Review, Problem 7

Prove that

$$(f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x)$$

where, as usual, $f^{(k)}(x)$ is the k^{th} derivative of f(x).

Proof: When n = 1 the formula is just the product rule for derivatives:

$$(f(x)g(x))^{(1)} = f^{(1)}(x)g(x) + f(x)g^{(1)}(x)$$

So suppose the result is true for n. Then, going on to the inductive step:

$$(f(x)g(x))^{(n+1)} = \frac{d((f(x)g(x))^{(n)}}{dx} =$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{df^{(n-k)}(x)g^{(k)}(x)}{dx} =$$

$$\sum_{k=0}^{n} \binom{n}{k} (f^{(n+1-k)}(x)g^{(k)}(x) + f^{(n-k)}(x)g^{(k+1)}(x)) =$$

$$\sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)}(x)g^{(k)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k+1)}(x)$$

Now, the very last sum can be re-indexed as

$$\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k+1)}(x) = \sum_{k=0}^{n} \binom{n}{k+1-1} f^{(n+1-(k+1))}(x)g^{(k+1)}(x) =$$

$$\sum_{k=1}^{n+1} \binom{n}{k-1} f^{(n+1-k)}(x)g^{(k)}(x)$$

So, assembling, we have

$$(f(x)g(x))^{(n+1)} =$$

$$\sum_{k=0}^{n}\binom{n}{k}f^{(n+1-k)}(x)g^{(k)}(x)+\sum_{k=1}^{n+1}\binom{n}{k-1}f^{(n+1-k)}(x)g^{(k)}(x)=$$

$$f^{(n+1)}(x)g(x)+\sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k-1}\right]f^{(n+1-k)}(x)g^{(k)}(x)+f(x)g^{(n+1)}(x)$$

$$\sum_{k=1}^{n+1}\binom{n+1}{k}f^{(n+1-k)}(x)g^{(k)}(x)$$

which completes the proof of the inductive step