5. Let $\{a_n\}$ be a convegent sequence, converging to a. Let

$$b_n = \frac{a_1 + a_2 + \ \dots\ + a_n}{n}.$$

Prove that $\{b_n\}$ converges to a.

Proof: First, let M be a bound for $\{a_n\}$, so $|a_n| \le M$ for all n. Second, choose N so that $|a_n - a| \le \varepsilon$ for all $n \ge N$.

Now, for P > N

(1) $$|a_1 + \ \dots\ + a_{N-1}| \le (N-1)M$$

(2) $$\begin{aligned} &|a_N + \quad + a_P - (P-(N-1))a| \le \\ &|a_N - a + \ a_P - a| < (P - N + 1)\varepsilon \end{aligned}$$

Then

$$\begin{aligned} &|b_P - a| = \\ &\left|\frac{a_1 + \ + a_{N-1} + a_N + \quad + a_P}{P} - a\right| \le \\ &\left|\frac{a_1 + \ + a_{N-1}}{P}\right| + \frac{|a_N + \quad + a_P - Pa}{P}| \le \\ &\frac{(N-1)M}{P} + \frac{|a_N + \quad + a_P - (P-N+1)a - (N-1)a}{P}| \le \\ &\frac{(N-1)M}{P} + \frac{|a_N + \quad + a_P - (P-N+1)a|}{P} + \frac{|(N-1)a|}{P} \end{aligned}$$

Using (1) and (2) we get

$$|b_P - a| \le \frac{(N-1)M}{P} + \frac{(P-N+1)\varepsilon}{P} + + \frac{|(N-1)a|}{P}.$$

So choose P > N so that

$$\frac{(N-1)M}{P} < \varepsilon, \quad \frac{(P-N+1)}{P} < 2, \quad \frac{|(N-1)a|}{P} < \varepsilon$$

Then

$$|b_P - a| < \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon.$$

Since ε was arbitrary, it follows that $\{b_n\}$ converges to a.