of F (indexed by the category I) — cf Example 1.13. The difference comes from the fact that Sets is not equivalent to $\mathrm{Sets}^{\mathrm{opp}}$: an equivalence between Sets and $\mathrm{Sets}^{\mathrm{opp}}$ sends $\emptyset$, an initial object of Sets, to a set with one element, a final object of Sets.

But $\mathrm{Hom}_{\mathrm{Sets}}(M, \emptyset) = \emptyset$ for all $M \neq \emptyset$ while $\mathrm{Hom}_{\mathrm{Sets}}(\{x\}, M) \neq \emptyset$ for $M \neq \emptyset$, hence there is no such equivalence.

Example 1.13. Take for I a discrete category (*i.e.*, the only arrows are the identities). Then, lim is called a *product*, denoted by $\prod_I$, and colim a *coproduct*, denote by $\coprod_I$. When F is in addition constant with value M, we denote by M^I the limit of F and by $M^{(I)}$ its colimit.

Example 1.14. Consider $I = \begin{array}{ccc} 3 & \longrightarrow & 1 \\ \downarrow & & \\ 2 & & \end{array}$. Given $G : I \to \mathcal{C}$ a functor, we call $\operatorname{colim} G$ the *fibered coproduct* of $G(1)$ by $G(2)$ above $G(3)$ and we denote it by $G(1) \sqcup_{G(3)} G(1)$.

Given a functor $F : I^{\mathrm{opp}} \to \mathcal{C}$, we call $\lim F$ the *fibered product* of $F(1)$ by $F(2)$ above $F(3)$ and we denote it by $F(1) \times_{F(3)} F(2)$.

Example 1.15. Consider the category $I = \bullet \rightrightarrows \bullet$. The data of a functor $F : I^{\mathrm{opp}} \to \mathcal{C}$ or of a functor $G : I \to \mathcal{C}$ corresponds to the data of two arrows $M \overset{f}{\underset{g}{\rightrightarrows}} N$. A limit of F (resp. a colimit of G) is called a *kernel*, or an equalizer, (resp. a *cokernel*, or a coequalizer) of $M \overset{f}{\underset{g}{\rightrightarrows}} N$.

When $\mathcal{C}$ is an additive category and $g = 0$, this is called a *kernel* (resp. a *cokernel*) of f.

1.3.3. *Description as (co)kernels.* We will now explain how to compute limits (resp. colimits) from kernels (resp. cokernels) and products (resp. coproducts). We leave the proofs of the following two propositions to the reader.

Let $F : I^{\mathrm{opp}} \to \mathcal{C}$ be a functor. Given $f : i \to j$ in I, the morphisms $\mathrm{id}_{F(i)} : F(i) \to F(i)$ and $F(f) : F(j) \to F(i)$ induce two morphisms $F(i) \times F(j) \overset{\mathrm{id}_{F(i)}}{\underset{F(f)}{\rightrightarrows}} F(i)$. We deduce a morphism

$$\textstyle\prod_{i\in \mathrm{Ob}(I)} F(i) \rightrightarrows \prod_{f\in \mathrm{Ar}(I)} F(\mathrm{source}(f)) .$$

Proposition 1.16. *The limit of F is isomorphic to the kernel of*

$$\textstyle\prod_{i\in \mathrm{Ob}(I)} F(i) \rightrightarrows \prod_{f\in \mathrm{Ar}(I)} F(\mathrm{source}(f)) .$$

Let $G : I \to \mathcal{C}$ be a functor. Given $f : i \to j$ in I, the morphisms $\mathrm{id}_{G(i)} : G(i) \to G(i)$ and $G(f) : G(i) \to G(j)$ induce two morphisms $G(i) \underset{G(f)}{\overset{\mathrm{id}_{G(i)}}{\rightrightarrows}} G(i) \coprod G(j)$.

We deduce a morphism

$$\coprod_{f \in \mathrm{Ar}(I)} G(\mathrm{source}(f)) \rightrightarrows \coprod_{i \in \mathrm{Ob}(I)} G(i) .$$

Proposition 1.17. *The colimit of G is isomorphic to the cokernel of*

$$\coprod_{f \in \mathrm{Ar}(I)} G(\mathrm{source}(f)) \rightrightarrows \coprod_{i \in \mathrm{Ob}(I)} G(i) .$$

1.3.4. *Colimits in* Sets. We have explained in §1.3.1 how to compute the limit of a functor $F : I^{\mathrm{opp}} \to \mathrm{Sets}$.

