

4. Let  $f(x)$  be a non-negative function defined on  $[0,1]$  (that is,  $f(x) \geq 0$  for all  $x$  in  $[0,1]$ ). Suppose that for any choice of  $n$  distinct numbers  $x_1, x_2, \dots, x_n$  in  $[0,1]$

$$f(x_1) + f(x_2) + \dots + f(x_n) \leq 2,345,678.$$

Prove that the number of  $x$  in  $[0,1]$  for which  $f(x) > 0$  is finite or countable.

Proof: For each positive integer  $m$ , let

$$X_m = \{x \text{ in } [0,1], f(x) > 1/m\}.$$

Suppose there are at least  $n$  numbers in  $X_m$ . Then, by the assumption of the problem

$$f(x_1) + f(x_2) + \dots + f(x_n) \leq 2,345,678.$$

Furthermore, since each  $x_i$  is in  $X_m$ ,

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq (1/m + 1/m + \dots + 1/m) = n/m.$$

Combining these two inequalities we obtain  $n/m \leq 2,345,678$ . That is

$$n \leq 2,345,678 * m.$$

This says that the number of elements in  $X_m$  is finite.

Consequently, if  $X = \{x \text{ in } [0, 1] : f(x) > 0\}$  then

$$X = \bigcup_{m=1}^{\infty} X_m,$$

being the union of a countable number of sets, each of which is finite, is countable.