9. Give a direct proof that f(x) = sin(x) is uniformly continuous on $(-\infty,\infty)$. That is, given an $\varepsilon > 0$ find a $\delta > 0$ such that

if $|x - y| < \delta$ then $|\sin(x) - \sin(y)| < \varepsilon$.

Proof: Recall first that $|\sin(h)| < |h|$. for small h.

Recall next

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$
$$\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$$
$$\Rightarrow$$
$$\sin(A+B) - \sin(A-B) = 2\sin(B)\cos(A)$$

Set

$$A+B = x$$
$$A-B = y$$

Then 2A = x+y and 2B = x-y, and

$$\sin(A+B)-\sin(A-B) = \sin(x)-\sin(y) = 2\sin((x-y)/2)\cos((x+y)/2).$$

That is,

$$|\sin(x)-\sin(y)| = |2\sin((x-y)/2)\cos((x+y)/2)| \le |x - y|.$$

So, given $\varepsilon > 0$, set $\delta = \varepsilon$. Then, for any x and y,

$$|x - y| < \delta = \varepsilon \Rightarrow |\sin(x) - \sin(y)| < \varepsilon,$$

So, sin(x) is uniformly continuous on $(-\infty,\infty)$.