Let $G : I \to \mathrm{Sets}$ be a functor. We define an equivalence relation $\sim$ on $\coprod_{i \in I} G(i)$ as the relation generated by $x \sim G(f)(x)$ for $i \in I$, $x \in G(i)$ and $f : i \to j$.

Let M be a set. Consider the composition of canonical morphisms

$$\mathrm{Hom}((\coprod_i G(i))/\sim, M) \hookrightarrow \mathrm{Hom}(\coprod_i G(i), M) \xrightarrow{\sim} \prod_i \mathrm{Hom}(G(i), M)).$$

Its image is contained in $\lim_i \mathrm{Hom}(G(i), M)$ and we obtain a bijection

$$\mathrm{Hom}((\coprod_i G(i))/\sim, M) \xrightarrow{\sim} \lim_i \mathrm{Hom}(G(i), M).$$

We deduce the following Proposition:

Proposition 1.18. *Let $G : I \to \mathrm{Sets}$ be a functor. Then*

$$\mathrm{colim}\, G \xrightarrow{\sim} \left(\coprod_{i \in I} G(i) \right) / \sim$$

where $\sim$ is the equivalence relation generated by $x \sim G(f)(x)$ for $i \in I$, $x \in G(i)$ and $f : i \to j$.

1.3.5. *Filtrant colimits.* We will see that under certain assumptions, the equivalence relation above has a more direct description.

We say that the category I is *filtrant* if

- for any i, j objects of I, there exists $i \to k$ and $j \to k$ two arrows in I : $\begin{matrix} i \searrow \\ \quad k \\ j \nearrow \end{matrix}$
- for any arrows $f, g : i \to j$ in I, there exists $h : j \to k$ such that $hf = hg : i \rightrightarrows j \longrightarrow k$.

Assume I is filtrant. Then the equivalence relation above has the following description. Let $x_i \in G(i)$ and $x_j \in G(j)$. Then $x_i \sim x_j$ if and only if there exists $f : i \to k$ and $g : j \to k$ such that $G(f)(x_i) = G(g)(x_j)$.

Example 1.19. Let $\mathcal{C} = \mathrm{Sets}$. The kernel of $M \underset{g}{\overset{f}{\rightrightarrows}} N$ is $\{m \in M | f(m) = g(m)\}$. Its cokernel is the quotient of N by the equivalence relation generated by $f(m) \sim g(m)$ for $m \in M$.

The product in Sets is the product of sets. The coproduct in Sets is the disjoint union.

2. Additive categories

2.1. Additive and k-linear categories.

2.1.1. Let k be a commutative ring. A *k-linear category* is a category $\mathcal{C}$ such that

- given $X, Y \in \mathrm{Ob}(\mathcal{C})$, the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is equipped with a structure of k-module and the compositions $\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$ are k-bilinear
- $\mathcal{C}$ has a zero object 0
- Given $X, Y \in \mathrm{Ob}(\mathcal{C})$, there is an object Z, the *direct sum* of X and Y, together with maps $p : Z \to X$, $i : X \to Z$, $q : Z \to Y$ and $j : Y \to Z$ such that $pi = \mathbf{1}_X$, $qj = \mathbf{1}_Y$ and $ip + jq = \mathbf{1}_Z$.

A $\mathbf{Z}$-linear category is called an *additive category.* Via the canonical morphism of rings $\mathbf{Z} \to k$, a k-linear category can be viewed as an additive category.

Being an additive category is a property of a category, as shown by the following proposition.

Proposition 2.1. *Let $\mathcal{C}$ be a category. Assume $\mathcal{C}$ has a zero object, has finite coproducts and finite products, and the canonical map $X \coprod Y \to X \prod Y$ is an isomorphism for all $X, Y \in \mathcal{C}$.*

Then, given $X, Y \in \mathcal{C}$, there is a unique abelian group structure on $\mathrm{Hom}(X,Y)$ *that make $\mathcal{C}$ into an additive category.*

Example 2.2. Let A be a k-algebra. The category of free A-modules of finite rank is a k-linear category. It is equivalent to the category Mat_k with objects the non-negative integers and with $\mathrm{Hom}_{\mathrm{Mat}_k}(m,n)$ the set of $m \times n$ matrices over A.

