3. Let

$$A = \{m + n\pi : m = 0, \pm 1, \pm 2, \ \ldots; \ \ n = 0, \pm 1, \pm 2, \ldots\}$$

and

$$a = \inf(A \cap (0, \infty))$$
.

Prove that a = 0.

Proof: Suppose that a > 0. Then, given any ε > 0 there is an element x in A such that

(1) a ≤ x < a + ε.

One of the hard parts of this proof is in showing that no element of the form $m + n\pi$, $m + n\pi$ > 0 can be the inf of $A \cap (0, \infty)$.

Suppose that the element $x + y\pi$ = a = inf $A \cap (0, \infty)$. Let $m + n\pi$ be any element with $m + n\pi$ > 0. Then there is an integer k such that ka ≤ $m + n\pi$ < (k+1)a. Thus 0 ≤ $m + n\pi$ -ka < a. This implies that $m + n\pi$ = ka; every element in $A \cap (0, \infty)$ is a multiple of a = $x + y\pi$,

But $x + 1 + y\pi$ is in $A \cap (0, \infty)$ and $x + 1 + y\pi$ is not an integral multiple of $x + y\pi$. (The fact that π is irrational is neede here)

Statement (1) can now be rephrased as: Suppose that a > 0. Then, given any ε > 0 there is an element x in A such that

(2) a < x < a + ε.

To continue, set ε = 1/n, where n is chosen so that 1/n < a. Then there is an element $x + y\pi$ such that

a < $x + y\pi$ < a + 1/n.

Next set ε = $x + y\pi$ -a . Then there is an element $u + v\pi$ such that

a < $u + v\pi$ < a +ε = $x + y\pi$ < 1/n .

Subtracting, we get

-1/n < ($x + y\pi$)-($u + v\pi$)< < 1/n.

But ($x + y\pi$)-($u + v\pi$) > 0 so , since n was selected to have 1/n < a,

0 < ($x + y\pi$)-($u + v\pi$)< 1/n < a.

This contradicts the definition of a, a = inf($A \cap (0, \infty)$).

The contradiction was to the assumption a > 0. Thus a = 0.