Let $\mathcal{C}$ be an additive category.

The coproducts in $\mathcal{C}$ are called *direct sums* and denoted by $\bigoplus$. Note that finite direct sums exist in $\mathcal{C}$ are are canonically isomorphic to the corresponding finite products.

An object of $\mathcal{C}$ is *zero* if it is isomorphic to 0. The category $\mathcal{C}$ is said to be a zero category if all its objects are zero.

We say that an object $X \in \mathcal{C}$ is *indecomposable* if $X \simeq X_1 \oplus X_2$ implies that X_1 or X_2 is zero.

2.1.2. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor of k-linear categories. The *kernel* of F, denoted by $\ker F$, is the full subcategory of $\mathcal{C}$ with objects those M such that $F(M) \simeq 0$. This is a strictly full k-linear subcategory of $\mathcal{C}$.

2.1.3. Let $\mathcal{C}$ be a k-linear category. Let $\mathcal{I}$ be a full subcategory of $\mathcal{C}$ closed under finite direct sums. We define the equivalence relation $\sim_{\mathcal{I}}$ by $f \sim_{\mathcal{I}} f'$ if $f - f'$ factors through an object of $\mathcal{I}$. The category $\mathcal{C}/\sim_{\mathcal{I}}$ is called the *k-linear quotient category* of $\mathcal{C}$ by $\mathcal{I}$ and denoted by $\mathcal{C}/^a\mathcal{I}$ (the exponent "a" refers to the additive structure). We have

$$\mathrm{Hom}_{\mathcal{C}/^a\mathcal{I}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)/\mathrm{Hom}_{\mathcal{C}}^{\mathcal{I}}(X,Y)$$

where $\mathrm{Hom}_{\mathcal{C}}^{\mathcal{I}}(X,Y)$ is the k-submodule of arrows from X to Y that factor through an object of $\mathcal{I}$.

The category $\mathcal{C}/^a\mathcal{I}$, together with the quotient functor $\mathrm{can} : \mathcal{C} \to \mathcal{C}/^a\mathcal{I}$, is the solution of the following universal problem:

Given $M \in \mathcal{I}$, then $\mathrm{can}(M) \simeq 0$ and given any k-linear functor $F : \mathcal{C} \to \mathcal{C}'$ such that $\mathcal{I} \subseteq \ker F$, there a functor $G : \mathcal{C}/^a\mathcal{I} \to \mathcal{C}'$ with $F \simeq G \circ \mathrm{can}$ and G is unique up to isomorphism.

3. Homotopy category of an additive category

3.1. Graded objects. Let $\mathcal{C}$ be a k-linear category. We denote by $\mathcal{C}$-gr the category of *graded objects* of $\mathcal{C}$: its objects are families $X = \{X^n\}_{n\in\mathbf{Z}}$ with X^n an object of $\mathcal{C}$ and $\operatorname{Hom}_{\mathcal{C}\text{-gr}}(X, X') = \prod_{n\in\mathbf{Z}} \operatorname{Hom}_{\mathcal{C}}(X^n, Y^n)$.

Given $m \in \mathbf{Z}$ and X a graded object of $\mathcal{C}$, we define a new graded object $X[m]$ by $(X[m])^i = X^{m+i}$. Given $f \in \operatorname{Hom}_{\mathcal{C}\text{-gr}}(X, X')$, we define $f[m]$
in $\operatorname{Hom}_{\mathcal{C}\text{-gr}}(X[m], X'[m])$ by $(f[m])^i = f^{m+i}$. This defines an automorphism $[m] : \mathcal{C}\text{-gr} \xrightarrow{\sim} \mathcal{C}\text{-gr}$.

Given Y, Y two graded objects of $\mathcal{C}$, we define a graded k-module $\operatorname{Hom}^\bullet_{\mathcal{C}}(X, Y)$ by

$$(\operatorname{Hom}^\bullet_{\mathcal{C}}(X, Y))^i = \operatorname{Hom}^i_{\mathcal{C}}(X, Y) = \prod_{n\in\mathbf{Z}} \operatorname{Hom}_{\mathcal{C}}(X^n, Y^{n+i}).$$

3.2. Complexes. We denote by $\operatorname{Comp}(\mathcal{C})$ the category of *complexes* of objects of $\mathcal{C}$. Its objects are families (X, d_X) where X is an object of $\mathcal{C}$-gr and $d_X : X \to X\langle 1\rangle$ satisfies $d_X^2 = 0$. The k-module $\operatorname{Hom}_{\operatorname{Comp}(\mathcal{C})}((X, d_X), (X', d_{X'}))$ is the sub-k-module of $\operatorname{Hom}_{\mathcal{C}\text{-gr}}(X, X')$ of those maps f such that $f d_X = d_{X'} f$.

Given $X \in \operatorname{Comp}(\mathcal{C})$ and $m \in \mathbf{Z}$, we can equip $X[m]$ with a structure of complex with differential $d^i_{X[m]} = (-1)^m d_X^{m+i}$. This defines an automorphism $[m]$ of $\operatorname{Comp}(\mathcal{C})$.

We denote by $\omega : \operatorname{Comp}(\mathcal{C}) \to \mathcal{C}\text{-gr}$ the forgetful functor.

We have a canonical fully faithful functor $\mathcal{C} \to \operatorname{Comp}(\mathcal{C})$ sending M to the complex C defined by $C^i = 0$ for $i \neq 0$ and $C^0 = M$.

Let $\mathcal{A}$ be an abelian category and let $C \in \operatorname{Comp}(\mathcal{A})$. Given $i \in \mathbf{Z}$, we define $H^i(C) = \ker d^i_C / \operatorname{im} d^{i-1}_C$. This provides an additive functor $H^i : \operatorname{Comp}(\mathcal{A}) \to \mathcal{A}$.

3.3. Homotopy. Let X and Y be two complexes of $\mathcal{C}$. We say that $f : X \to Y$ is *null-homotopic* if there exists a morphism $s : \omega(X) \to \omega(Y)[-1]$ such that $f = s d_X + d_Y s$. We say that s is a *homotopy* from f to g.

We say that f and g are *homotopic* if $f - g$ is null-homotopic.

The *homotopy category* of $\mathcal{C}$, denoted by $\operatorname{Ho}(\mathcal{C})$, is the quotient category of $\operatorname{Comp}(\mathcal{C})$ by the homotopy relation: its objects are complexes of $\mathcal{C}$ and $\operatorname{Hom}_{\operatorname{Ho}(\mathcal{C})}(X, Y) = \operatorname{Hom}_{\operatorname{Comp}(\mathcal{C})}(X, Y)/I$ where I is the sub-k-module of null-homotopic maps.

There is an exact sequence of k-modules

$$\begin{gathered}\operatorname{Hom}_{\mathcal{C}\text{-gr}}(\omega X, \omega Y[-1]) \to \operatorname{Hom}_{\operatorname{Comp}(\mathcal{C})}(X, Y) \to \operatorname{Hom}_{\operatorname{Ho}(\mathcal{C})}(X, Y) \to 0 \\ s \mapsto sd + ds\end{gathered}$$

Note that given $M, N \in \mathcal{C}$ and $i \in \mathbf{Z}$, we have $\operatorname{Hom}_{\operatorname{Ho}(\mathcal{C})}(M, N[i]) = \delta_{i0} \operatorname{Hom}_{\mathcal{C}}(M, N)$.

Remark 3.1. Classically the letter "K" is used instead of "Ho" for the homotopy category. We reserve the use of the letter "K" for the Grothendieck groups.

We say that X is *contractible* if id_X is null-homotopic.

Lemma 3.2. *Let $f : X \to Y$ be a morphism of complexes. Then, the following assertions are equivalent*

- *f is null-homotopic*

- *f factors through the canonical map $X \to \operatorname{cone}(\mathrm{id}_X)$*
- *f factors through the canonical map $\operatorname{cone}(\mathrm{id}_{Y[-1]}) \to Y$*
- *f factors through a contractible complex.*

So, the category $\mathrm{Ho}(\mathcal{C})$ is canonically isomorphic to the k-linear quotient of $\mathrm{Comp}(\mathcal{C})$ by the subcategory of contractible complexes.

We say that two complexes of $\mathcal{C}$ are *homotopy equivalent* if they are isomorphic as objects of $\mathrm{Ho}(\mathcal{C})$. In particular, a complex is contractible if and only if it is homotopy equivalent to 0.

3.4. Hom**-complex.** Let $C, D \in \mathrm{Comp}(\mathcal{C})$. We define a complex $\mathrm{Hom}^\bullet_{\mathcal{C}}(C, C')$ by putting a differential on $\mathrm{Hom}^\bullet_{\mathcal{C}}(\omega C, \omega C')$. Given $f = (f_i)_{i \in \mathbf{Z}}$, we define $d(f)$ by

$$d(f)_i = (-1)^i (d_{C'} f_i - f_i d_C).$$

This provides a functor

$$\mathrm{Hom}^\bullet_{\mathcal{C}}(?, ?') : \mathrm{Comp}(\mathcal{C})^{\mathrm{opp}} \times \mathrm{Comp}(\mathcal{C}) \to \mathrm{Comp}(k\text{-Mod}).$$

PROPOSITION 3.3. *Let $X, Y \in \mathrm{Comp}(\mathcal{C})$. There is a canonical isomorphism $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) \xrightarrow{\sim} H^0(\mathrm{Hom}^\bullet_{\mathcal{C}}(X, Y))$.*

3.5. Cones. Let $f \in \mathrm{Hom}_{\mathrm{Comp}(\mathcal{C})}(X, Y)$. The *cone* of f is the complex

$$\operatorname{cone}(f) = (X[1] \oplus Y, \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}).$$

There are canonical morphisms

$$X \xrightarrow{f} Y \xrightarrow{0 \oplus \mathrm{id}_Y} \operatorname{cone}(f) \xrightarrow{\mathrm{id}_{X[1]} \oplus 0} X[1] \tag{1}$$

3.6. Chainwise split exact sequences. A *chainwise split exact sequence* of $\mathrm{Comp}(\mathcal{C})$ is a sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ of maps of $\mathrm{Comp}(\mathcal{C})$ such that for all $n \in \mathbf{Z}$, the sequence $0 \to X^n \to Y^n \to Z^n \to 0$ is a split short exact sequence.

EXAMPLE 3.4. Given $f \in \mathrm{Hom}_{\mathrm{Comp}(\mathcal{C})}(X, Y)$, the sequence constructed above $0 \to Y \to \operatorname{cone}(f) \to X[1] \to 0$ is a chainwise split exact sequence of $\mathrm{Comp}(\mathcal{C})$.

Consider a chainwise split exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$. Let us construct a morphism $h : Z \to X[1]$. Fix $s^i : Z^i \to Y^i$ such that $g^i s^i = \mathrm{id}$. Note that $d(s) = ds - sd$ defines a morphism of complexes $Z \to Y[1]$. Since $g[1] \circ d(s) = 0$, it follows that $d(s)$ factors as $d(s) = fh$ for some morphism of complexes $h : Z \to X[1]$. The image of h in $\mathrm{Ho}(\mathcal{C})$ depends is independent of the choices.

We define a morphism of complexes $\phi : Z \to \operatorname{cone}(f)$ by $\phi^i = (h^i, s^i)$. The image of ϕ in $\mathrm{Ho}(\mathcal{C})$ is an isomorphism and there is a commutative diagram in $\mathrm{Ho}(\mathcal{C})$

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\
X & \xrightarrow[f]{} & Y & \xrightarrow[(0,\mathrm{id})]{} & \operatorname{cone}(f) & \xrightarrow[(\mathrm{id},0)]{} & X[1]
\end{array}$$

Lemma 3.5. *Consider a chainwise split exact sequence* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$. *Given* $C \in \mathrm{Comp}(\mathcal{C})$, *there is a chainwise split exact sequence of* $\mathrm{Comp}(k\text{-Mod})$

$$0 \to \mathrm{Hom}^\bullet_{\mathcal{C}}(C, X) \to \mathrm{Hom}^\bullet_{\mathcal{C}}(C, Y) \to \mathrm{Hom}^\bullet_{\mathcal{C}}(C, Z) \to 0.$$

Lemma 3.6. *Assume* $\mathcal{C}$ *is an abelian category and* $0 \to X \to Y \to Z \to 0$ *is a chainwise split exact sequence of* $\mathrm{Comp}(\mathcal{C})$. *Then we have a long exact sequence*

$$\cdots H^{i-1}(Z) \to H^i(X) \to H^i(Y) \to H^i(Z) \to \cdots$$

3.7. Distinguished triangles. A *triangle* of $\mathrm{Ho}(\mathcal{C})$ is a sequence $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{h} X_1[1]$. An *isomorphism of triangles* from $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{h} X_1[1]$ to $X'_1 \xrightarrow{f'} X'_2 \xrightarrow{g'} X'_3 \xrightarrow{h'} X'_1[1]$ is the data of $\alpha_i \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X_i, X'_i)$ invertible for $i \in \{1, 2, 3\}$ such that the following diagram commutes:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f} & X_2 & \xrightarrow{g} & X_3 & \xrightarrow{h} & X_1[1] \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_1[1]} \\ X'_1 & \xrightarrow[f']{} & X'_2 & \xrightarrow[g']{} & X'_3 & \xrightarrow[h']{} & X'_1[1] \end{array}$$

A *standard distinguished triangle* of $\mathrm{Ho}(\mathcal{C})$ is the image of a sequence of $\mathrm{Comp}(\mathcal{C})$ of the form $X \xrightarrow{f} Y \xrightarrow{(0,\mathrm{id}_Y)} \mathrm{cone}(f) \xrightarrow{(\mathrm{id}_{X[1]},0)} X[1]$.

Definition 3.7. *A* distinguished triangle *of* $\mathrm{Ho}(\mathcal{C})$ *is a triangle isomorphic to a standard distinguished triangle.*

A key operation on distinguished triangles is the rotation.

Proposition 3.8. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *be a distinguished triangle. Then* $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{f[1]} Y[1]$ *and* $Z[-1] \xrightarrow{h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$ *are distinguished triangles.*

Proposition 3.9. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *be a distinguished triangle of* $\mathrm{Ho}(\mathcal{C})$ *and let* $C \in \mathrm{Ho}(\mathcal{C})$. *There is a long exact sequence*

$$\cdots \mathrm{Hom}(C, X[i]) \xrightarrow{\mathrm{Hom}(C,f[i])} \mathrm{Hom}(C, Y[i]) \xrightarrow{\mathrm{Hom}(C,g[i])} \mathrm{Hom}(C, Z[i]) \xrightarrow{\mathrm{Hom}(C,h[i])} \mathrm{Hom}(C, X[i+1]) \to \cdots$$

4. Abelian categories

4.1. Structure.

4.1.1. *Definitions.* Let $\mathcal{C}$ be an additive category. Let $f : c_1 \to c_2$ be a map in $\mathcal{C}$. Assume that f has a kernel and a cokernel: we have maps $f_1 : \ker f \to c_1$ and $f_2 : c_2 \to \mathrm{coker}\, f$. Assume further that f_1 has a cokernel and f_2 has a kernel. We have a canonical map $\mathrm{coker}\, f_1 \to \ker f_2$.

Definition 4.1. *A category is* $\mathcal{C}$ *abelian if*

- *it is additive*
- *every map has a kernel and a cokernel*
- *given* $f : c_1 \to c_2$, *the canonical map* $\mathrm{coker}\, f_1 \to \ker f_2$ *is an isomorphism, where* $f_1 : \ker f \to c_1$ *and* $f_2 : c_2 \to \mathrm{coker}\, f$ *are the canonical maps.*

Example 4.2. Let R be a ring. Then the category R-Mod of (left) R-modules is an abelian category.

If $\mathcal{C}$ is an abelian category, then $\mathcal{C}^{\text{opp}}$ is an abelian category. A kernel in $\mathcal{C}$ becomes a cokernel in $\mathcal{C}^{\text{opp}}$.

Let $\mathcal{A}$ be an abelian category.

We say that a sequence of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ is *exact* if $gf = 0$ and the canonical map $\operatorname{im} f \to \ker g$ is an isomorphism.

We say that a sequence $\cdots \to C^i \xrightarrow{f^i} C^{i+1} \to \cdots$ is *exact* if the subsequences $C^i \xrightarrow{f^i} C^{i+1} \xrightarrow{f^{i+1}} C^{i+2}$ are exact for all i.

4.1.2. *Subcategories and functors.* Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor.

We say that F is

- *left exact* if it preserves kernels
- *right exact* if it preserves cokernels
- *exact* if it is left and right exact.

Note that F is exact if and only if it send a short exact sequence to a short exact sequence.

Let $\mathcal{A}$ be an abelian category and $\mathcal{A}'$ a subcategory. We say that $\mathcal{A}'$ is an *abelian subcategory* of $\mathcal{A}$ if $\mathcal{A}'$ is an abelian category and if given any map f of $\mathcal{A}'$, then the kernel and the cokernel of f in $\mathcal{A}'$ coincide with the kernel and cokernel taken in $\mathcal{A}$. This is equivalent to the requirement that the inclusion functor $\mathcal{A}' \to \mathcal{A}$ is exact.

Let R be a ring. Denote by R-mod the full subcategory of R-Mod of finitely generated R-modules. We say that R is *left noetherian* if every left ideal of R is finitely generated as an R-module. The following result is easy.

Proposition 4.3. *A ring R is left noetherian if and only if* R-mod *is an abelian subcategory of* R-Mod.

Exercise 4.1. Let R be a ring. We say that an R-module M is *finitely presented* if there exists an exact sequence $P \to P' \to M \to 0$, where P and P' are finitely generated free R-modules. Denote by R-Mfp the full subcategory of R-Mod of finitely presented R-modules.

We say that R is *left coherent* if every finitely generated left ideal of R is finitely presented as an R-module.

Show that R is left coherent if and only if R-Mfp is an abelian subcategory of R-Mod.

The following theorem, for which we give no proof, shows that every (small) abelian category is equivalent to a full abelian subcategory of the category of modules over a ring.

Theorem 4.4 (Freyd-Mitchell embedding). *Let $\mathcal{A}$ be a small abelian category. There exists a ring R and an exact and fully faithful functor $\mathcal{A} \to R$*-Mod.

4.1.3. Ext *as long exact sequences.* Let $\mathcal{A}$ be a k-linear abelian category.

Let $M, N \in \mathcal{A}$ and $n \geq 1$. An *n-extension* of M by N is an exact sequence

$$0 \to N \to N_n \to \cdots \to N_1 \to M \to 0.$$

Consider a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & N & \longrightarrow & N_n & \longrightarrow & \cdots & \longrightarrow & N_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & N & \longrightarrow & N'_n & \longrightarrow & \cdots & \longrightarrow & N'_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are exact. We say that the sequence $0 \to N \to N_n \to \cdots \to N_1 \to M \to 0$ is linked to $0 \to N \to N'_n \to \cdots \to N'_1 \to M \to 0$. This generates an equivalence relation on n-extensions of M by N (we take the symmetric and transitive closure of the relation above).

DEFINITION 4.5. $\mathrm{Ext}^n_{\mathcal{A}}(M,N)$ *is defined to be the set of equivalence classes of n-extensions of M by N.*

There is a 0 element, given by the equivalence class of $0 \to N \xrightarrow{(\mathrm{id},0)} N \oplus M \xrightarrow{(0,\mathrm{id})} M \xrightarrow{\mathrm{id}} M \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} M \to 0$. reprsentant une flche $N_{i+1} \to N_i$ nulle.

Let us endow $\mathrm{Ext}^n_{\mathcal{A}}(M,N)$ with a k-module stucture.

Let $0 \to N \to N_n \to \cdots \to N_1 \to M \to 0$ and $0 \to N \to N'_n \to \cdots \to N'_1 \to M \to 0$ be two exact sequences.

Let $N''_1 = N_1 \times_M N'_1$ and $N''_n = N_n \sqcup_N N'_n$. The sequence

$$0 \to N \to N''_n \to N_{n-1} \oplus N'_{n-1} \to \cdots \to N_2 \oplus N'_2 \to N''_1 \to M \to 0$$

is exact, and it is defined to be the sum of the sequences $0 \to N \to N_n \to \cdots \to N_1 \to M \to 0$ and $0 \to N \to N'_n \to \cdots \to N'_1 \to M \to 0$. One checks easily that this induces an operation on $\mathrm{Ext}^n_{\mathcal{A}}(M,N)$.

Let $\alpha \in k$ and let $0 \to N \xrightarrow{f_n} N_n \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} N_1 \xrightarrow{f_0} M \to 0$. We define a new exact sequence $0 \to N \xrightarrow{\alpha f_n} N_n \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} N_1 \xrightarrow{f_0} M \to 0$. This induces an action of k on $\mathrm{Ext}^n_{\mathcal{A}}(M,N)$ and one checks easily that these endow $\mathrm{Ext}^n_{\mathcal{A}}(M,N)$ with a structure of k-module.

EXERCISE 4.2. Show that the short exact sequence $0 \to N \to N_1 \to M \to 0$ is equivalent to $0 \to N \to N'_1 \to M \to 0$ if and only if there is an isomorphism $N_1 \xrightarrow{\sim} N'_1$ making the following diagram commutative

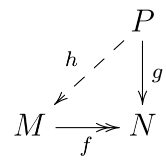
$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & N_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & N & \longrightarrow & N'_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Show that $\mathrm{Ext}^1_{\mathcal{A}}(M,N) = 0$ if and only if every short exact sequence $0 \to N \to N_1 \to M \to 0$ is split.

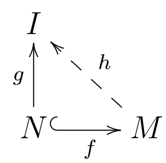
EXERCISE 4.3. Let $\mathcal{A}$ be an abelian category such that $\mathrm{Ext}^n_{\mathcal{A}}(M,N) = 0$ for all objects M, N. Show that $\mathrm{Ext}^i_{\mathcal{A}}(M,N) = 0$ for all objects M, N and all $i \geq n$.

4.1.4. *Projective and injective objects.*

Definition 4.6. *An object P of $\mathcal{A}$ is* projective *if given any surjection $f : M \twoheadrightarrow N$ and any map $g : P \to N$, there is a map $h : P \to M$ such that $g = fh$.*



An object I of $\mathcal{A}$ is injective *if given any injection $f : N \hookrightarrow M$ and any map $g : N \to I$, there is a map $h : M \to I$ such that $g = hf$.*



Note that an object P of $\mathcal{A}$ is projective if and only if it is injective as an object of $\mathcal{A}^{\mathrm{opp}}$.

Definition 4.7. *Let S be a family of projective objects of $\mathcal{A}$. We say that S* generates *$\mathcal{A}$ if every object of $\mathcal{A}$ is isomorphic to a quotient of a (possibly infinite) direct sum of objects of S. We say that an object P of $\mathcal{A}$ is a* progenerator *of $\mathcal{A}$ if P is projective and $\{P\}$ generates $\mathcal{A}$.*

Let S be a family of injective objects of $\mathcal{A}$. We say that S cogenerates *$\mathcal{A}$ if every object of $\mathcal{A}$ is isomorphic to a subobject of a (possibly infinite) product of objects of S.*

4.1.5. *Module categories.* Let R be a ring.

The following proposition has an easy proof.

Proposition 4.8. *An R-module is projective if and only if it is a direct summand of a free R-module.*

*The R-module R is a progenerator for the category R-*Mod.

Proposition 4.9. *Let $\mathcal{A}$ be an abelian category. The category $\mathcal{A}$ is equivalent to R-*Mod *for some ring R if and only if $\mathcal{A}$ admits arbitrary direct sums and $\mathcal{A}$ has a progenerator.*

Proposition 4.10. *Given R a ring, the category R-*Mod *has enough injectives.*

Proof. Given M an R-module, let $M^\circ = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$. This is an R^{opp}-module, and $?^\circ$ defines a functor R-Mod $\to R^{\mathrm{opp}}$-Mod. If M is projective, then M° is injective. If f is surjective, then f° is injective.

Note also that there is a canonical injection $M \hookrightarrow (M^\circ)^\circ$ for $M \in R$-Mod.

Let $M \in R$-Mod. Fix a surjection $P \twoheadrightarrow M^\circ$ of R^{opp}-modules, with P projective. We obtain an injection $(M^\circ)^\circ \to P^\circ$, hence an injection $M \hookrightarrow P^\circ$ with P° an injective R-module. $\square$